Dissipative Anomalies in Singular Euler Flows

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Euler Equations: 250 Years On
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Energy Dissipation at Zero Viscosity

Experiments

H. L. Dryden, Q. Appl. Maths 1, 7 (1943)
K. R. Sreenivasan, Phys. Fluids 27, 1048 (1984)
O. Cadot et al. Phys. Rev. E 56, 427 (1997)
R. B. Pearson et al. Phys. Fluids 14, 1288 (2002)

Simulations

K. R. Sreenivasan, Phys. Fluids 10, 528 (1998)

Y. Kaneda et al., Phys. Fluids 15, L21 (2003)

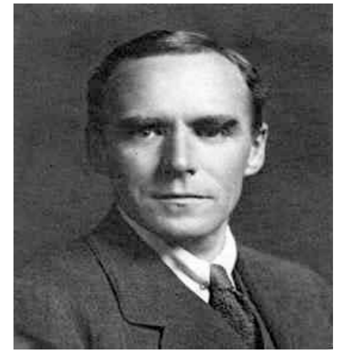
Energy dissipation rate in various turbulent flows seems to remain positive as Reynolds number tends to infinity.

It has been shown by v. Karman that if the surface stress in a pipe is expressed in the form $\tau = \rho v_{\times}^2$ then

$$\frac{U_c - U}{v_{\times}} = f\left(\frac{r}{a}\right), \tag{54}$$

where U_c is the maximum velocity in the middle of the pipe and U is the velocity at radius r. This relationship is associated with the conception that the Reynolds's stresses are proportional to the squares of the turbulent components of velocity. It seems that the rate of dissipation of energy in such a system must be proportional, so far as changes in linear dimensions, velocity, and density are concerned, to $\rho u'^3/l$, where l is some linear dimension defining the scale of the system.

Taylor, "Statistical Theory of Turbulence" (1935)



Geoffrey Ingram Taylor (1886-1975)

Non-vanishing of mean dissipation at infinite Reynolds number was a fundamental hypothesis of the Kolmogorov similarity theory (1941).

Andrei Nikolaevich Kolmogorov (1903-1987)

High-Reynolds Asymptotics

As suggested by G. I. Taylor (1935), turbulent energy dissipation ε scales as

$$\varepsilon \sim U^3/L,$$

where U is rms velocity and L is the integral length, so that

$$D(Re) = \frac{\varepsilon}{U^3/L} \to D_* > 0$$

as $Re \to \infty$. E.g. the compilation of DNS data of Y. Kaneda et al. (2003):

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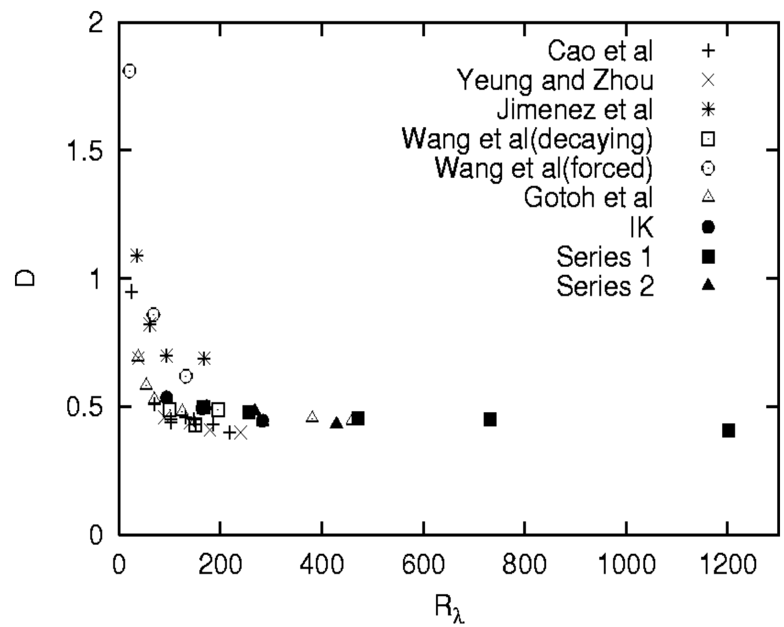
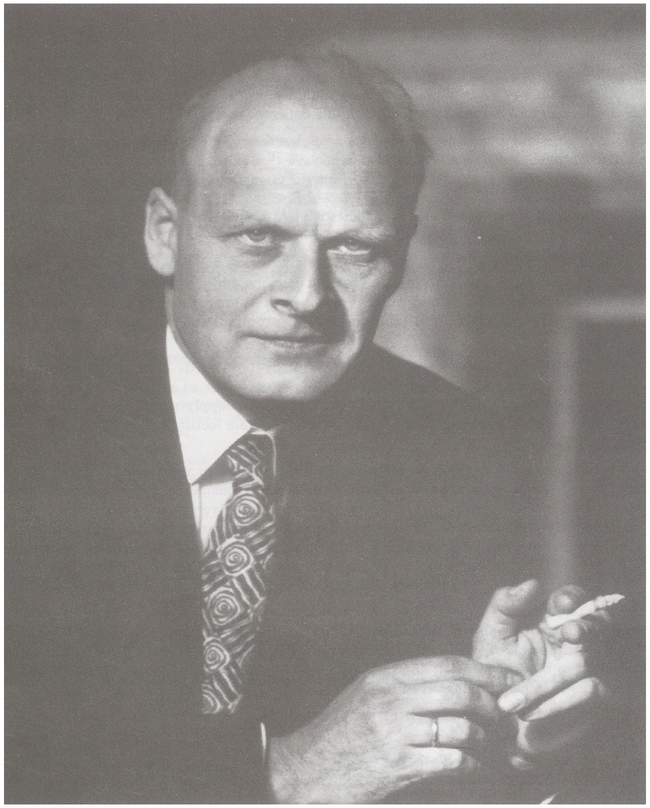


