# Short-time complex singularities for 2D MHD and 3D Euler equations

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5th December 2006

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*Abstract submitted to EE250*

Recently, the analytic structure of 2D periodic Euler flows in the asymptotic régime $t \to 0_+$ has been extensively studied in a series of papers [1, 2, 3]. In particular, it has been found that 2D Euler flows with initial conditions which are trigonometric polynomials possess complex singularities which are located on smooth objects (manifolds). Quite unexpectedly, the nature of these singularities depends on the initial conditions; this may however be an exceptional feature of two-dimensional flows.

The aim of the present work is to study the analytic structure of solutions of the 2D MHD equations ($\Psi$: stream function, $A$: magnetic potential)

$$\partial_t \nabla^2 \Psi = J(\Psi, \nabla^2 \Psi) - J(A, \nabla^2 A)\,, \qquad \partial_t A = J(\Psi, A), \tag{1}$$

and of the 3D Euler equations in the short-time asymptotic régime. As I have shown in [3], in this régime, solutions of the 2D Euler equations with initial conditions which are trigonometric polynomials can be reduced to suitable combinations of solutions with only two initial modes. Similar results apply to the 2D MHD and to the 3D Euler equations, the number of required modes being always equal to the number of spacial dimensions.

One essential feature of the short-time 2D Euler flows is the positivity of all the Fourier coefficients (except one); as explained in [3], this implies that there is a whole curve of singularities in an "imaginary plane" located "over" a particular real point of symmetry of the flow. The case of the 2D MHD equations is much richer. For some - rather exceptional - initial conditions, the positivity is preserved. The nature of the singularities can then be studied by the same techniques as for 2D Euler. In particular one can determine the nature of the singularities by analyzing the algebraic prefactor of the Fourier coefficients of the stream function. In the 2D Euler case its exponent depends on the angle between the initial modes but is always between $-5/2$ and $-8/3$; for the 2D MHD case I find it to be close to $-3/2$. Beyond such exceptional cases, the more general behaviour in 2D MHD is that the Fourier coefficients of both $\Psi$ and $A$ display oscillations at high wavenumbers, which I also found for the short-time version of the 2D Navier-Stokes equation (see Figure). In such oscillatory situations, the analysis of the singularities can in principle be done by the recently developed Borel–Pólya–Hoeven technique [4], for which I hope to have results by the time I prepare a paper for the Conference proceedings (if selected).

I turn now to the 3D Euler case. The short-time régime is then reducible to studying problems with three basic modes. This constitutes a very large class not easily amenable to careful numerical investigation. I have thus limited myself to studying simple examples, mostly concentrating on cases having the same kind of symmetry as the Kida–Pelz initial condition [5], which has been much investigated in connection with (hypothetical) blow-up. I found that the *Permutation Flow* with the initial condition

$$u_1 = \sin x_2 + \sin x_3, \qquad u_2 = \sin x_1 + \sin x_3 \qquad u_3 = \sin x_1 + \sin x_2,$$

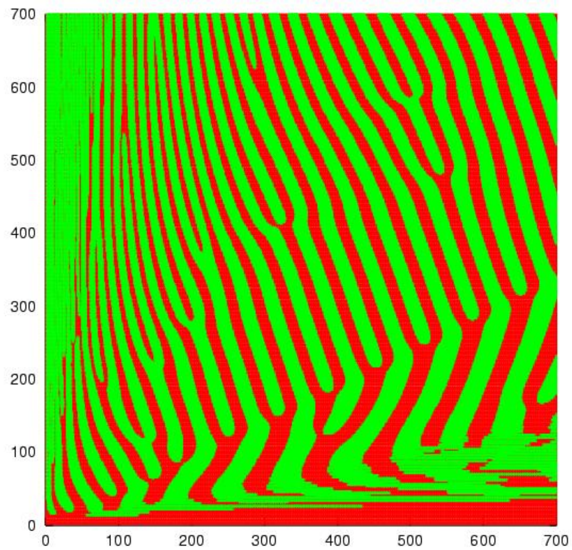


Figure 1: Color-coded sign of the Fourier coefficients in **k**-space for 2D Navier-Stokes flow in the short-time régime.

displays no sign oscillations of its Fourier coefficients along any (rational) direction in Fourier space. Scaling exponents can then be determined rather easily, provided the resolution is high enough. A the moment, because of the need to use quadruple precision, I have not more than $128^3$ mode but this will soon be much increased using new fast algorithms developed in collaboration with J. van der Hoeven and D. Mitra. It will be interesting to see if the 3D case is more universal than the 2D case. I finally notice that the *Permutation Flow* is easily studied beyond the short-time régime, using Taylor expansions in the time variable, a spectral method for the space variables and a relaxed multiplication algorithm to reduce the complexity of the evaluation of the Taylor coefficients [6].

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