The Hanle effect
Beyond the single scattering approximation.

Single scattering approximation

$$\boldsymbol{I}_{\rm obs} \simeq \oint \int R(x, \boldsymbol{\Omega}, x', \boldsymbol{\Omega}'; \boldsymbol{B}) \boldsymbol{I}_{\rm inc}(\tau, x', \boldsymbol{\Omega}') \, dx' \, \frac{d\Omega'}{4\pi}$$

$$|Q/I| \propto (1-\mu^2) \times \text{depol. collisions} \times \text{Hanle depol.} \times \text{aniso. rad. field}$$

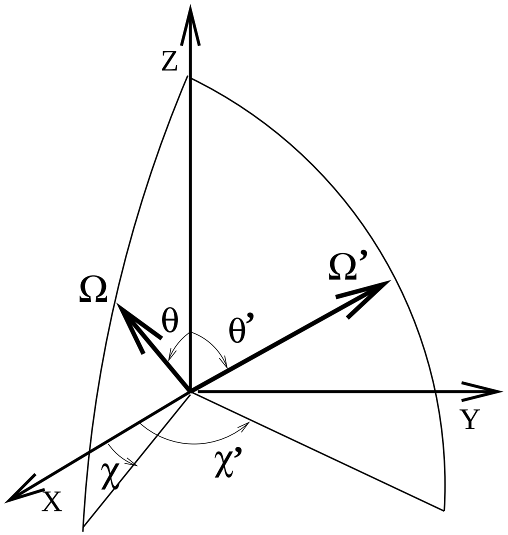
Transfer equation for the Stokes vector

$$\mu \frac{\partial \boldsymbol{I}}{\partial \tau} = \varphi(x)[\boldsymbol{I} - \boldsymbol{S}]$$

$$\boldsymbol{S}(\tau, \boldsymbol{\Omega}) = \boldsymbol{G}(\tau) + \oint \int \varphi(x') P_{\rm H}(\boldsymbol{\Omega}, \boldsymbol{\Omega}'; \boldsymbol{B}) \boldsymbol{I}(\tau, x', \boldsymbol{\Omega}') \, dx' \, \frac{d\Omega'}{4\pi}$$

$\boldsymbol{I} = (I, Q, U, V) = \{I_i\}, \quad i = 0, \ldots, 3, \quad \boldsymbol{S} = \{S_i\}, \quad \boldsymbol{G} = \{G_i\},$
complete frequency redistribution, $\quad P_{\rm H}$: Hanle phase matrix
$\boldsymbol{G}(\tau)$ unpolarized $\Rightarrow$ $G_0 \neq 0$ other $G_i = 0, i = 1, 2, 3$

Decomposition in irreducible spherical components



$$S_i(\tau, \mathbf{\Omega}) = \sum_{KQ} \mathcal{T}^K_Q(i, \mathbf{\Omega}) S^K_Q(\tau)$$

$$I_i(\tau, x, \mathbf{\Omega}) = \sum_{KQ} \mathcal{T}^K_Q(i, \theta, \chi) I^K_Q(\tau, x, \cos\theta)$$

$$G^0_0(\tau) = G_0(\tau) \quad \text{other} \quad G^K_Q = 0$$

$$I(\tau, x, \mathbf{\Omega}) = I^0_0(\tau, x, \mu) + \frac{1}{2\sqrt{2}}(3\cos^2\theta - 1)\, I^2_0(\tau, x, \mu)$$

$$+ \quad \text{terms depending on } : I^2_Q,\ \theta,\ \mathrm{e}^{Q\mathrm{i}\,\chi} \quad \text{with} \quad Q = \pm 1, \pm 2$$

$$Q(\tau, x, \mathbf{\Omega}) = -\frac{3}{2\sqrt{2}}(1 - \cos^2\theta)\, I^2_0(\tau, x, \mu)$$

$$+ \quad \text{terms depending on } : I^2_Q,\ \theta,\ \mathrm{e}^{Q\mathrm{i}\,\chi} \quad \text{with} \quad Q = \pm 1, \pm 2$$

$$U(\tau, x, \mathbf{\Omega}) = \sqrt{3}\sin\theta\, \frac{\mathrm{i}}{2}[(I^2_1)^* \mathrm{e}^{-\mathrm{i}\,\chi} - I^2_1 \mathrm{e}^{\mathrm{i}\,\chi}] + \sqrt{3}\cos\theta\, \frac{\mathrm{i}}{2}[(I^2_2)^* \mathrm{e}^{-2\mathrm{i}\,\chi} - I^2_2 \mathrm{e}^{2\mathrm{i}\,\chi}]$$