FIG. 3. Normalized energy dissipation rate D versus R_λ from Ref. 5 (data up to $R_\lambda = 250$), Ref. 3 (△,●), and the present DNS databases (■,▲).



Lars Onsager (1903-1976)

For an in-depth historical discussion, see Eyink & Sreenivasan (2006).

"It is of some interest to note that in principle, turbulent dissipation as described could take place just as readily without the final assistance by viscosity. In the absence of viscosity, the standard proof of the conservation of energy does not apply, because the velocity field does not remain differentiable! In fact it is possible to show that the velocity field in such "ideal" turbulence cannot obey any LIPSCHITZ condition of the form

(26) $|\mathbf{v}(\mathbf{r}'+\mathbf{r})-\mathbf{v}(\mathbf{r}')| < (\text{const.})r^n$

for any order n greater than 1/3; otherwise the energy is conserved. Of course, under the circumstances, the ordinary formulation of the laws of motion in terms of differential equations becomes inadequate and must be replaced by a more general description...

"Statistical Hydrodynamics" (1949)

Effective "Coarse-Grained" Equations

Starting with the incompressible Navier-Stokes equation

$$\partial_t \mathbf{u} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla} p + \nu \triangle \mathbf{u}, \quad \boldsymbol{\nabla}\cdot\mathbf{u} = 0$$

the proof of Onsager's theorem (following Constantin, E & Titi (1994), Eyink (1995)) considers a coarse-grained (low-pass filtered) velocity

$$\overline{\mathbf{u}}_\ell(\mathbf{x}) = \int d\mathbf{r}\ G_\ell(\mathbf{r})\mathbf{u}(\mathbf{x}+\mathbf{r}).$$

This yields effective equations at a continuum of length-scales ℓ :

$$\partial_t \overline{\mathbf{u}}_\ell + \boldsymbol{\nabla}\cdot[\overline{\mathbf{u}}_\ell\overline{\mathbf{u}}_\ell + \boldsymbol{\tau}_\ell] = -\boldsymbol{\nabla}\overline{p}_\ell + \nu \triangle \overline{\mathbf{u}}_\ell, \quad \boldsymbol{\nabla}\cdot\overline{\mathbf{u}}_\ell = 0$$

where $\boldsymbol{\tau}_\ell$ is the *subscale stress tensor*

$$\boldsymbol{\tau}_\ell = \overline{(\mathbf{u}\otimes\mathbf{u})}_\ell - \overline{\mathbf{u}}_\ell \otimes \overline{\mathbf{u}}_\ell,$$

from the eliminated modes.

This is similar to what is called Wilson-Kadanoff renormalization group (RG). The same approach is used in Large-Eddy Simulation (LES) of turbulent flow, where a closure equation is employed for the stress tensor $\boldsymbol{\tau}_\ell$.

Inertial Range at High-Reynolds-Number

A simple estimate of the viscous diffusion term is

$$\|\nu \triangle \overline{\mathbf{u}}_\ell\|_2 \leq (\text{const.})(\nu/\ell^2)\|\mathbf{u}\|_2$$

where

$$\|\mathbf{u}\|_2^2 = \int_0^T dt \int d\mathbf{x}\, |\mathbf{u}(\mathbf{x},t)|^2$$

is (twice) the time-average kinetic energy. Thus, this term is negligible for small ν or large ℓ (irrelevant) and can be dropped.

Simpler effective equations result for the *inertial-range* of length-scales ℓ:

$$\partial_t \overline{\mathbf{u}}_\ell + \boldsymbol{\nabla}\boldsymbol{\cdot}[\overline{\mathbf{u}}_\ell \overline{\mathbf{u}}_\ell + \boldsymbol{\tau}_\ell] = -\boldsymbol{\nabla}\overline{p}_\ell, \quad \boldsymbol{\nabla}\boldsymbol{\cdot}\overline{\mathbf{u}}_\ell = 0$$

retaining only the contributions from the nonlinearity.

E.g., these hold if the Navier-Stokes solution ($\mathbf{u}^\nu$ for viscosity ν) converges $\mathbf{u}^\nu \to \mathbf{u}$ in L^2 norm as $\nu \to 0$, i.e. the residual energy in $\mathbf{u} - \mathbf{u}^\nu$ vanishes.

