Monge-Ampère-Kantorovich Reconstruction

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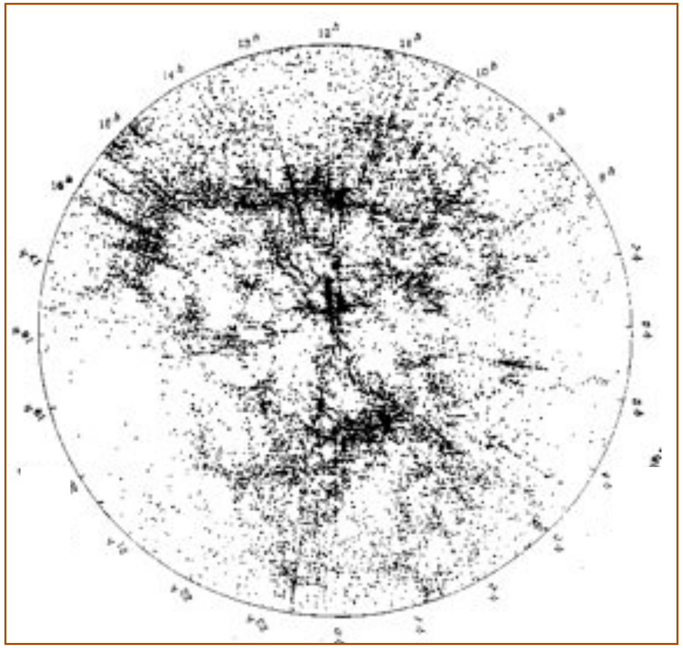
Also: Yann Brenier, Michel Hénon, Grégoire Loèper, Sabino Matarrese

Objectives



Galaxies are tracers of an underlying **dark matter fluid**

"positions" observed for ~1000 000 galaxies
"velocities" observed for ~1000 "test" galaxies

"optimal" mass density parameter

adjustement of mass density parameter

comparison with observed velocities of test galaxies

Dynamical history of our Universe (spatial distribution of mass now and 10^9 years ago)

"positions"

Reconstruction with a mass density parameter

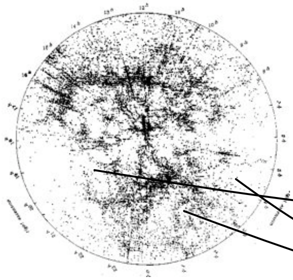
velocity

Minimisation of Euler-Lagrangian action

(Peebles 1989)

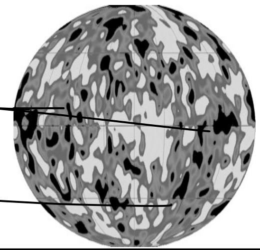
Trajectoires of N *galaxies minimise* the Euler-Lagrange action

Boundary condition 1:
τ = now
positions of galaxies=x



$$I \sim \int d\tau \sum m_i (x'^2_i + \alpha\ m_j / |x_i - x_j| + \beta\ x_i^2)$$

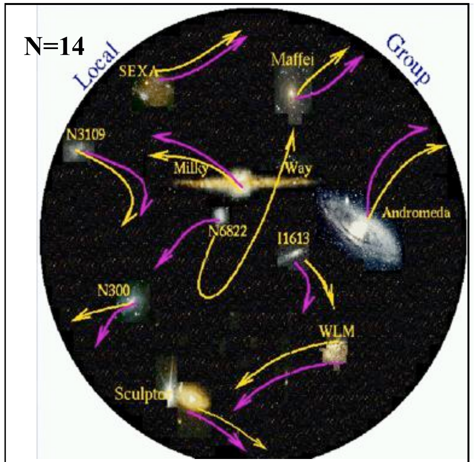
Boundary condition 2:
τ _ 0
Initial positions q
are uniform