$$\text{Remark} : I^0_0 \gg I^2_0 \gg I^2_Q$$

Stokes parameters decomposition

$$I(\tau, x, \mathbf{\Omega}) = I_0^0(\tau, x, \mu) + \frac{1}{2\sqrt{2}}(3\cos^2\theta - 1)\, I_0^2(\tau, x, \mu)$$
$$-\sqrt{3}\cos\theta\sin\theta\,(I_1^{\mathrm{x}2}\cos\chi - I_1^{\mathrm{y}2}\sin\chi)$$
$$+\frac{\sqrt{3}}{2}(1-\cos^2\theta)(I_2^{\mathrm{x}2}\cos 2\chi - I_2^{\mathrm{y}2}\sin 2\chi)$$
$$Q(\tau, x, \mathbf{\Omega}) = -\frac{3}{2\sqrt{2}}(1-\cos^2\theta)\, I_0^2(\tau, x, \mu)$$
$$-\sqrt{3}\cos\theta\sin\theta\,(I_1^{\mathrm{x}2}\cos\chi - I_1^{\mathrm{y}2}\sin\chi)$$
$$-\frac{\sqrt{3}}{2}(1+\cos^2\theta)(I_2^{\mathrm{x}2}\cos 2\chi - I_2^{\mathrm{y}2}\sin 2\chi)$$
$$U(\tau, x, \mathbf{\Omega}) = \sqrt{3}\sin\theta\,(I_1^{\mathrm{x}2}\sin\chi + I_1^{\mathrm{y}2}\cos\chi)$$
$$+\sqrt{3}\cos\theta\,(I_2^{\mathrm{x}2}\sin 2\chi + I_2^{\mathrm{y}2}\cos 2\chi)$$

$$\text{notation}: I_Q^{xK} = \Re[I_Q^K(\tau, x, \mu)], \quad I_Q^{yK} = \Im[I_Q^K(\tau, x, \mu)]$$

New non-LTE Transfer equation

$$\mu \frac{\partial \boldsymbol{\mathcal{I}}}{\partial \tau} = \varphi(x)[\boldsymbol{\mathcal{I}} - \boldsymbol{\mathcal{S}}]$$

$$\boldsymbol{\mathcal{S}}(\tau) = \boldsymbol{\mathcal{G}}(\tau) + \hat{N}(\boldsymbol{B}) \int \frac{1}{2} \int_{-1}^{+1} \hat{\Psi}(\mu) \boldsymbol{\mathcal{I}}(\tau, x, \mu)\, d\mu\, dx$$

for the formal vectors $\boldsymbol{\mathcal{S}} = \{S_Q^K\}, \quad \boldsymbol{\mathcal{I}} = \{I_Q^K\}, \quad \boldsymbol{\mathcal{G}} = \{G_Q^K\}$

Linear polarization only $\Rightarrow \quad K = 0, 2 \quad \Rightarrow \quad$ 6 components I_Q^K

$$\hat{N}(\boldsymbol{B}) = \begin{bmatrix} \times & 0 & 0 & 0 & 0 & 0 \\ 0 & \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times & \times \end{bmatrix} \qquad \hat{\Psi}(\mu) = \begin{bmatrix} 1 & \times & 0 & 0 & 0 & 0 \\ \times & \otimes & 0 & 0 & 0 & 0 \\ 0 & 0 & \triangle & 0 & 0 & 0 \\ 0 & 0 & 0 & \triangle & 0 & 0 \\ 0 & 0 & 0 & 0 & \diamondsuit & 0 \\ 0 & 0 & 0 & 0 & 0 & \diamondsuit \end{bmatrix}$$

$$\hat{N} = \{(1 - \epsilon^{(K)})\mathcal{N}_{QQ'}^K\}, \times = (1 - \epsilon^{(2)})\mathcal{N}_{00}^2 \qquad \hat{\Psi} = \{\Psi_Q^{KK'}\}, \times = \Psi_0^{20}, \otimes = \Psi_0^{22}$$

Scalar non-LTE equation for each I^K_Q

- Stokes I not affected by polarization and magnetic field

$$I(\tau, x, \boldsymbol{\Omega}) \simeq I^0_0(\tau, x, \mu)$$

$I^0_0(\tau, x, \mu)$ solution of a standard scalar transfer equation

- Non-LTE equation for I^2_0 (Stokes Q) $\Leftarrow$ keep only terms in $\hat{N}$ and $\hat{\Psi}$

$$S^2_0(\tau) \simeq (1-\epsilon^{(2)})\mathcal{N}^2_{00}(\boldsymbol{B}) \left[\bar{J}^2_0(\tau) + \int \frac{1}{2} \int_{-1}^{+1} \Psi^{22}_0(\mu) I^2_0(\tau, x, \mu)\, d\mu\, dx \right]$$

$$\bar{J}^2_0(\tau) = \int \frac{1}{2} \int_{-1}^{+1} \Psi^{20}_0(\mu) I^0_0(\tau, x, \mu)\, d\mu\, dx$$