Large-Scale Energy Balance

Large-scale energy density (per mass)

$$e_\ell = \frac{1}{2}|\overline{\mathbf{u}}_\ell|^2$$

satisfies a local balance equation

$$\partial_t e_\ell + \boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{J}_\ell = -\Pi_\ell$$

where

$$\mathbf{J}_\ell = (e_\ell + \overline{p}_\ell)\overline{\mathbf{u}}_\ell + \overline{\mathbf{u}}_\ell\boldsymbol{\cdot}\boldsymbol{\tau}_\ell$$

is *space transport* of large-scale energy and

$$\Pi_\ell = -\boldsymbol{\nabla}\overline{\mathbf{u}}_\ell\boldsymbol{:}\boldsymbol{\tau}_\ell$$

is the *rate of work* of the large-scale velocity-gradient against the small-scale stress, or "deformation work" (Tennekes & Lumley, 1972).

Turbulent energy cascade is the dynamical transfer of kinetic energy from large-scales to small-scales via the "energy flux" Π_ℓ through the inertial-range.

Onsager's Basic Estimate

Onsager realized that energy flux Π_ℓ depends only upon *velocity-increments*

$$\delta\mathbf{u}(\mathbf{x};\mathbf{r}) \equiv \mathbf{u}(\mathbf{x}+\mathbf{r}) - \mathbf{u}(\mathbf{x}).$$

In particular,

$$\boldsymbol{\tau}_\ell = \int d\mathbf{r}\ G_\ell(\mathbf{r})\delta\mathbf{u}(\mathbf{r}) \otimes \delta\mathbf{u}(\mathbf{r}) - \int d\mathbf{r}\ G_\ell(\mathbf{r})\delta\mathbf{u}(\mathbf{r}) \otimes \int d\mathbf{r}\ G_\ell(\mathbf{r})\delta\mathbf{u}(\mathbf{r})$$

and

$$\boldsymbol{\nabla}\overline{\mathbf{u}}_\ell = -(1/\ell)\int d\mathbf{r}\ (\boldsymbol{\nabla}G)_\ell(\mathbf{r})\delta\mathbf{u}(\mathbf{r})$$

Essentially,

$$\Pi_\ell = O(|\delta u(\ell)|^3/\ell).$$

It is then easy to see, for example, that

$$\Pi_\ell(\mathbf{x},t) = O(\ell^{3\alpha-1})$$

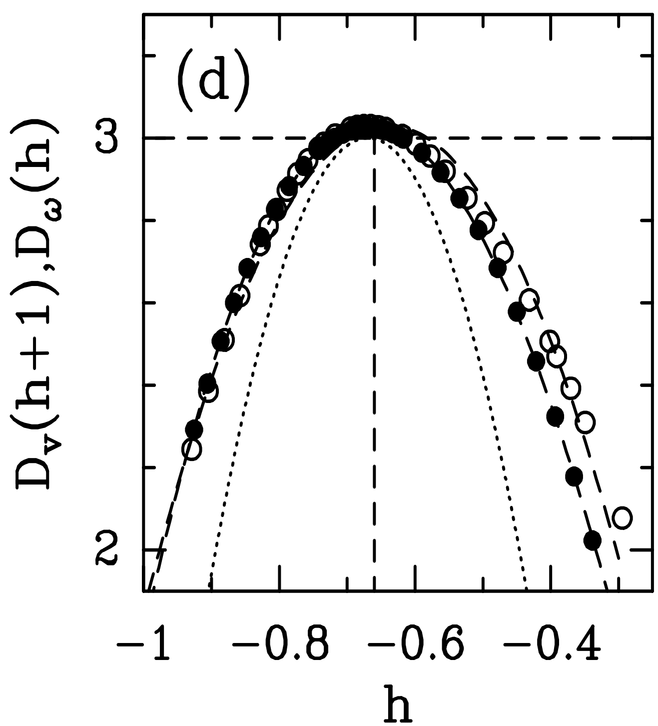
if $\mathbf{u}(t) \in C^\alpha(\mathbf{x})$, i.e. if $|\delta\mathbf{u}(\mathbf{x};\mathbf{r})| = O(r^\alpha)$, and $\Pi_\ell(\mathbf{x},t) \to 0$ for $\alpha > 1/3$.

Inviscid Dissipation Requires Singularities

The converse: to explain the observed energy dissipation requires $\alpha \leq 1/3$ in the infinite Reynolds number limit. Onsager's prediction of such (near) singularities has been confirmed by experiment and simulation:

J. F. Muzy et al. , Phys. Rev. Lett. 67, 3515 (1991)
A. Arneodo et al. , Physica A 213, 232 (1995)
P. Kestener and A. Arneodo, Phys. Rev. Lett. 93, 044501 (2004)

A great triumph of pure intellect!



Singular Euler Solutions

All the above considerations apply to a non-smooth velocity field $\mathbf{u}$ that satisfies the *incompressible Euler equations*

$$\partial_t \mathbf{u} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla} p, \quad \boldsymbol{\nabla}\cdot\mathbf{u} = 0$$

in the sense of distributions. That is, the equations are meaningful when smeared with smooth test functions φ.

E.g. if $\mathbf{u} \in L^3$, then energy balance holds in the form

$$\partial_t(\frac{1}{2}|\mathbf{u}|^2) + \boldsymbol{\nabla}\cdot \left[(\frac{1}{2}|\mathbf{u}|^2 + p)\right] = -D(\mathbf{u})$$

where the distribution $D(\mathbf{u})$ is defined by $D(\mathbf{u}) = \lim_{\ell\to 0} \Pi_\ell$, or, alternatively,

$$D(\mathbf{u}) = \lim_{\ell\to 0} \frac{1}{4\ell} \int d\mathbf{r}\, (\boldsymbol{\nabla} G)_\ell(\mathbf{r})\cdot \left[\delta\mathbf{u}(\mathbf{r})|\delta\mathbf{u}(\mathbf{r})|^2\right].$$

See Duchon & Robert (2000). This is Onsager's *dissipative anomaly*.

