Scattering line polarization and Hanle effect in weakly inhomogeneous atmospheres

Beaulieu 2007

The talk, briefly:

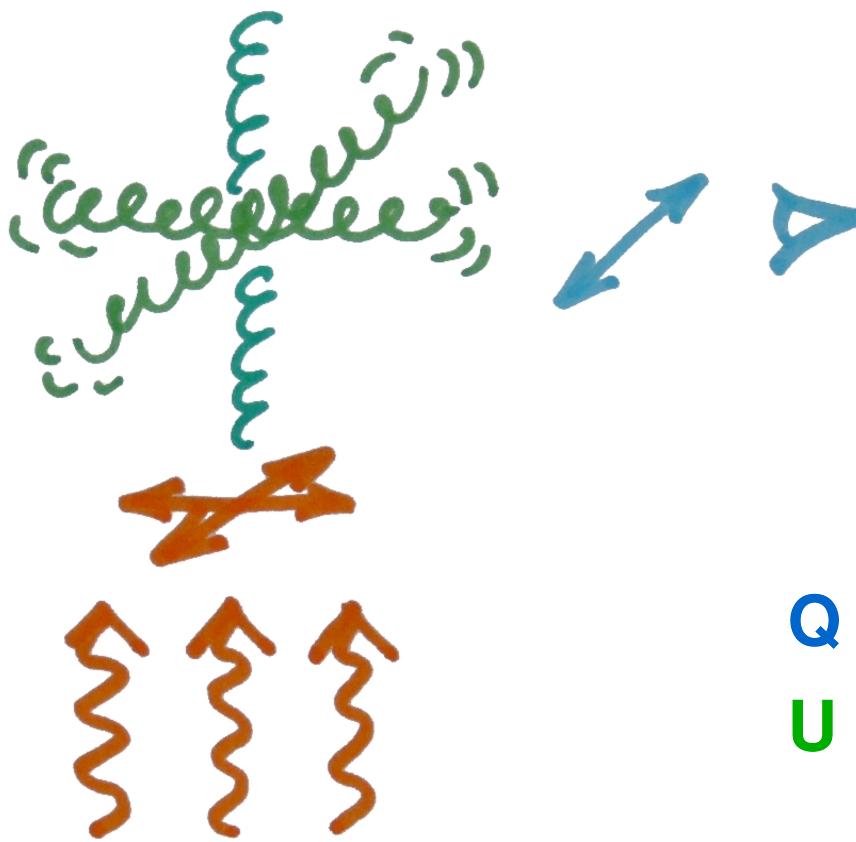
A gentle introduction: scattering line polarization, the Hanle effect and horizontal transfer effects in pictures

Equations, equations, equations...

The method: linearization, harmonic analysis, numerical solution

Conclusions: it works

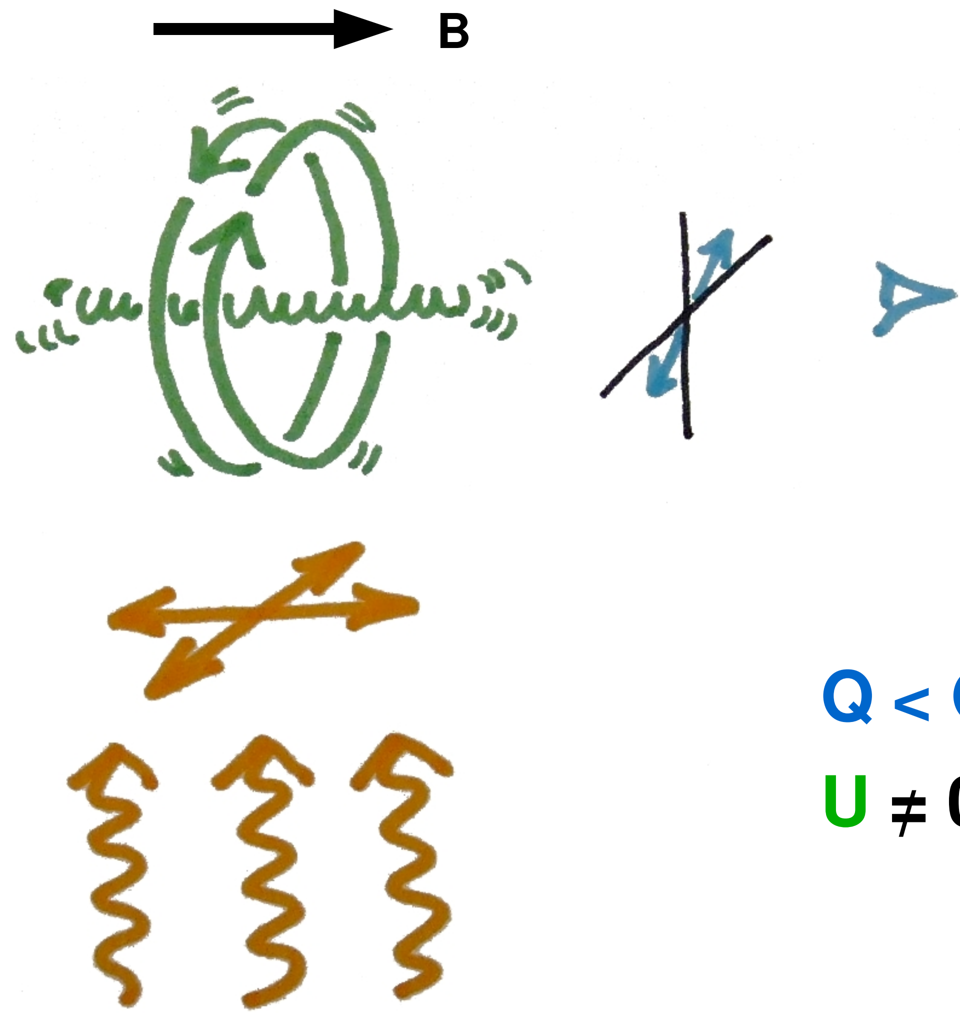
Scattering line polarization: the classical (springs) model



