Complex-space singularities of 2D Euler flow in Lagrangian frame

Takeshi Matsumoto[1], Jérémie Bec[2], Uriel Frisch[2]

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[1] Department of Physics, Kyoto University,
Kitashirakawa Oiwakecho Sakyoku Kyoto, 606-8502, Japan
[2] CNRS UMR 6202, Observatoire de la Côte d'Azur,
BP 4229, 06304 Nice Cedex 4, France

Abstract submitted to EE250

The main result reported here is that two-dimensional inviscid incompressible flow with simple initial condition has not only Eulerian but also Lagrangian complex singularities. The latter are closer to the real domain than the former.

To understand why we investigated this question, a few lines of context are needed. A solution to the incompressible Euler equation, starting from entire initial data (e.g. trigonometric polynomials), can be analytically continued to the complex space as long as it stays analytic in the real space. As is known since the seventies, initial real analyticity for periodic solutions is never lost in 2D [1]. The proven lower bound for the distance $\delta(t)$ to the real domain of the nearest singularities decreases as a double exponential. However numerical simulations at very high resolution (up to 8192^2) indicate that the decrease is much slower, close to a simple exponential. The discrepancy seems related to the strong depletion of nonlinearity which is systematically observed in 2D and 3D incompressible flow.

Recently it was shown by very precise simulations that in such 2D flow, the vorticity becomes infinite at complex singularities [2, 3]. This implies that the corresponding (complex) fluid particles must be located at infinity at $t = 0$. Could it be that, in Lagrangian coordinates, the solution has no other singularity than at complex infinity? This would mean that the solution is entire in Lagrangian coordinates. A simple counterexample is the AB flow $\psi = \sin x_1 \cos x_2$, which is a steady solution to the 2D Euler equation in Eulerian coordinates. The trajectories of fluid particles can then be integrated by elliptic functions and it was found in Ref. [4] that fluid particles initially at suitable finite complex locations can escape to infinity at any real positive time t.

What about flows which are not Eulerian steady-state solutions, such as the flow with the initial condition $\psi = \cos x_1 + \cos 2x_2$? This question was investigated numerically using spectral techniques with enough accuracy on high-order harmonics to allow the accurate determination of complex singularities [5]. Direct application of spectral techniques in Lagrangian coordinates is awkward. To calculate the solution in Lagrangian coordinates, we used the fact that the inverse Lagrangian map $\boldsymbol{a}(\boldsymbol{x}, t)$ satisfies the advection equation $\partial_t \boldsymbol{a} + \boldsymbol{u}(\boldsymbol{x}, t) \cdot \nabla \boldsymbol{a} = 0$, where $\boldsymbol{u}$ is the Eulerian velocity field. This equation was solved by Eulerian spectral techniques along with the 2D Eulerian equation for $\boldsymbol{u}$. Then the Lagrangian map (or more precisely the displacement $\boldsymbol{d} \equiv \boldsymbol{x} - \boldsymbol{a}$) was calculated in Lagrangian coordinates. The inversion involved

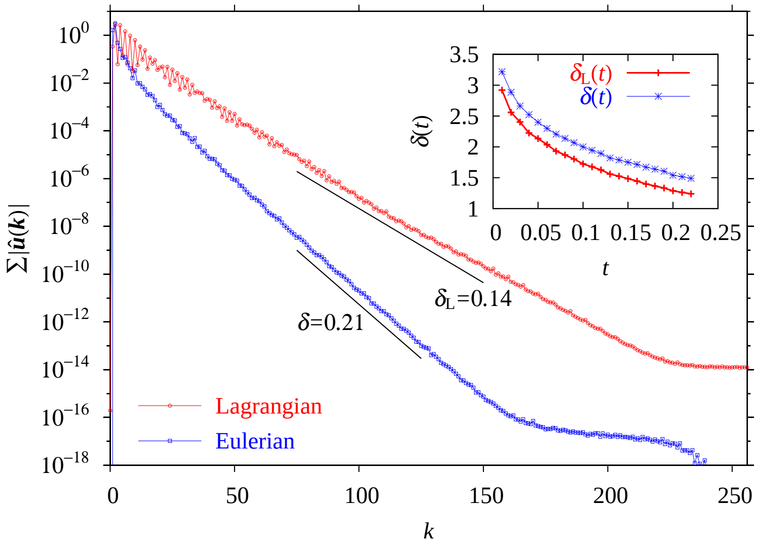

Figure 1: Shell-summed amplitudes for Eulerian and Lagrangian velocities at time $t = 1.245$ in lin-log coordinates. Inset: time variation of Eulerian $\delta(t)$ and Lagrangian $\delta_{\rm L}(t)$ at short times. The relation $\delta_{\rm L}(t) < \delta(t)$ holds up to $t = 1.245$.

the use of two uniform grids, a coarse Lagrangian and a finer Eulerian grid, together with a Newton iteration (details will be reported in the paper if the contribution is selected).

By composing the solution $\boldsymbol{u}(\boldsymbol{x}, t)$ with the Lagrangian map we obtained the Lagrangian velocity $\boldsymbol{u}_{\rm L}(\boldsymbol{a}, t)$. From its 2D Fourier transform the Lagrangian width of the analyticity strip $\delta_{\rm L}(t)$ was recovered in a standard way, by analyzing the high-wavenumber dependence of shell-summed Fourier amplitudes [2, 5].

The figure shows the wavenumber dependence of shell-summed amplitudes for the Eulerian and Lagrangian velocities at the same instant. They both exhibit exponential decay from which the Eulerian $\delta(t)$ and its Lagrangian counterpart $\delta_{\rm L}(t)$ are measured. It is seen that the Lagrangian δ is always smaller than the Eulerian one. This could be a signature of the weaker depletion of nonlinearity in Lagrangian coordinates, compared to the Eulerian case (this observation was suggested to us by S. Orszag).

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