If a smooth Navier-Stokes solution $\mathbf{u}^\nu \to \mathbf{u}$ in L^3 norm, then furthermore

$$D(\mathbf{u}) = \lim_{\nu\to 0} \nu|\boldsymbol{\nabla}\mathbf{u}^\nu|^2 \geq 0.$$

This is the multifractal measure studied, e.g. by Meneveau & Sreenivasan.

PROOF: Using a smooth point-splitting regularization of the energy density

$$e_\ell^* \equiv \frac{1}{2}\mathbf{u}\cdot\overline{\mathbf{u}}_\ell = \frac{1}{2}\int d\mathbf{r}\, G_\ell(\mathbf{r})\, \mathbf{u}(\mathbf{x},t)\cdot\mathbf{u}(\mathbf{x}+\mathbf{r},t)$$

gives a local energy balance identity

$$\partial_t e_\ell^* + \boldsymbol{\nabla}\cdot\left[e_\ell^*\mathbf{u} + \frac{1}{2}(\overline{p}_\ell\mathbf{u} + p\overline{\mathbf{u}}_\ell) + \frac{1}{2}\left(\overline{(|\mathbf{u}|^2\mathbf{u})}_\ell - \overline{|\mathbf{u}|^2}_\ell\mathbf{u}\right)\right] = -D_\ell(\mathbf{u})$$

with

$$D_\ell(\mathbf{u}) = \frac{1}{4\ell}\int d\mathbf{r}\,(\boldsymbol{\nabla}G)_\ell(\mathbf{r})\cdot\left[\delta\mathbf{u}(\mathbf{r})|\delta\mathbf{u}(\mathbf{r})|^2\right].$$

Take the limit $\ell \to 0$.

* This was Onsager's own proof! The above identity (in a space-integrated form) was communicated by Onsager to C. C. Lin in a letter in 1945. See Eyink & Sreenivasan (2006).

* This is a *local form of the Kolmogorov 4/5-law*, derived with no assumptions of homogeneity or isotropy, nor any statistical averaging. See Duchon & Robert (2000), Eyink (2003).

Open PDE Questions on Dissipative Euler Solutions

Nearly everything! Existence, uniqueness & regularity are all essentially open.

Existence

Shnirelman (2000) has constructed an example of a $\mathbf{u} \in L^2$ in 3D which is a distributional solution of the Euler equations for which

$$E(t) = \frac{1}{2}\int d\mathbf{x}\,|\mathbf{u}(\mathbf{x},t)|^2$$

is monotone decreasing in time.

However, this solution is *not* obtained from a Leray solution $\mathbf{u}^\nu$ of the Navier-Stokes equation in the limit $\nu \to 0$.

It also lacks the regularity expected of a turbulent solution (below).

Zero-Viscosity Limit

Zero-viscosity limits of Navier-Stokes solutions have only been shown to exist and to give "Euler solutions" in some more generalized sense.

DiPerna & Majda (1987) show, roughly speaking, that $\mathbf{u}^\nu$ converges weakly (along a suitable subsequence) to a Young measure $P_{\mathrm{x},t}(d\mathbf{u})$ which is a *measure-valued Euler solution*, i.e.

$$\partial_t \langle \mathbf{u} \rangle_{\mathrm{x},t} + \boldsymbol{\nabla} \langle \mathbf{u}\mathbf{u} \rangle_{\mathrm{x},t} = -\boldsymbol{\nabla} p(\mathbf{x}, t)$$

for some distribution p, where $\langle \cdot \rangle_{\mathrm{x},t}$ is average with respect to $P_{\mathrm{x},t}$.

P.-L. Lions (1996) defines $\mathbf{u} \in L^2$, $\boldsymbol{\nabla}\cdot\mathbf{u} = 0$ to be a *dissipative Euler solution* if $\int d\mathbf{x}\, \frac{1}{2}|\mathbf{u}(t) - \mathbf{v}(t)|^2$ satisfies a suitable upper bound for all "nice" $\mathbf{v}$ with $\boldsymbol{\nabla}\cdot\mathbf{v} = 0$. Lions proves that "dissipative solutions" coincide with classical Euler solutions, when those exist, and can always be obtained as suitable weak limits of Leray solutions $\mathbf{u}^\nu$ as $\nu \to 0$ (along a subsequence).

Uniqueness

Distributional Euler solutions are *not* unique!

Scheffer (1993) has constructed a 2D solution $\mathbf{u} \in L^2$ with compact support in spacetime: with initial condition $\mathbf{u}_0 \equiv \mathbf{0}$ the solution $\mathbf{u}$ has nontrivial evolution and then comes again to rest in finite time! A unique classical solution exists for all time ($\mathbf{u} \equiv \mathbf{0}$) but many other pathological "Euler solutions" in the sense of distributions exist as well.

See also Shnirelman (1997). de Lellis & Székelyhidi, Jr. (2007) show that such weird solutions with compact spacetime support exist even with more regularity, $\mathbf{u}, p \in L^\infty$, for any dimension d.