Q > 0

U = 0

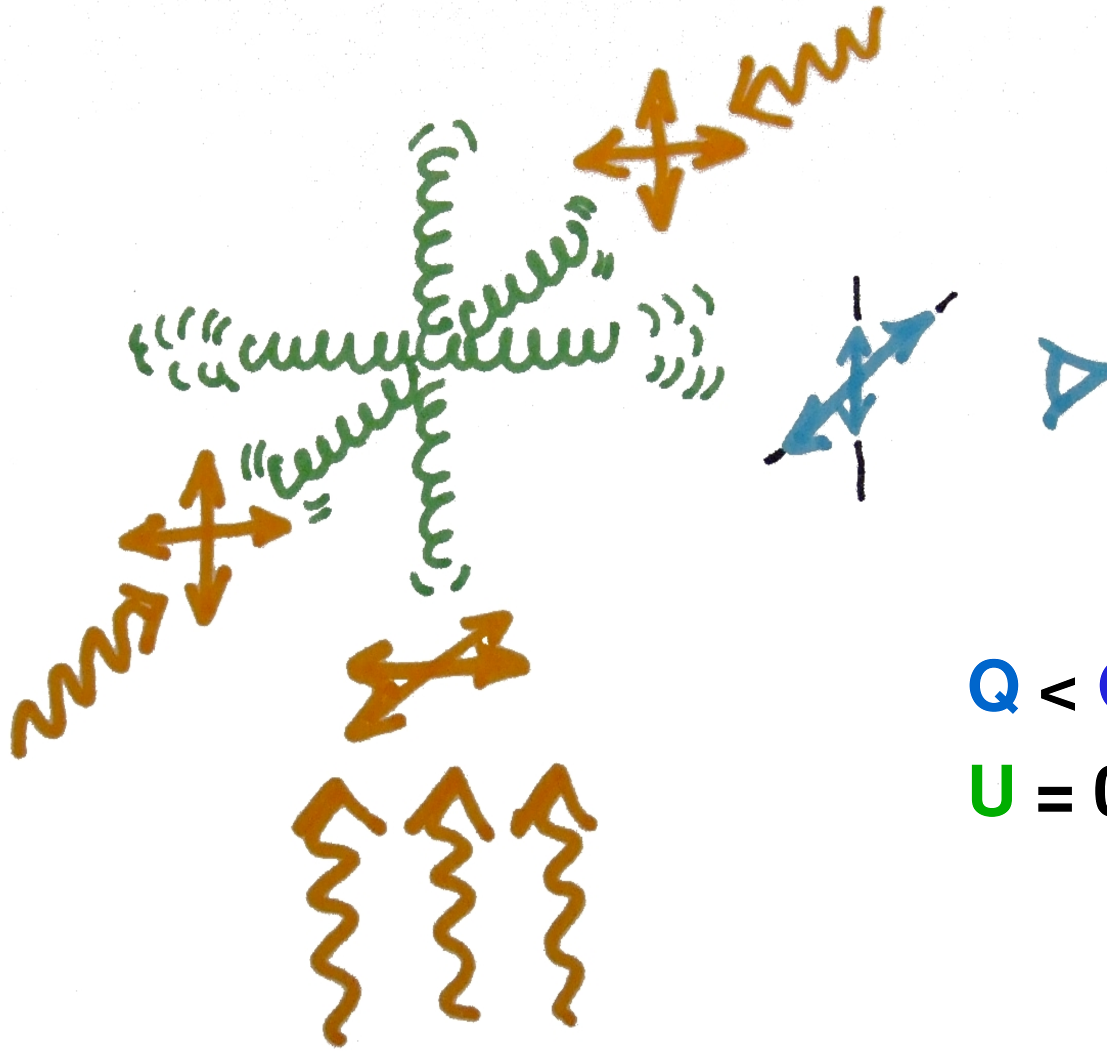
The Hanle effect: the classical (spring & hops) model