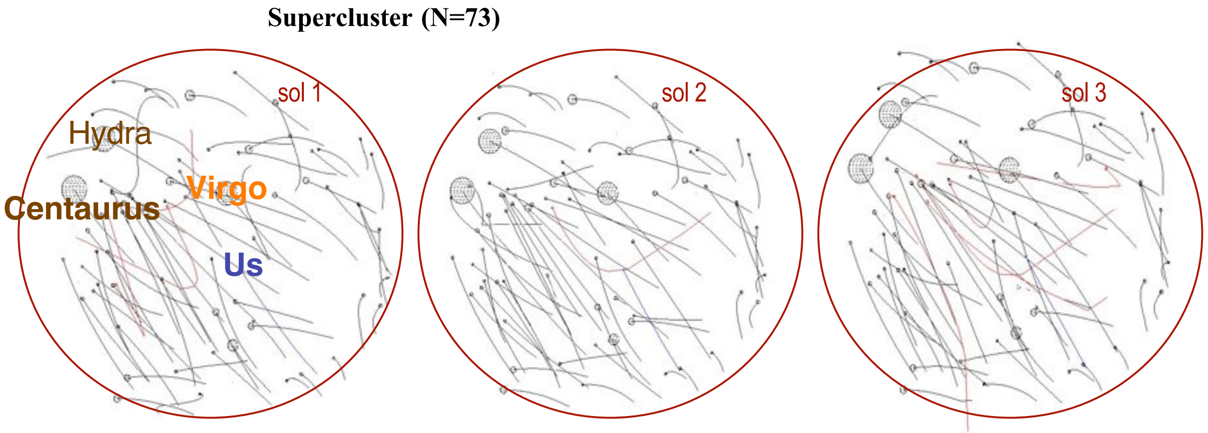


But there are many
Ensembles of paths !

Complex landscape !

Lack of uniqueness !



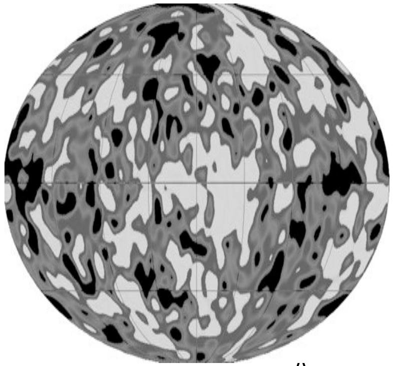


etc...

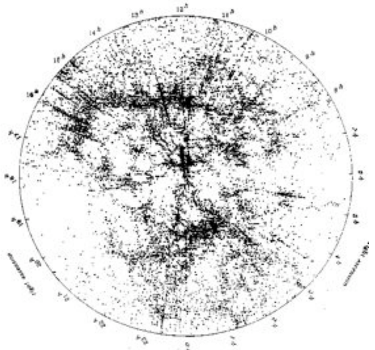
Monge-Ampère-Kantorovich Reconstruction

Frisch et al Nature 2002, 417, 260 ; Brenier et al MNRAS 2003,346,501

Initial (q)



$x=\nabla_q\Phi(q)$ $q=\nabla_x\Theta(x)$



present (x)

Hypothesis: Potential flow + Convexity

$x=\nabla_q\Phi(q)$

⇩

Legendre transform : $\Theta(x)=\underset{q}{\mathrm{Max}}\,(x.q-\Phi(q))$

$q=\nabla_x\Theta(x)$

⇩

Conservation of mass **$\rho_\iota(q)dq = \rho(x)dx$**

⇩

Monge-Ampère Equation

$$\det\left[\partial_{x_i}\partial_{x_j}\Theta(x)\right]=\rho(x)/\rho_\iota(q)$$

⇩

Solution: Variational approach à la Kantorovich

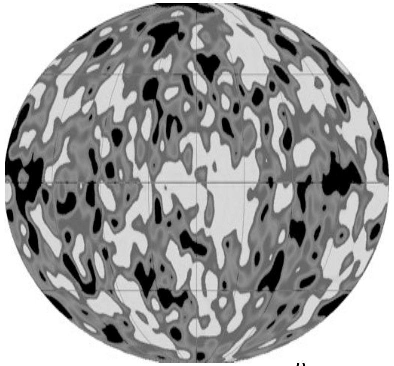
(Y. Brenier 1984)

The solution is UNIQUE

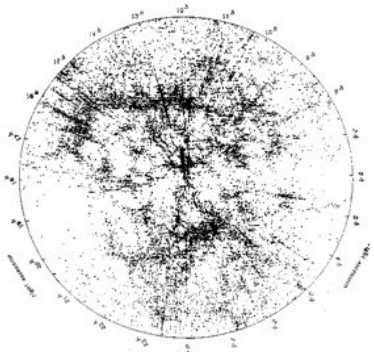
Reconstruction of velocity and initial density fields

Frisch et al Nature 2002, 417, 260 ; Brenier et al MNRAS 2003,346,501

Initial (q)



$x=\nabla_q\Phi(q)$ $q=\nabla_x\Theta(x)$



present (x)

Hypothesis: Potential flow + Convexity
$x=\nabla_q\Phi(q)$

Why !