Question: Is there a natural selection criterion to guarantee uniqueness? A notion of viscosity solution? An entropy principle? Physics suggests some intriguing possibilities....

Regularity

Experimental measurements and simulations of high Reynolds-number turbulence show that, for all $p \geq 1$ and r in the inertial-range $\eta \ll r \ll L$

$$\langle|\delta\mathbf{u}(\mathbf{r})|^p\rangle^{1/p} \sim r^{\sigma_p}$$

for some $0 < \sigma_p < 1$, i.e. *multifractal scaling*.

This suggests that Euler solutions relevant to infinite-Reynolds turbulence have $\mathbf{u} \in B_p^{\sigma_p}$, where B_p^s is the so-called *Besov space* consisting of $\mathbf{u} \in L^p$ with

$$\sup_{|\mathbf{r}|<L} \frac{\|\delta\mathbf{u}(\mathbf{r})\|_{L^p}}{|\mathbf{r}|^s} < \infty.$$

No PDE theory of such solutions exist.

Constantin, E, Titi (1994) generalized Onsager's result to show that Euler solutions with $\mathbf{u} \in B_p^s$ for $p \geq 3$ and $s > 1/3$ will conserve energy.

Turbulent solutions of Euler appear to have the least degree of singularity consistent with positive dissipation. This suggests a generalized "energy estimate" to prove regularity....

2D Enstrophy Cascade

Smooth solutions of 2D Euler conserve also the *enstrophy*:

$$\Omega(t) = \frac{1}{2}\int d^2\mathbf{x}\,\omega^2(\mathbf{x},t).$$

It was suggested by Kraichnan (1967) and Batchelor (1969) that there can be a forward cascade of enstrophy in 2D (and an inverse cascade of energy!)

The "coarse-grained" 2D Euler equation in vorticity form is

$$\partial_t\overline{\omega}_\ell + \boldsymbol{\nabla}\boldsymbol{\cdot}[\overline{\mathbf{u}}_\ell\overline{\omega}_\ell + \boldsymbol{\sigma}_\ell] = 0$$

where $\boldsymbol{\sigma}_\ell = \overline{(\mathbf{u}\omega)}_\ell - \overline{\mathbf{u}}_\ell\overline{\omega}_\ell$ is the *turbulent vorticity transport*. The large-scale vorticity does not move with the large-scale velocity, but has a relative "drift velocity" $\boldsymbol{\Delta}\mathbf{u}_\ell = \boldsymbol{\sigma}_\ell/\overline{\omega}_\ell$.

The large-scale enstrophy density $\eta_\ell = (1/2)|\overline{\omega}_\ell|^2$ satisfies the balance:

$$\partial_t\eta_\ell + \boldsymbol{\nabla}\boldsymbol{\cdot}[\eta_\ell\overline{\mathbf{u}}_\ell + \overline{\omega}_\ell\boldsymbol{\sigma}_\ell] = -Z_\ell, \qquad Z_\ell = -\boldsymbol{\nabla}\overline{\omega}_\ell\boldsymbol{\cdot}\boldsymbol{\sigma}_\ell$$

Enstrophy suffers "ideal dissipation" when the the enstrophy transport $\boldsymbol{\sigma}_\ell$ tends to be down the vorticity-gradient $\boldsymbol{\nabla}\overline{\omega}_\ell$, or $\boldsymbol{\sigma}_\ell \propto -\boldsymbol{\nabla}\overline{\omega}_\ell$.

Anomalous Dissipation of Enstrophy?

Since the enstrophy flux satisfies an "Onsager-type" bound

$$Z_\ell = O(|\delta u(\ell)/\ell|^3) = O(|\delta\omega(\ell)|^3),$$

very modest smoothness of ω implies $\lim_{\ell\to 0} Z_\ell = 0$. E.g. if the vorticity is Hölder continuous, $\omega \in C^s$ for any small $s > 0$, then enstrophy is conserved.

In fact, using the DiPerna & Lions (1989) theory of "renormalized solutions," it can be shown that *any solution of 2D Euler equations with finite enstrophy must conserve enstrophy!* See Lions (1996), Eyink (2000), Lopes-Filho, Mazzucato & Nussenzveig-Lopes (2006).

Note that the Kraichnan-Batchelor theory predicts an enstrophy spectrum $\Omega(k) \sim k^{-1}$ (with log-correction) having infinite total enstrophy as $\nu \to 0$.

Much more is known in 2D about existence, zero-viscosity limit, uniqueness and regularity of Euler solutions than in 3D.

3D Helicity Cascade

Smooth solutions of 3D Euler conserve also the *helicity:*

$$H(t) = \int d^3\mathbf{x}\,\boldsymbol{\omega}(\mathbf{x},t){\boldsymbol\cdot}\mathbf{u}(\mathbf{x},t),$$

as noted by J. J. Moreau (1961), H. K. Moffatt (1969). Brissaud et al. (1973) proposed that in reflection-nonsymmetric turbulence there should be a forward cascade of helicity, coexisting with the forward energy cascade.