Q < Q0

U ≠ 0

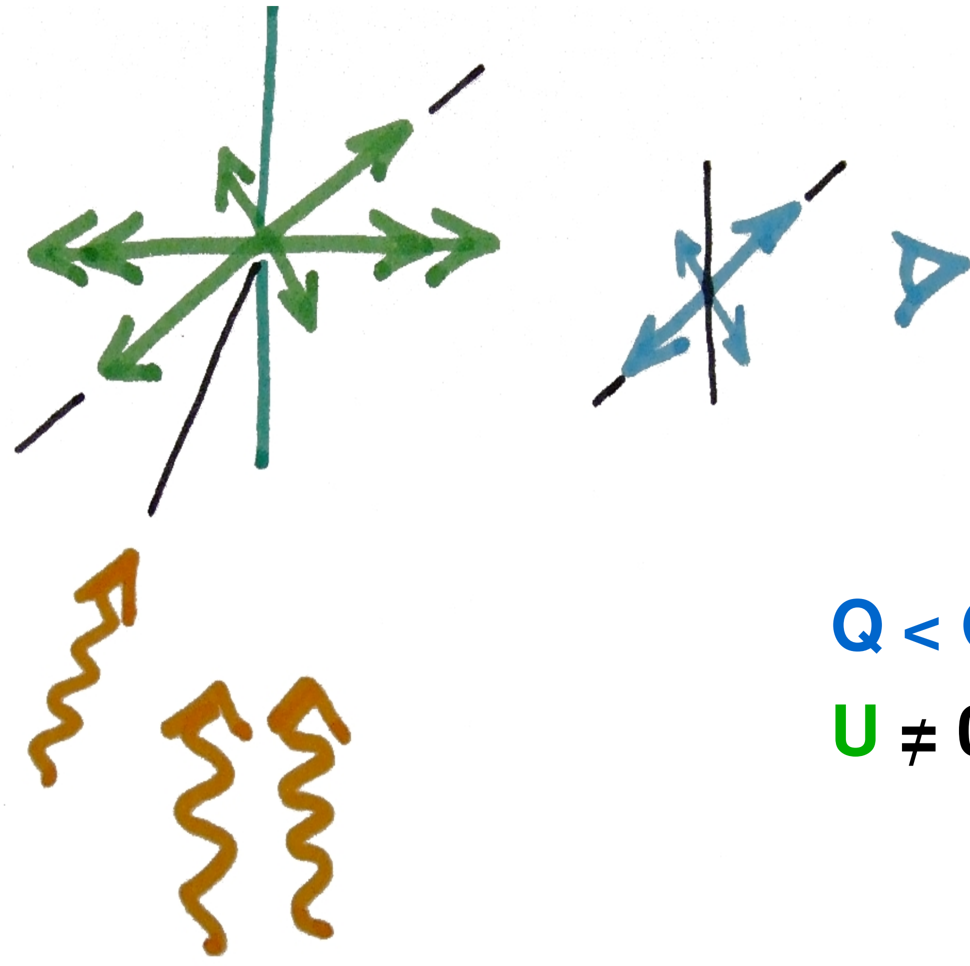
Scattering line polarization: horizontal illumination



Q < Q0

U = 0

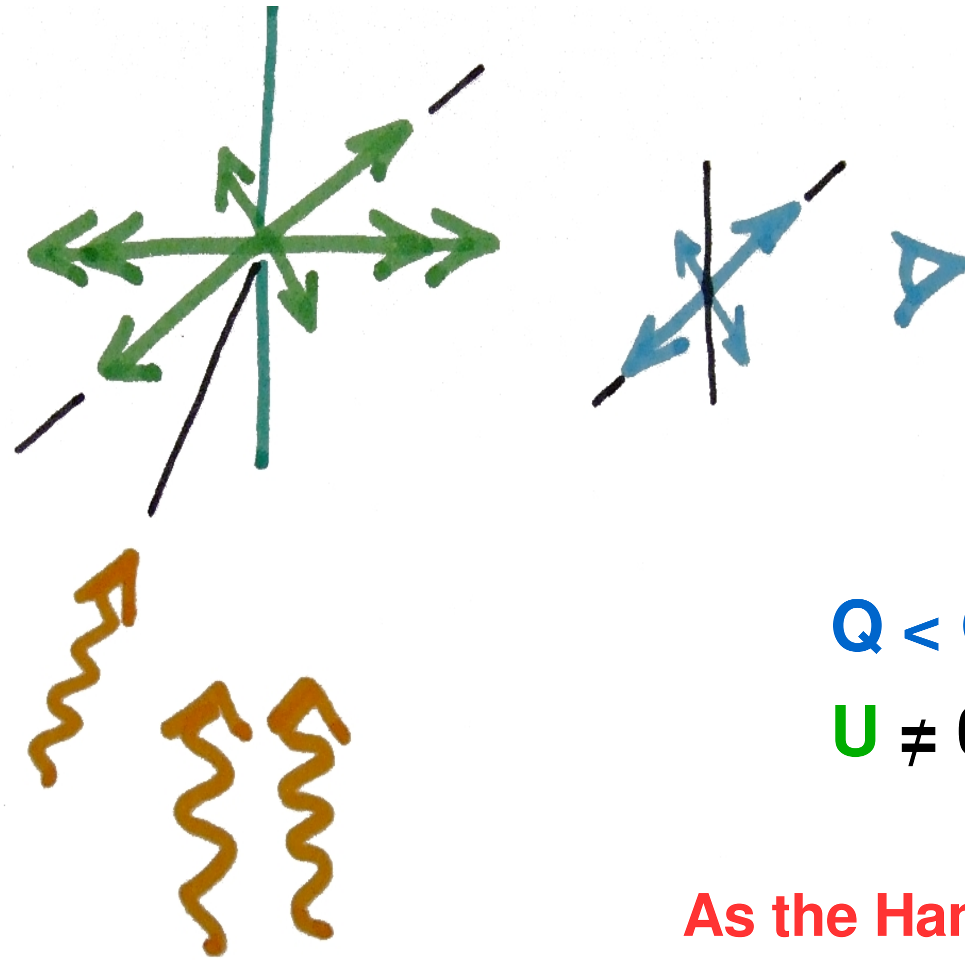
Scattering polarization: oblique illumination



$Q < Q0$

$U \neq 0$

Scattering polarization: oblique illumination



Q < Q0

U ≠ 0

As the Hanle effect!