Legendre transform : $\Theta(x)=\underset{q}{\mathrm{Max}}\,(x.q-\Phi(q))$

$q=\nabla_x\Theta(x)$

Conservation of mass $\boldsymbol{\rho_\iota(q)dq = \rho(x)dx}$

Monge-Ampère Equation

$$\det\,[\partial_{x_i}\partial_{x_j}\Theta(x)]=\rho(x)/\rho_\iota(q)$$

Solution: Variational approach à la Kantorovich
(Y. Brenier 1984)

The solution is UNIQUE

"A" Universe filled by dark matter fluid

Klimontovich density

$$f = m\sum \delta[r-r_i(t)]\,\delta[v-v_i(t)]$$

coarse graining
+
collisionless

Vlasov-Poisson equation

$$\frac{\partial\langle f\rangle}{\partial t} + \frac{\partial\langle f\rangle}{\partial r}\frac{dr}{dt} + \frac{\partial\langle f\rangle}{\partial v}\frac{dv}{dt} = 0$$

$$\nabla^2\phi \sim \int \langle f\rangle\, dv$$

Take the moments
+
velocity dispersion tensor=0

Fluid equations

Fluid dynamics in expanding Universe

EULER :

$\partial_\tau \upsilon + (\upsilon.\nabla_x)\upsilon = (-3/(2\tau))\ (\upsilon+\nabla_x\varphi_g)$

Mass Conservation :

$\partial_\tau \rho + \nabla_X.(\rho\upsilon) = 0$

Poisson:

$\nabla_X{}^2\varphi_g = (\rho-1)/\tau$

Fluid dynamics in expanding Universe

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$\tau(t) \rightarrow 0$:

$\upsilon_i = -\nabla\varphi^i_g$

$\rho_i = 1$

Fluid dynamics in expanding Universe

EULER (compressible) :

$$\partial_\tau \upsilon + (\upsilon.\nabla_{\mathbf{x}})\upsilon = (-3/2\tau)\ (\upsilon + \nabla_x \varphi_g)$$

Mass Conservation :

$$\partial_\tau \rho + \nabla_x.(\rho\upsilon) = 0$$

Poisson :

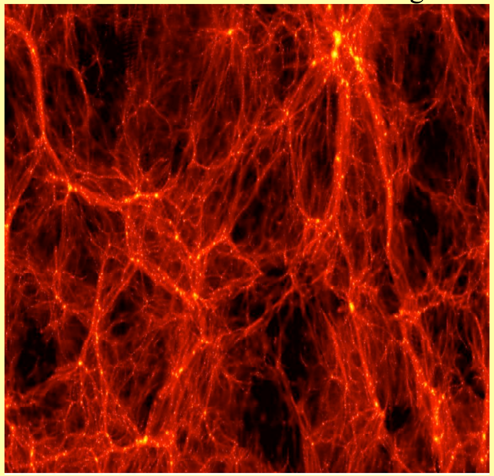
$$\nabla_x^{\ 2}\varphi_g = (\rho - 1)/\tau$$