Using the "coarse-grained" vorticity equation in 3D

$$\partial_t\overline{\boldsymbol{\omega}}_\ell = \boldsymbol{\nabla}\times(\overline{\mathbf{u}}_\ell\times\overline{\boldsymbol{\omega}}_\ell + \mathbf{f}_\ell)$$

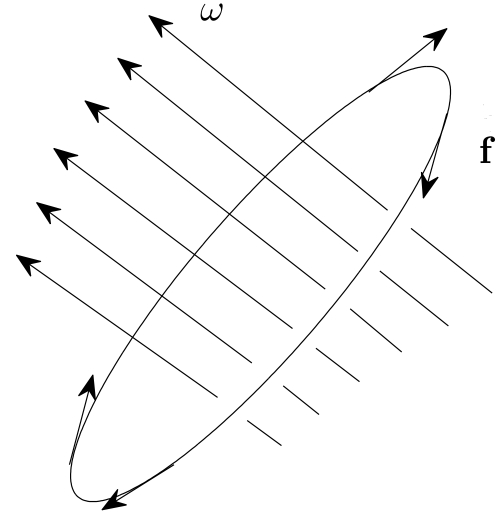
where $\mathbf{f}_\ell = -\boldsymbol{\nabla}{\boldsymbol\cdot}\boldsymbol{\tau}_\ell$ is the *turbulent (subscale) force*, one derives a balance equation for the large-scale *helicity density* $h_\ell = \overline{\mathbf{u}}_\ell{\boldsymbol\cdot}\overline{\boldsymbol{\omega}}_\ell$:

$$\partial_t h_\ell + \boldsymbol{\nabla}{\boldsymbol\cdot}[h_\ell\overline{\mathbf{u}}_\ell + (\overline{p}_\ell - e_\ell)\overline{\boldsymbol{\omega}}_\ell + \overline{\mathbf{u}}_\ell\times\mathbf{f}_\ell] = -\Lambda_\ell$$

with scale-to-scale helicity flux

$$\Lambda_\ell = -2\overline{\boldsymbol{\omega}}_\ell{\boldsymbol\cdot}\mathbf{f}_\ell.$$

Mechanism of Helicity Cascade



The component of the turbulent force $\mathbf{f}_\ell$ parallel to $\overline{\boldsymbol{\omega}}_\ell$ accelerates fluid about closed vortex loops L, generating circulation around them. Vorticity-flux is thus created through the vortex-loop.

According to a theorem of Arnold (1986), helicity H is the average self-linking number of the vortex-lines.

Anomalous Dissipation of Helicity?

The helicity flux can easily be shown to satisfy an "Onsager bound"

$$\Lambda_\ell = -2\overline{\boldsymbol{\omega}}_\ell \cdot \mathbf{f}_\ell = O(|\delta u(\ell)|^3/\ell^2).$$

This suggests conservation if $\delta u(\ell) \sim \ell^s$ with $s > 2/3$.

Cheskidov et al. (2007) have proved that helicity is conserved for any distributional solution of 3D Euler with $\mathbf{u} \in B_3^s \cap H^{1/2}$ for $s > 2/3$, improving an earlier result of Chae (2003).

More regularity is required for conservation of helicity than of energy. These results are consistent with constant helicity flux coexisting with constant energy flux in a $k^{-5/3}$-type inertial range.

Note that Cheskidov et al. (2007) in fact prove somewhat sharper results, and also improve slightly upon earlier results for energy conservation in any dimension and enstrophy conservation in 2D.

Cascade of Circulations?

Large-scale circulation:

$$\overline{\Gamma}_\ell(C,t) = \oint_{\overline{C}_\ell(t)} \overline{\mathbf{u}}_\ell(t)\cdot d\mathbf{x} = \int_{\overline{S}_\ell(t)} \overline{\boldsymbol{\omega}}_\ell(t)\cdot d\mathbf{S}$$

where $\overline{C}_\ell(t)$ and $\overline{S}_\ell(t)$ are advected by $\overline{\mathbf{u}}_\ell$, which generates a flow of diffeomorphisms with $\mathbf{u} \in L^2$. Then the balance holds:

$$(d/dt)\overline{\Gamma}_\ell(C,t) = \oint_{\overline{C}_\ell(t)} \mathbf{f}^*_\ell(t)\cdot d\mathbf{x}$$

where

$$\mathbf{f}^*_\ell = \overline{(\mathbf{u}\times\boldsymbol{\omega})}_\ell - \overline{\mathbf{u}}_\ell\times\overline{\boldsymbol{\omega}}_\ell = \mathbf{f}_\ell + \boldsymbol{\nabla} k_\ell$$

is the *turbulent vortex-force* and $k_\ell = (1/2)\mathrm{Tr}\,\boldsymbol{\tau}_\ell$ is the subgrid kinetic energy.

Define *loop-torque* $K_\ell(C) \equiv -\oint_C \mathbf{f}^*_\ell\cdot d\mathbf{x}$ (or with $\mathbf{f}^*_\ell \to \mathbf{f}_\ell$). If velocity $\mathbf{u} \in C^\alpha$ and $L(C)$ is the length of C

$$|K_\ell(C)| \leq (\mathrm{const.}) L(C)\ell^{2\alpha-1}$$

See Eyink (2006). The key estimate on the vortex-force is that $\mathbf{f}_\ell = O(|\delta u(\ell)|^2/\ell)$.

This allows violation of Kelvin Theorem in the inertial range, if $\alpha \leq 1/2$.