Scattering line polarization: radiative transfer equations

$$\frac{d}{d\tau} I = I - S_I$$

$$\frac{d}{d\tau} Q = Q - S_Q$$

$$\frac{d}{d\tau} U = U - S_U$$

(unpolarized lower level)

Scattering line polarization: radiative transfer equations

$$S_I = S_0^0 + \frac{1}{2\sqrt{2}}(3\mu^2-1)\,S_0^2 - \sqrt{3}\,\mu\sqrt{1-\mu^2}\left(\cos\chi\,\tilde{S}_1^2 - \sin\chi\,\hat{S}_1^2\right)$$
$$+ \frac{\sqrt{3}}{2}(1-\mu^2)\left(\cos 2\chi\,\tilde{S}_2^2 - \sin 2\chi\,\hat{S}_2^2\right)$$

$$S_Q = \frac{3}{2\sqrt{2}}(\mu^2-1)\,S_0^2 - \sqrt{3}\,\mu\sqrt{1-\mu^2}\left(\cos\chi\,\tilde{S}_1^2 - \sin\chi\,\hat{S}_1^2\right)$$
$$- \frac{\sqrt{3}}{2}(1+\mu^2)\left(\cos 2\chi\,\tilde{S}_2^2 - \sin 2\chi\,\hat{S}_2^2\right)$$

$$S_U = \sqrt{3}\sqrt{1-\mu^2}\left(\sin\chi\,\tilde{S}_1^2 + \cos\chi\,\hat{S}_1^2\right)$$
$$+ \sqrt{3}\,\mu\left(\sin 2\chi\,\tilde{S}_2^2 + \cos 2\chi\,\hat{S}_2^2\right)$$

Scattering line polarization: statistical equilibrium

$$S_0^0 = (1-\varepsilon) J_0^0 + \varepsilon B$$

$$[1+\delta(1-\varepsilon)] S_0^2 = (1-\varepsilon) J_0^2$$

$$[1+\delta(1-\varepsilon)] \tilde{S}_1^2 = (1-\varepsilon) \tilde{J}_1^2$$

$$[1+\delta(1-\varepsilon)] \hat{S}_1^2 = -(1-\varepsilon) \hat{J}_1^2$$

$$[1+\delta(1-\varepsilon)] \tilde{S}_2^2 = (1-\varepsilon) \tilde{J}_2^2$$

$$[1+\delta(1-\varepsilon)] \hat{S}_2^2 = -(1-\varepsilon) \hat{J}_2^2$$

Radiation field tensors

$$J^0_0 = \oint \frac{d\Omega}{4\pi}\, I$$

$$J^2_0 = \oint \frac{d\Omega}{4\pi}\, \frac{1}{2\sqrt{2}} \left[(3\mu^2 - 1) I + 3(\mu^2 - 1) Q \right]$$

$$\tilde{J}^2_1 = \oint \frac{d\Omega}{4\pi}\, \frac{\sqrt{3}}{2} \sqrt{1-\mu^2} \left[-\mu (I+Q) \cos\chi + \sin\chi\, U \right]$$

$$\hat{J}^2_1 = \oint \frac{d\Omega}{4\pi}\, \frac{\sqrt{3}}{2} \sqrt{1-\mu^2} \left[-\mu (I+Q) \sin\chi - \cos\chi\, U \right]$$

$$\tilde{J}^2_2 = \oint \frac{d\Omega}{4\pi}\, \frac{\sqrt{3}}{2} \left\{ \left[\frac{1}{2}(1-\mu^2) I - \frac{1}{2}(1+\mu^2) Q \right] \cos 2\chi + \mu U \sin 2\chi \right\}$$

$$\hat{J}^2_2 = \oint \frac{d\Omega}{4\pi}\, \frac{\sqrt{3}}{2} \left\{ \left[\frac{1}{2}(1-\mu^2) I - \frac{1}{2}(1+\mu^2) Q \right] \sin 2\chi - \mu U \cos 2\chi \right\}$$

Planeparallel geometry: We know how to solve it (either using of statistical tensors, or with equivalent phase matrix formalism)

2D & 3D geometries are hard, but it can be done nonetheless (Manso Sainz 2002, Manso Sainz & Trujillo Bueno 1998)

Here we consider a restricted case (easier, more transparent, more fun): 2D case with weak fluctuations