$\bar{J}^2_0(\tau)$: anisotropy of Stokes I $\Leftarrow$ $\Psi^{20}_0(\mu) = \frac{1}{2\sqrt{2}}(3\mu^2 - 1)$

- Integral equation for $S^2_0(\tau)$

$$S^2_0(\tau) \simeq (1-\epsilon^{(2)})\mathcal{N}^2_{00}(\boldsymbol{B}) \left[\bar{J}^2_0(\tau) + \int_0^T K_{22}(\tau-\tau') S^2_0(\tau')\, d\tau' \right]$$

$\epsilon^{(2)} = \frac{\epsilon+\delta^{(2)}}{1+\epsilon+\delta^{(2)}}$: rate of depolarizing collisions; $\int_{-\infty}^{+\infty} K_{22}(\tau)\, d\tau = \frac{7}{10}$

Neuman series expansion

$$S(\tau) = G(\tau) + (1-\epsilon)\int_0^T K(\tau-\tau')S(\tau')\,d\tau' \qquad \text{(scalar)}$$

Expansion

$$S^{(0)}(\tau) = G(\tau)$$

$$S^{(1)}(\tau) = G(\tau) + (1-\epsilon)\int_0^T K(\tau-\tau')G(\tau')\,d\tau'$$

$$\dots\dots\dots$$

$$S^{(n)}(\tau) = G(\tau) + \sum_{k=1}^{n}(1-\epsilon)^k \int_0^T d\tau_1\,K(\tau-\tau_1)\int_0^T d\tau_2\,K(\tau_1-\tau_2)$$

$$\dots\int_0^T d\tau_k\,K(\tau_{k-1}-\tau_k)G(\tau_k)$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

Convergence

- ϵ very small (10^{-4}) norm $\int K(\tau)\,d\tau = 1$ $\Rightarrow$ slow convergence when T large
- for $S_0^2(\tau) : \epsilon^{(2)} \simeq 10^{-1}$, norm $\int K_{22}(\tau)\,d\tau = 0.7$ $\Rightarrow$ a few terms sufficient

Approximate expressions for S^2_0 and S^2_Q

- Stokes Q

$$S^2_0(\tau) \simeq (1-\epsilon^{(2)})\,\mathcal{N}^2_{00}(\boldsymbol{B}(\tau))\bar{J}^2_0(\tau) \qquad \Rightarrow \textit{single scattering}$$

$$+\,(1-\epsilon^{(2)})^2\,\mathcal{N}^2_{00}(\boldsymbol{B}(\tau))\int_0^T K_{22}(\tau-\tau')\,\mathcal{N}^2_{00}(\boldsymbol{B}(\tau'))\bar{J}^2_0(\tau')\,d\tau'$$

$$+\;\ldots\ldots$$

Two level atom with unpolarized ground level

$$\mathcal{N}^2_{00}(\boldsymbol{B}) = 1 - 3\cos^2\theta_{\rm B}\,H^2\left[\frac{\cos^2\theta_{\rm B}}{1+H^2} + \frac{\sin^2\theta_{\rm B}}{1+4H^2}\right]$$

$H = 2\pi\nu_{\rm L}/\Gamma$: Hanle efficiency factor $\theta_{\rm B}$: inclination of the magnetic field vector

"Microturbulence"(constant B, isotropic distribution) : $\langle\mathcal{N}^2_{00}(B)\rangle = 1 - \frac{2}{5}\left[\frac{H^2}{1+H^2} + \frac{4H^2}{1+4H^2}\right]$

- Stokes U

Keep only the terms (second column) in the matrix $\hat{N}(\boldsymbol{B})$

$$S^2_Q(\tau) \simeq S^2_0(\tau)\mathcal{N}^2_{0Q}(\boldsymbol{B})/\mathcal{N}^2_{00}(\boldsymbol{B})$$

Concluding remarks (qualitative)

- Scattering expansion applicable to the calculation of Stokes I
- Useful for lines with small optical depth (a few unity)
- Scattering expansion applicable to the calculation of linear polarization
- Useful for lines with small optical depth (a few unity) may be also for lines with very large optical depth
- Scattering expansion applicable to partial frequency redistribution

Multipolar expansion of the Hanle phase matrix

Atmospheric reference frame

$$[P_{\rm H}]_{ij}(\boldsymbol{\Omega},\boldsymbol{\Omega}';\boldsymbol{B}) = \sum_{KQ} \mathcal{T}^K_Q(i,\boldsymbol{\Omega}) \sum_{Q'} \mathcal{N}^K_{QQ'}(\boldsymbol{B})(-1)^{Q'} \mathcal{T}^K_{-Q'}(j,\boldsymbol{\Omega}')$$

$i,j = 0,\ldots,3$ Stokes parameter index, $K = 0,1,2$, $-K \le Q \le +K$

$\mathcal{T}^K_Q(i,\boldsymbol{\Omega})$: spherical tensors for polarimetry (LL 2004 book)

