Stability of uniformly rotating configurations of point vortices

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It is well-known that two-dimensional Euler equations for inviscid fluid can be written in terms of the stream function $\psi(x,y,t)$ and the z-component $\zeta(x,y,t)$ of vorticity vector:

$$\frac{\partial\zeta}{\partial t}+\frac{\partial\psi}{\partial y}\frac{\partial\zeta}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial\zeta}{\partial y}=0,\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=-\zeta. \tag{1}$$

This system admits a weak solution in form of N point vortices in an infinite plane:

$$\begin{aligned}\zeta(x,y,t)&=\sum_{\alpha=1}^{N}\Gamma_\alpha\delta(x-x_\alpha(t))\delta(y-y_\alpha(t)),\\ \psi(x,y,t)&=-\frac{1}{4\pi}\sum_{\alpha=1}^{N}\Gamma_\alpha\ln[(x-x_\alpha(t))^2+(y-y_\alpha(t))^2],\end{aligned} \tag{2}$$

provided that positions $Z_\alpha(t)=X_\alpha(t)+Y_\alpha(t),\alpha=1,.....,N$ of α-vortex satisfy the system of N ordinary differential equations

$$\frac{dz_\alpha^*}{dt}=\frac{\Gamma}{2\pi i}{\sum_{\beta=1}^{N}}'\frac{1}{z_\alpha-z_\beta},\quad \alpha=1,.....,N\ , \tag{3}$$

where the asterisk denotes complex conjugation, and the prime on the summation indicates omission of the singular term $\beta=\alpha$.

The vortex-atom model introduced by W. Thomson in 1867 and especially developed by J.J. Thomson in 1883 based up on brilliant experiments of A.M. Mayer on floating magnets [1] leads to search and study of uniformly rotating configurations of N identical point vortices of intensity $\Gamma_\alpha=\Gamma$. Simple equilibria such as regular polygon are known since classical studies of J.J. Thomson [2] and T. Havelock [3]. A major numerical exploration was undertaken by Campbell & Ziff [4] resulting in what is known as Los Alamos Catalog. Some of these vortex arrays were observed experimentally [5] in Helium. Numerical studies by Aref and co-authors [1,6] revealed a great number of nonsymmetrical relative equilibria configurations.

If we are looking for the system of uniformly rotating with some constant angular velocity ω point vortices of equal intensity $\Gamma_\alpha=2\pi\omega$, the substitution

$z_\alpha(t) = Z_\alpha e^{i\omega t}$ leads to the system of nonlinear algebraic equations with complex constant Z_α:

$$Z_\alpha^* = \sum_{\beta=1}^{N}{}' \frac{1}{Z_\alpha - Z_\beta}, \alpha = 1...N, \quad \alpha = 1,....., N. \tag{4}$$

The main goal of the present talk is to find all such configurations and to analyze their Routh and Lyapunov stability.

We used the approach developed in [1,6]. In addition to configurations presented in there works we have found some new ones (Fig.1) for $N = 9$ and $N = 10$.

We analyzed the trajectories of motion of the systems of point vortices without and with small initial linear perturbation. In such a way, we presented that two of four invariant of motion remained constant.

The detailed analysis of stability is also conducted for Routh's asymmetric configurations of point vortices. In addition, although majority from them appear unsteady in relation to small linear perturbations, such vortex systems play an important role in formation of new vortex models.

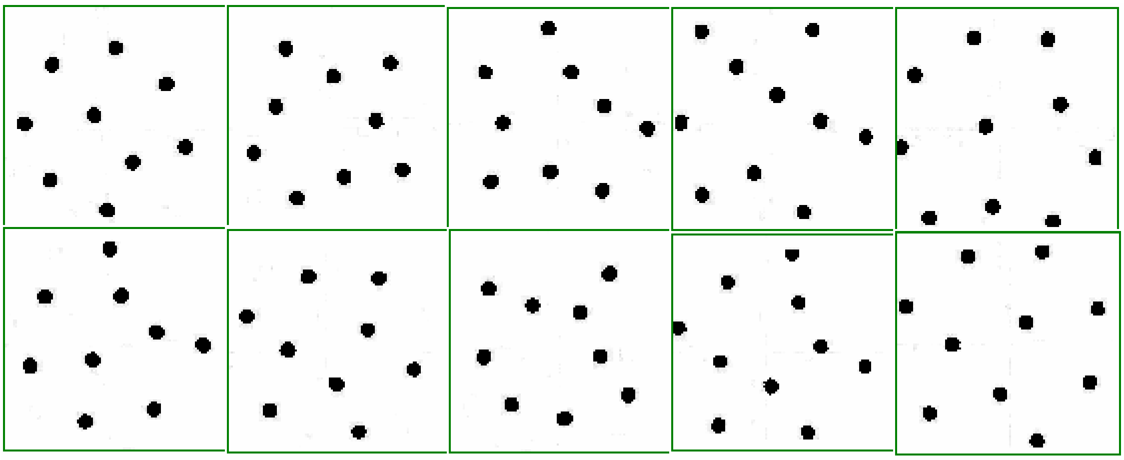

Fig.1

References

[1] Aref H., Newton P.K., Stremler M., Tokieda T., Vainchtein D.L., *Vortex crystals,* Adv. Appl. Mech. **29** (2003), 1-79.

[2] Thomson J.J., *A treatise on the motion of vortex rings*, London: Macmillan (1883),1-124.

[3] Havelock T., *The stability of motion of rectilinear vortices in ring formation*, Phil. Mag. Ser. 7, **11** (1931),617-633.

[4] Aref H., Vainchtein D.L., *Point vortices exhibit asymmetric equilibria*, Nature **392** (1998), 769-770.

[5] Donnelly R.J., *Quantized vortices in Helium II,* Cambridge University Press (1991).

[6] Campbell L., Ziff R., *A catalog of two-dimensional vortex patterns,* Los Alamos Scientific Laboratory Report №. La-7384-MS (1978).