1. **Linear problem**
2. **3D problem can be reduced to this 2D case**

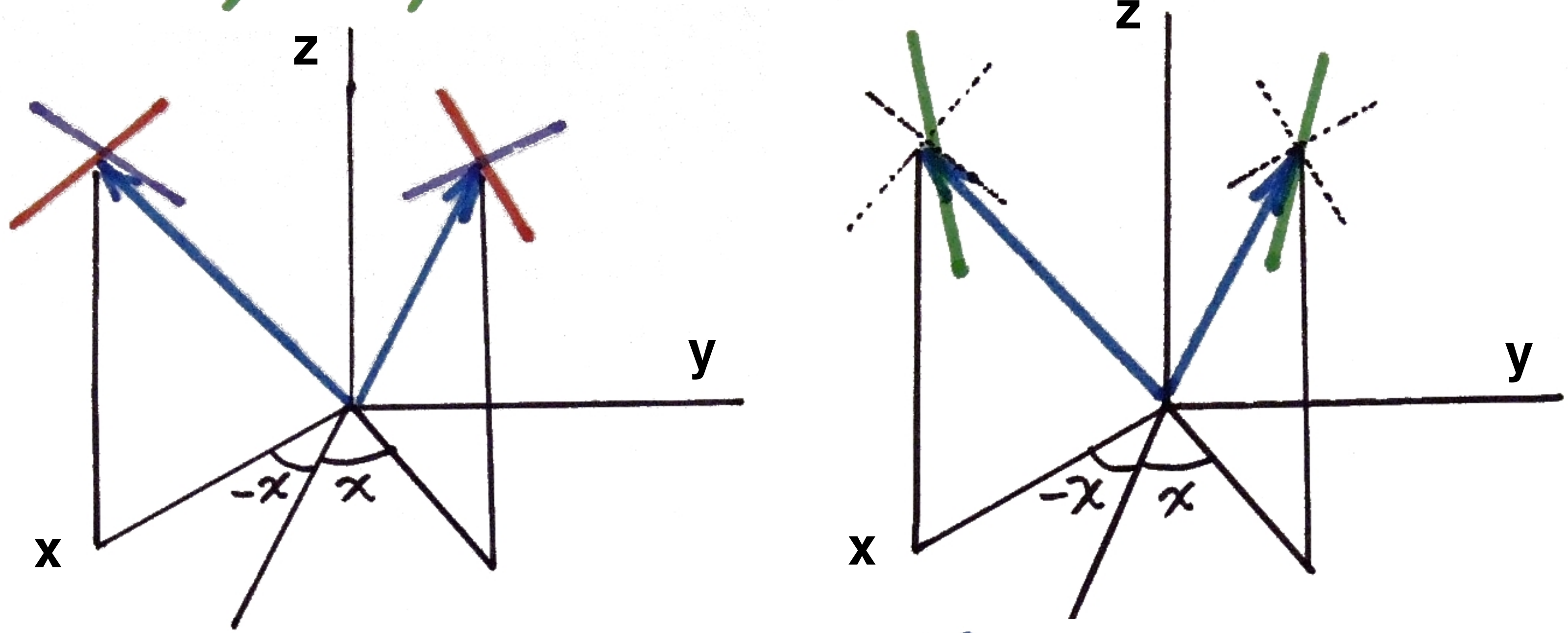
A note on symmetry of a 2D scattering atmosphere

Let y be the invariant direction (i.e., things vary along x), then a reflexion on the y-z plane leaves the system invariant. Therefore:

$$I_{\mu\chi} = I_{\mu-\chi}$$

$$Q_{\mu\chi} = Q_{\mu-\chi}$$

$$U_{\mu\chi} = -U_{\mu-\chi}$$



Therefore: $\hat{j}^{z}_{1} = \hat{j}^{z}_{2} = 0$

Radiation field tensors

$$J^0_0 = \oint \frac{d\Omega}{4\pi} I$$

$$J^2_0 = \oint \frac{d\Omega}{4\pi} \frac{1}{2\sqrt{2}} \left[(3\mu^2-1) I + 3(\mu^2-1) Q\right]$$

$$\tilde{J}^2_1 = \oint \frac{d\Omega}{4\pi} \frac{\sqrt{3}}{2} \sqrt{1-\mu^2} \left[-\mu (I+Q) \cos\chi + \sin\chi \, U\right]$$

$$\hat{J}^2_1 = \oint \frac{d\Omega}{4\pi} \frac{\sqrt{3}}{2} \sqrt{1-\mu^2} \left[-\mu (I+Q) \sin\chi \boxed{- \cos\chi \, U}\right]$$

$$\tilde{J}^2_2 = \oint \frac{d\Omega}{4\pi} \frac{\sqrt{3}}{2} \left\{\left[\frac{1}{2}(1-\mu^2) I - \frac{1}{2}(1+\mu^2) Q\right] \cos 2\chi + \mu U \sin 2\chi\right\}$$

$$\hat{J}^2_2 = \oint \frac{d\Omega}{4\pi} \frac{\sqrt{3}}{2} \left\{\left[\frac{1}{2}(1-\mu^2) I - \frac{1}{2}(1+\mu^2) Q\right] \sin 2\chi \boxed{- \mu U \cos 2\chi}\right\}$$

The RTE for intensity:

$$\frac{d}{ds} I = -\kappa (I - S_I)$$

Assuming *weak* horizontal fluctuations:

$$\kappa(x,y,z) = \bar{\kappa}(z) + \delta\kappa(x,y,z)$$

$$S_I(x,y,z) = \bar{S}_I(z) + \delta S_I(x,y,z)$$

$$I(x,y,z) = \bar{I}(z) + I(x,y,z)$$

$$\frac{\delta\kappa}{\bar{\kappa}}, \frac{\delta S_I}{\bar{S}_I}, \frac{\delta I}{\bar{I}} \ll 1$$

Then:

$$\frac{d}{ds} \delta I = -\bar{\kappa} (\delta I - \delta S_I^{\mathrm{df}})$$

$$\delta S_I^{\mathrm{df}} = \delta S_I - \frac{\delta\kappa}{\bar{\kappa}} (\bar{I} - \bar{S}_I)$$