Energy Dissipation and Kelvin Theorem

But conservation of circulations and the Helmholtz laws of vortex motion are believed to be essential for turbulent energy dissipation in 3D! For example,

> "Two-dimensional convection, which merely redistributes vorticity, cannot account for the rapid dissipation which one observes. However, as pointed out by G. I. TAYLOR [7], convection in three dimensions will tend to increase the total vorticity. *Since the circulation of a vortex tube is conserved*, the vorticity will increase whenever a vortex tube is stretched. Now it is very reasonable to expect that a vortex line —or any line which is deformed by the motion of the liquid— will tend to increase in length as a result of more or less haphazard motion. This process tends to make the texture of the motion ever finer, and greatly accelerates the viscous dissipation." —L. Onsager, "Statistical Hydrodynamics" (1949)

See also Taylor & Green (1937), going back to Taylor (1917). Nevertheless, the Kelvin Theorem is unlikely to hold in its usual form as $\nu \to 0$, since:

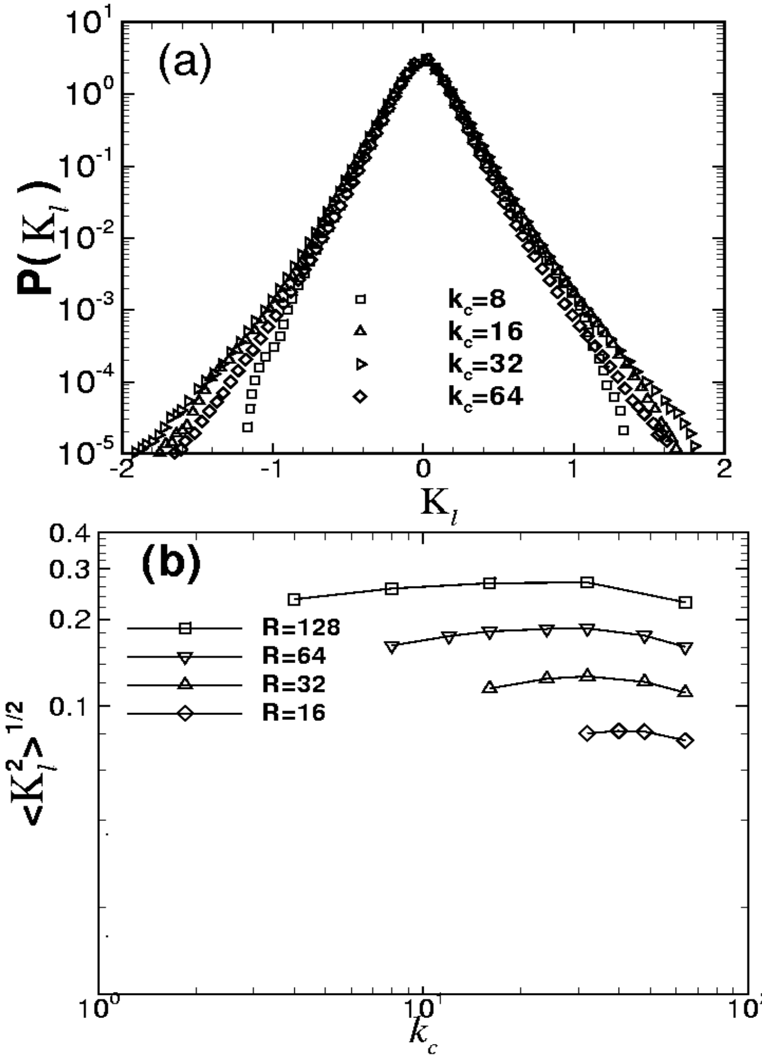
(1) The K41 exponent $\alpha = 1/3$ is less than than the $1/2$ required for vanishing of the vortex force $\mathbf{f}_\ell^*$ as $\ell \to 0$, and

(2) Loops $C(t)$ advected by the rough velocity field should become fractal, and thus $L(\overline{C}_\ell(t)) \to \infty$ as $\ell \to 0$.

Circulation-Cascade: Numerical Results

(a) PDF and (b) RMS of the subscale *loop-torque* $K_\ell(C)$ for square loops C of edge-length 64 in 1024^3 DNS of forced 3D hydrodynamic turbulence. (S. Chen et al., PRL, 2006)

PDF & RMS of subscale torque are nearly independent of $k_c = 2\pi/\ell$ in the turbulent inertial-range: the cascade of circulations is persistent in scale!



Breakdown of the Helmholtz Laws

The turbulent vortex force may be decomposed into components longitudinal and transverse to vortex lines:

$$\mathbf{f}_\ell = \alpha_\ell \overline{\boldsymbol{\omega}}_\ell + (\boldsymbol{\Delta}\mathbf{u}_\ell)\times\overline{\boldsymbol{\omega}}_\ell$$

where

$$\alpha_\ell = \overline{\boldsymbol{\omega}}_\ell\cdot\mathbf{f}_\ell/|\overline{\boldsymbol{\omega}}_\ell|^2, \quad \boldsymbol{\Delta}\mathbf{u}_\ell = \overline{\boldsymbol{\omega}}_\ell\times\mathbf{f}_\ell/|\overline{\boldsymbol{\omega}}_\ell|^2$$

We have seen that the longitudinal force is responsible for helicity cascade and is a hydrodynamic analogue of the MHD α-effect (cf. Frisch, She & Sulem, 1987).