Zeldovich approximation

$$\upsilon = -\nabla_{\mathbf{x}}\varphi_g$$

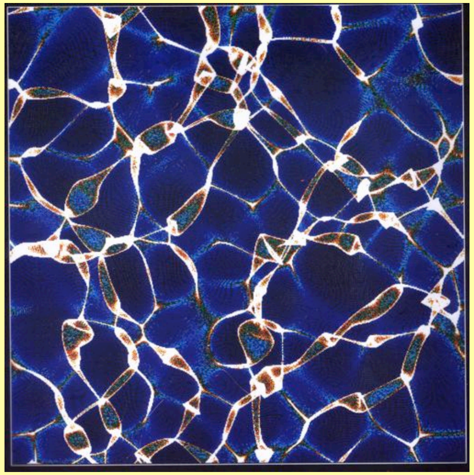
Full dynamical equation

$$D_\tau \upsilon = (-3/2\tau)\,(\upsilon + \nabla_x \varphi_g)$$



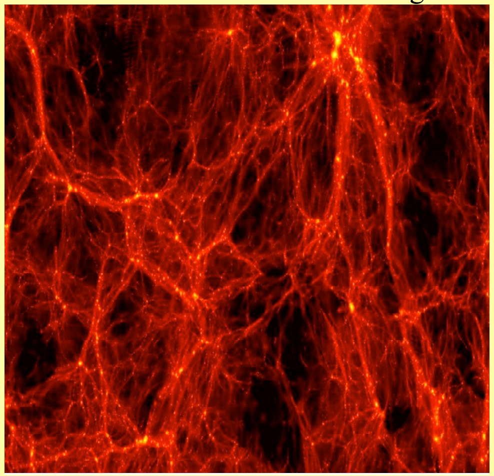
"Zeldovich" approximation

$$D_\tau \upsilon = 0$$



Full dynamical equation

$D_\tau \upsilon = (-3/2\tau)\,(\upsilon + \nabla_x \varphi_g)$



"Zeldovich" approximation

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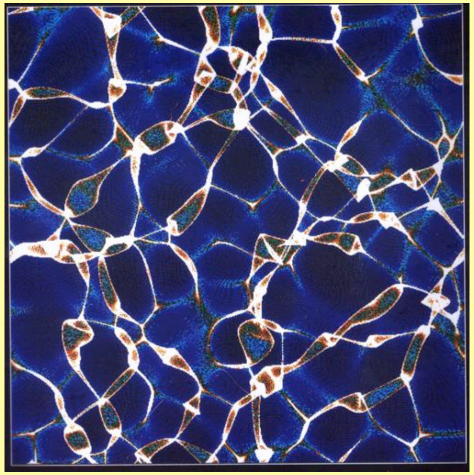
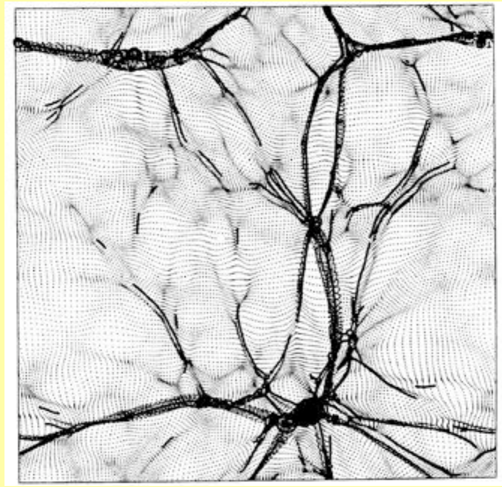


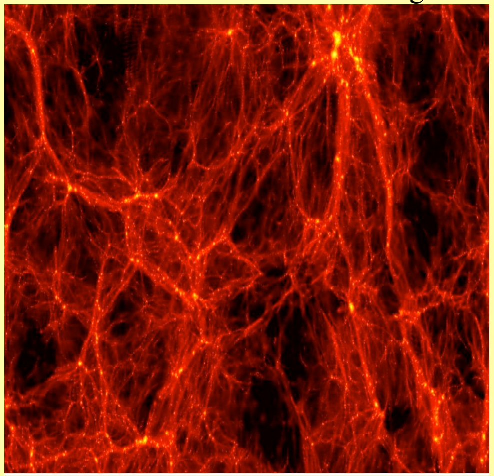
"Burgers" equation

$D_\tau \upsilon = \nu \nabla^2 \upsilon$



Full dynamical equation

$D_\tau \upsilon = (-3/2\tau)\,(\upsilon + \nabla_x \varphi_g)$



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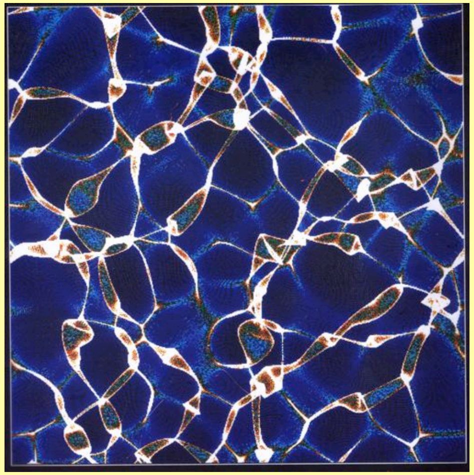