The RTE for Stokes Q:

$$\frac{d}{ds}Q = -\kappa(Q - S_Q)$$

Assuming *weak* horizontal fluctuations:

$$\kappa(x,y,z) = \bar{\kappa}(z) + \delta\kappa(x,y,z)$$

$$S_Q(x,y,z) = \bar{S}_Q(z) + \delta S_Q(x,y,z) \qquad \frac{\delta\kappa}{\bar{\kappa}}, \frac{\delta S_Q}{\bar{S}_Q}, \frac{\delta Q}{\bar{Q}} \ll 1$$

$$Q(x,y,z) = \bar{Q}(z) + \delta Q(x,y,z)$$

Then:

$$\frac{d}{ds}\delta Q = -\bar{\kappa}(\delta Q - \delta S_Q^{eff})$$

$$\delta S_Q^{eff} = \delta S_Q - \frac{\delta\kappa}{\bar{\kappa}}(\bar{I} - \bar{S}_Q)$$

The RTE for intensity:

$$\frac{d}{ds}U = -\kappa\left(U - S_U\right)$$

Assuming *weak* horizontal fluctuations:

$$\kappa(x,y,z) = \bar{\kappa}(z) + \delta\kappa(x,y,z)$$

$$S_U(x,y,z) = \bar{S}(z) + \delta S(x,y,z)$$

$$U(x,y,z) = \bar{U}(z) + \delta U(x,y,z)$$

$$\frac{\delta\kappa}{\bar{\kappa}}, \frac{\delta S_U}{\bar{S}_U}, \frac{\delta U}{\bar{U}} \ll 1$$

Then:

$$\frac{d}{ds}\delta U = -\bar{\kappa}\left(\delta U - \delta S_U^{\mathrm{eff}}\right)$$

$$\delta S_U^{\mathrm{eff}} = \delta S_U - \frac{\delta\kappa}{\bar{\kappa}}\left(\bar{U} - \bar{S}_U\right)$$

Harmonic Analysis

Let's assume:

$$B(x,y,z) = \bar{B}(z) + \Delta B(z) \cos Kx$$

$$\kappa(x,y,z) = \bar{\kappa}(z)\left[1 + \alpha \cos Kx\right]$$

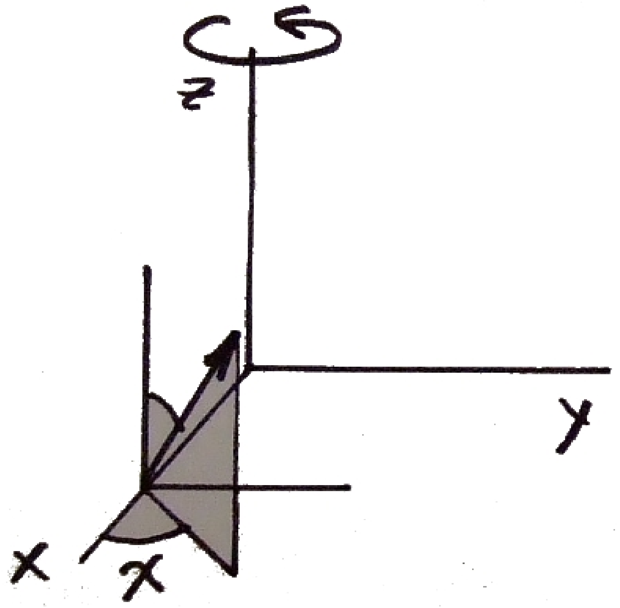
Then, due to linearity, every Stokes parameter I, Q & U satisfies:

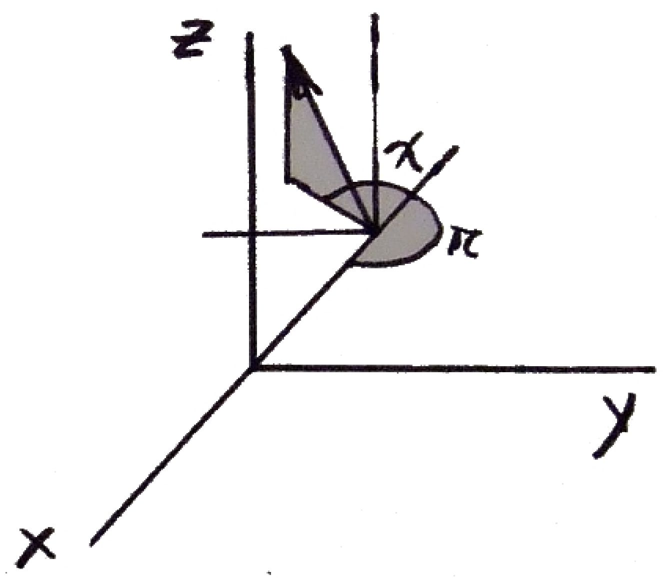
$$\delta I_i = \Delta_1 I_i(z) \cos Kx + \Delta_2 I_i(z) \sin Kx$$

A note on the symmetries of the sinusoidal fluctuations

$$I_{\mu\chi}(x) = I_{\mu\,\chi+\pi}(-x)$$

Dem.:





Therefore:

$$\delta I = \Delta_1 I \cos Kx + \Delta_2 I \sin Kx$$

$$\delta I_{\mu x}(x) = \delta I_{\mu x + \pi}(-x)$$

$$\boxed{\begin{aligned} \Delta_1 I_{\mu x} &= \Delta_1 I_{\mu x+\pi} \\ \Delta_2 I_{\mu x} &= -\Delta_2 I_{\mu x+\pi} \end{aligned}}$$

(and analogously for Q and U)