The transverse part corresponds to a "drift" of the vortex-lines through the background turbulence, with relative velocity $\boldsymbol{\Delta}\mathbf{u}_\ell$. The transverse force can be interpreted as a *turbulent Magnus force* associated to this motion.

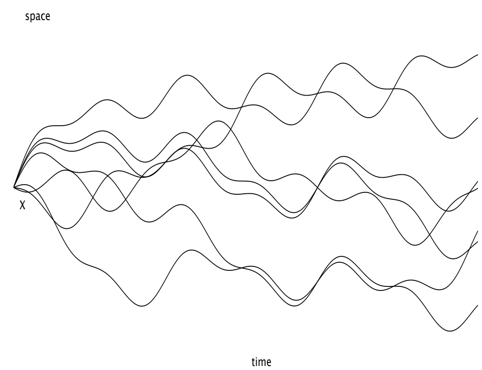
Note that, formally,

$$\alpha_\ell \sim \boldsymbol{\Delta}\mathbf{u}_\ell \sim \delta u(\ell) \to 0$$

as $\ell \to 0$. Is there any sense in which the Kelvin-Helmholtz results can be valid in the infinite-Reynolds inertial range?

Spontaneous Stochasticity

Another complication: Lagrangian trajectories are expected to be non-unique and stochastic for a fixed realization of a rough (Hölder) velocity field!



This was discovered by Bernard, Gawędzki & Kupiainen (1998) in the Kraichnan model of random advection and rigorously proved there by LeJan & Raimond (2002). Physically, this corresponds to Richardson diffusion.

For a smooth ϕ_ρ with $\mathrm{supp}\,\phi_\rho \subset B(\mathbf{0},\rho)$, bounded, continuous ψ, and $t > t_0$:

$$\lim_{\rho\to 0}\lim_{\ell\to 0}\int d\mathbf{r}_0\,\phi_\rho(\mathbf{r}_0)\psi(\overline{\boldsymbol{\xi}}_\ell^{t,t_0}(\mathbf{x}_0+\mathbf{r}_0)) = \int P_{\mathbf{u}}(d\mathbf{x},t|\mathbf{x}_0,t_0)\psi(\mathbf{x}).$$

Here $\overline{\boldsymbol{\xi}}_\ell^{t,t_0}$ is the smooth flow generated by $\overline{\mathbf{u}}_\ell$.

A "Martingale" Hypothesis

In the Kraichnan model of random advection,

$$\partial_t \theta + (\mathbf{u} \circ \boldsymbol{\nabla})\theta = 0, \qquad \boldsymbol{\nabla}\cdot\mathbf{u} = 0$$

the unique dissipative solution is represented by averaging over this random ensemble of (backward) Lagrangian characteristics:

$$\theta(\mathbf{x}, t) = \int P_{\mathbf{x},t}(d\mathbf{x}'|\mathbf{u})\theta(\mathbf{x}'(t'), t').$$

This implies that, for $t > t'$,

$$\int d\mathbf{x}\, |\theta(\mathbf{x}, t)|^2 < \int d\mathbf{x}\, |\theta(\mathbf{x}, t')|^2.$$

The analogous conjecture for circulations is that, for $t > t'$,

$$\Gamma(C, t) = \int P_{C,t}(dC'|\mathbf{u})\Gamma(C'(t'), t').$$

See Eyink (2006, 2007). The circulations may be conserved on average by a generalized Euler flow (roughly in the sense of Brenier (1989)).

Classical Josephson-Anderson Relation: Pipe Flow

The *vorticity transport* of the azimuthal vorticity ω_θ in the radial direction r

$$\sigma_{r\theta} = (\mathbf{u}\times\boldsymbol{\omega} - \nu\boldsymbol{\nabla}\times\boldsymbol{\omega})_z$$

has constant (negative) mean

$$(\partial/\partial r)\langle\sigma_{r\theta}\rangle = 0,$$

which, multiplied by mean mass flux J, is energy dissipation per pipe length:

$$(1/L_z)\int_{\text{pipe}} dV\ \rho\,\varepsilon = J|\langle\sigma_{r\theta}\rangle|,$$

a classical analogue of the *Josephson-Anderson relation* in superfluids, relating cross-stream transport of vorticity and energy dissipation. See Anderson (1966), Huggins (1970), and—also—Taylor (1932)!

Energy dissipation in realistic inhomogeneous turbulent flows often requires *organized motion of vorticity!*

Conclusion

* If Onsager was correct, then the fundamental inertial-range dynamics of turbulent flow is given by singular solutions of the Euler fluid equations. Observational evidence and rigorous results are consistent with the idea.

* This subject is therefore of interest not only to mathematicians, but also to experimentalists and simulators. The foundations of the subject are empirical, and further laboratory and numerical investigations are necessary to shed light on many difficult and basic questions, still beyond the scope of analysis.

* In particular, a major open problem is how to relate turbulent dissipation of energy, precisely, to the inviscid motion of vortex lines.

* Leonhard Euler would doubtless be delighted to see that his equations are of vital interest to the problem of turbulence and remain at the forefront of engineering, physics and mathematics.