$\mathcal{T}^K_Q(i,\boldsymbol{\Omega}) = \tilde{\mathcal{T}}^K_Q(i,\theta)\,{\rm e}^{{\rm i}\,Q\chi}$ with $\tilde{\mathcal{T}}^K_Q(i,\theta)$ trigonometric functions of θ

Examples :

$$\mathcal{T}^2_0(I,\boldsymbol{\Omega}) = \frac{1}{2\sqrt{2}}(3\cos^2\theta - 1); \quad \mathcal{T}^2_0(Q,\boldsymbol{\Omega}) = -\frac{3}{2\sqrt{2}}\sin^2\theta; \quad \mathcal{T}^2_0(U,\boldsymbol{\Omega}) = 0$$

Matrix notation : $P_{\rm H}(\boldsymbol{\Omega},\boldsymbol{\Omega}';\boldsymbol{B}) = \hat{T}^{\rm out}(\boldsymbol{\Omega})\hat{N}(\boldsymbol{B})\hat{T}^{\rm in}(\boldsymbol{\Omega}')$

$\hat{T}^{\rm out}(\boldsymbol{\Omega})$: 4×9 matrix; $\hat{N}(\boldsymbol{B})$: 9×9 matrix; $\hat{T}^{\rm in}(\boldsymbol{\Omega}) = [\hat{T}^{\rm out}]^\dagger(\boldsymbol{\Omega})$: 9×4 matrix

Matrix $\hat{\Psi}(\mu) = \hat{T}^{\rm out}(\mathbf{\Omega})\hat{T}^{\rm in}(\mathbf{\Omega})$

$$\Psi_Q^{KK'}(\mu) = \sum_{i=0}^{i=3} (-1)^Q \mathcal{T}_{-Q}^{K}(i,\mathbf{\Omega})\mathcal{T}_{Q}^{K'}(i,\mathbf{\Omega})$$

Linear polarisation

$$\begin{bmatrix}
1 & \frac{1}{2\sqrt{2}}(3\mu^2-1) & 0 & 0 & 0 & 0 \\
\frac{1}{2\sqrt{2}}(3\mu^2-1) & \frac{1}{4}(5-12\mu^2+9\mu^4) & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{3}{4}(1-\mu^2)(1+2\mu)^2 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{3}{4}(1-\mu^2)(1+2\mu)^2 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{3}{8}(1+\mu^2)^2 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{3}{8}(1+\mu^2)^2
\end{bmatrix}$$

Spherical Tensors $\mathcal{T}^K_Q(i, \mathbf{\Omega})$ Reference angle $\gamma = 0$

Linear polarisation

K, Q	$i = 0$	$i = 1$	$i = 2$
0,0	1	0	0
2,0	$\frac{1}{2\sqrt{2}}(3\cos^2\theta - 1)$	$-\frac{3}{2\sqrt{2}}\sin^2\theta$	0
2,-1	$\frac{\sqrt{3}}{2}\sin\theta\cos\theta\,\mathrm{e}^{-\mathrm{i}\,\chi}$	$\frac{\sqrt{3}}{2}\sin\theta\cos\theta\,\mathrm{e}^{-\mathrm{i}\,\chi}$	$-\mathrm{i}\,\frac{\sqrt{3}}{2}\sin\theta\,\mathrm{e}^{-\mathrm{i}\,\chi}$
2,1	$-\frac{\sqrt{3}}{2}\sin\theta\cos\theta\,\mathrm{e}^{\mathrm{i}\,\chi}$	$-\frac{\sqrt{3}}{2}\sin\theta\cos\theta\,\mathrm{e}^{\mathrm{i}\,\chi}$	$-\mathrm{i}\,\frac{\sqrt{3}}{2}\sin\theta\,\mathrm{e}^{\mathrm{i}\,\chi}$
2,-2	$\frac{\sqrt{3}}{4}\sin^2\theta\,\mathrm{e}^{-2\mathrm{i}\,\chi}$	$-\frac{\sqrt{3}}{4}(1+\cos^2\theta)\,\mathrm{e}^{-2\mathrm{i}\,\chi}$	$2\mathrm{i}\,\frac{\sqrt{3}}{4}\cos\theta\,\mathrm{e}^{-2\mathrm{i}\,\chi}$
2,2	$\frac{\sqrt{3}}{4}\sin^2\theta\,\mathrm{e}^{2\mathrm{i}\,\chi}$	$-\frac{\sqrt{3}}{4}(1+\cos^2\theta)\,\mathrm{e}^{2\mathrm{i}\,\chi}$	$-2\mathrm{i}\,\frac{\sqrt{3}}{4}\cos\theta\,\mathrm{e}^{2\mathrm{i}\,\chi}$