Therefore:

$$\int_0^{2\pi} dx\,\Delta_2 I_x = \int_0^{\pi} dx\,\Delta_2 I_x + \int_{\pi}^{2\pi} dx\,\Delta_2 I_x$$

$$= \int_0^{\pi} dx\left(\Delta_2 I_x + \Delta_2 I_{x+\pi}\right)$$

$$= \int_0^{\pi} dx\left(\Delta_2 I_x - \Delta_2 I_x\right) \equiv 0$$

$$\int_0^{2\pi} dx\,\cos x\,\Delta_1 I_x = \int_0^{\pi} dx\left[\cos x\,\Delta_1 I_x + \cos(x+\pi)\,\Delta_1 I_{x+\pi}\right]$$

$$= \int_0^{\pi} dx\left[\cos x\,\Delta_1 I_x - \cos x\,\Delta_1 I_x\right] \equiv 0$$

$$\int_0^{2\pi} dx\,\sin x\,\Delta_1 I_x \equiv 0 \qquad \int_0^{2\pi} \cos 2x\,\Delta_1 I_x \equiv 0$$

$$\int_0^{2\pi} dx\,\sin 2x\,\Delta_2 I_x \equiv 0$$

Hence, for the sinusoidal fluctuation:

$$\delta J_0^0 = \Delta J_0^0(z) \cos Kx$$

$$\delta J_0^2 = \Delta J_0^2(z) \cos Kx$$

$$\delta \tilde{J}_1^2 = \Delta \tilde{J}_1^2(z) \sin Kx$$

$$\delta \tilde{J}_2^2 = \Delta \tilde{J}_2^2(z) \cos Kx$$

The 2D transfer equation for the fluctuations:

$$\boxed{\frac{d}{ds}\delta I = -\bar{\kappa}(\delta I - \delta S)}$$

$$\frac{d}{ds} = \mu\frac{\partial}{\partial z} + \lambda\frac{\partial}{\partial x}$$

$$\mu = \cos\theta \qquad \lambda = \sin\theta\cos\chi$$

Two coupled (pseudo) 1D transfer equations for the amplitudes:

$$\frac{d}{d\tau}\Delta_1 I = \Delta_1 I - \left[\Delta_1 S - \frac{\lambda k}{\bar{\kappa}}\Delta_2 I\right]$$

$$\frac{d}{d\tau}\Delta_2 I = \Delta_2 I - \left[\Delta_2 S + \frac{\lambda k}{\bar{\kappa}}\Delta_1 I\right]$$

$$d\tau = -\bar{\kappa}\,\frac{dz}{\mu}$$

Derivation:

$$\frac{d}{ds}\delta I = -\bar{k}(\delta I - \delta S)$$

$$\left(\mu\frac{\partial}{\partial z} + \lambda\frac{\partial}{\partial x}\right)(\Delta_1 I \cos kx + \Delta_2 I \sin kx)$$

$$= -\bar{k}\,(\Delta_1 I \cos kx + \Delta_2 I \sin kx$$

$$- \Delta_1 S \cos kx - \Delta_2 S \sin kx\,)$$

$$\mu\frac{\partial}{\partial z}\Delta_1 I \cos kx + \mu\frac{\partial}{\partial z}\Delta_2 I \sin kx$$

$$-\lambda \Delta_1 I\, k \sin kx + \lambda k \Delta_2 I \cos kx$$

$$= -\bar{k}\Delta_1 I \cos kx - \bar{k}\Delta_2 I \sin kx + \bar{k}\Delta_1 S \cos kx$$

$$+ \bar{k}\Delta_2 S \sin kx$$

Identify cos & sin terms.