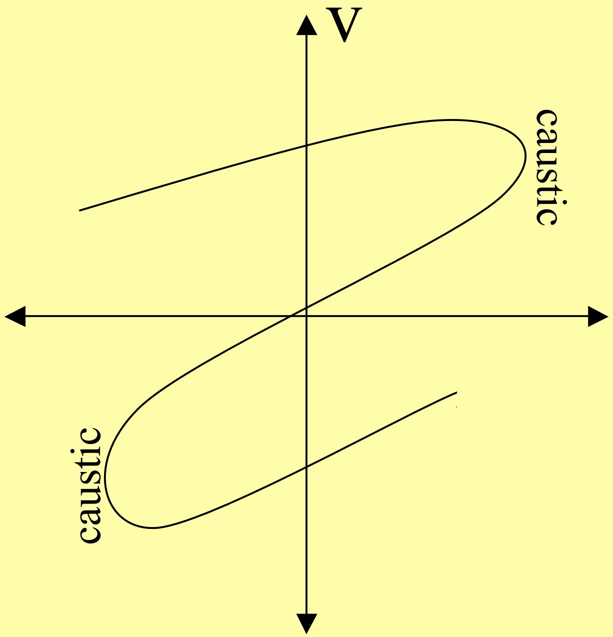

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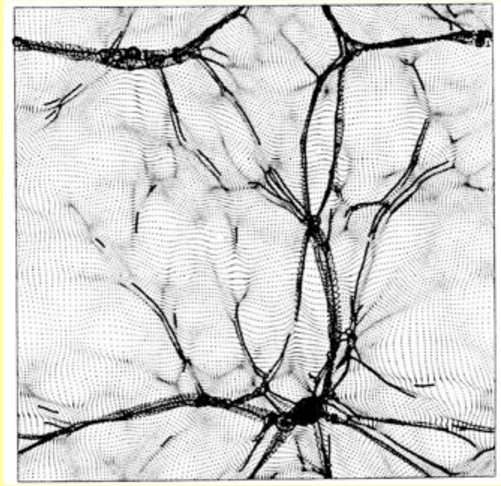
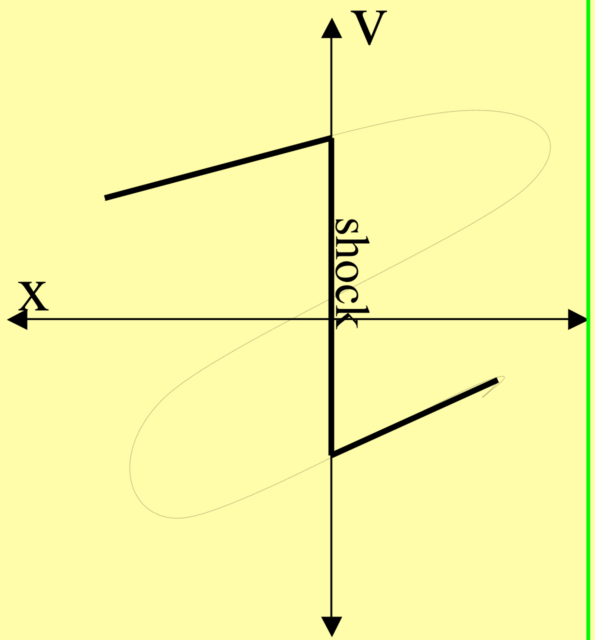