QED

Numerical method of solution

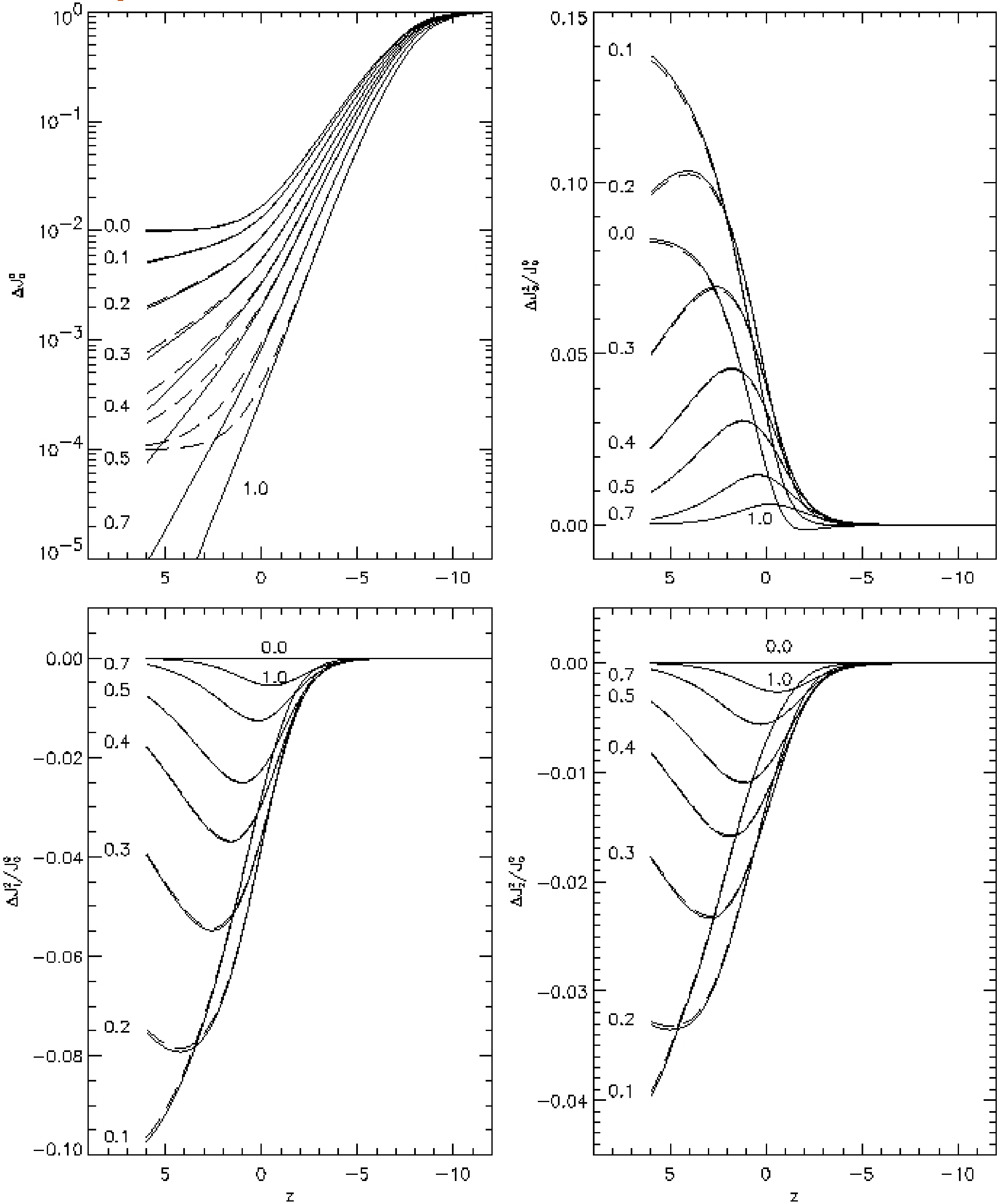
Formal solution based on short-characteristics integration (but for two coupled equations)

Angular quadrature: Gaussian for inclination (as always), ***and*** trapezoidal rule for azimuth

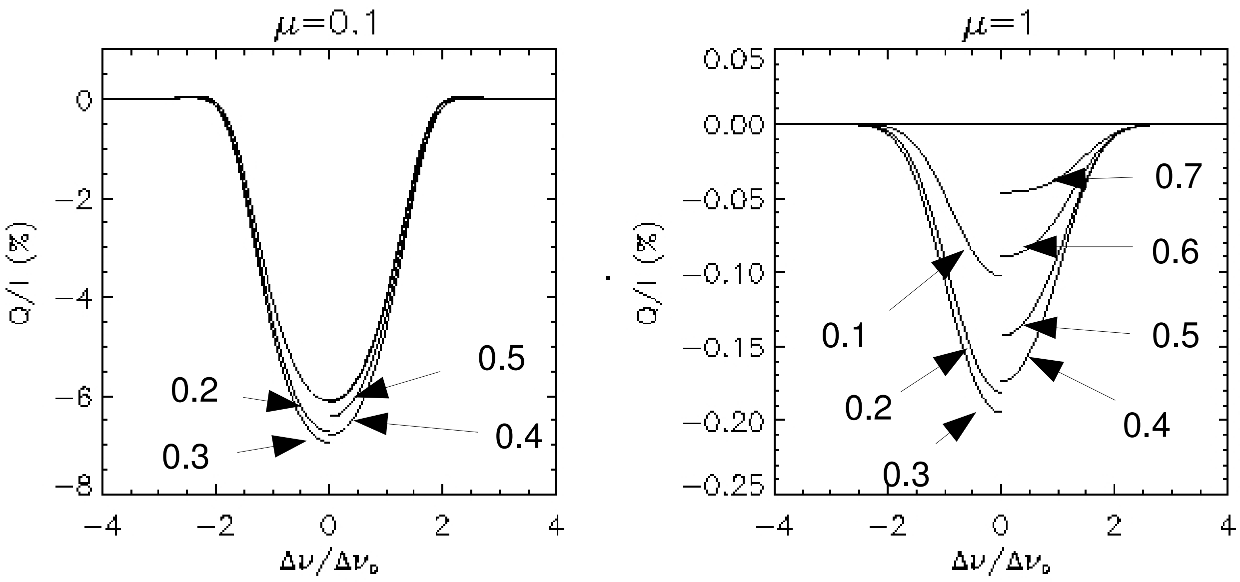
Iterative method: variation on ALI or GS including inversion of a tridiagonal matrix

More details: soon

Amplitudes of the radiation field tensor fluctuations



Emergent profiles along the sinusoidal fluctuations



$\Delta B(z)/\bar{B}(z)$ = 0.1

Summarizing...

As long as we keep linear:

A 3D problem reduces to several 2D problems (Fourier transform on the horizontal plane + rotations of radiation tensors)

A 2D problem reduces to 2 coupled plane-parallel problems

The 1D problem is easily (fast and accurate) solved numerically

So, now what?

Further generalizations:

Magnetic field (Hanle effect) ***easy***

Perturbation of other parameters (other than B and k) ***easy***

From normal Zeeman triplet to transitions with lower level atomic polarization ***ha!***