Lagrangian map : a convex potential

Zeldovich/Burger equations $D_\tau \upsilon = 0$ □ $x = q + \tau\, \upsilon^i(q)$

$$= q - \tau \nabla_q \varphi\,(q)$$

$$= \nabla_q \left(q^2/2 - \tau\varphi(q) \right)$$

$$= \nabla_q \Phi(q)$$

If $\Phi(q)$ is convex :

$$X = \nabla_q \Phi(q)$$

$$\Phi(q) = \mathrm{Max} \left(x.q - \Theta(x) \right)$$

Legendre transform

$$\Theta(x) = \mathrm{Max} \left(x.q - \Phi(q) \right)$$

$$q = \nabla_x \Theta(x)$$

Monge-Ampère equation

Inverse lagrangian map

$q = \nabla_x \Theta(x)$

Mass conservation

$\boldsymbol{\rho_\iota(q)dq = \rho(x)dx}$

Monge-Ampère equation

$\det [\partial_{x_i} \partial_{x_j} \Theta(x)] = \rho(x)/\rho_\iota(q)$

Monge-Ampère equation: optimal assignment problem

Brenier 1987 , 1991

Minimize over all x(q) & q(x)

$$\int |x(q)-q|^2 \rho(x)\ dx = \int |x-q(x)|^2 \rho_\iota(q)\ dq$$

The minimizing maps are : $q = \nabla_x \Theta(x)$ $\quad x = \nabla_q \phi(q)$

Discretize

$\rho(x)=\Sigma\delta(x-x_i)$; $\rho_\iota(q)=\Sigma\delta(q-q_j)$

Minimize Over all $i \rightarrow j(i)$

$$I = \sum_{i=1}^{N} |x_i - q_{j(i)}|^2$$

Complexity of optimization code :

Brute force: $N!$

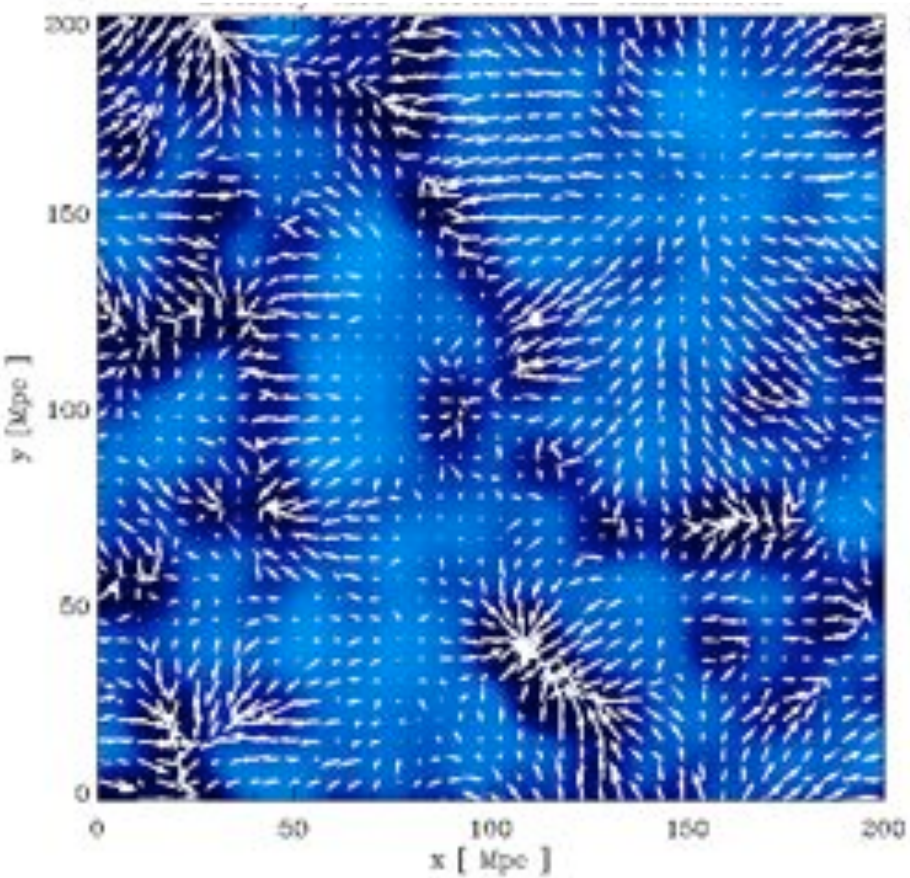
worst case: N^3

best case : $N^{2.25}$

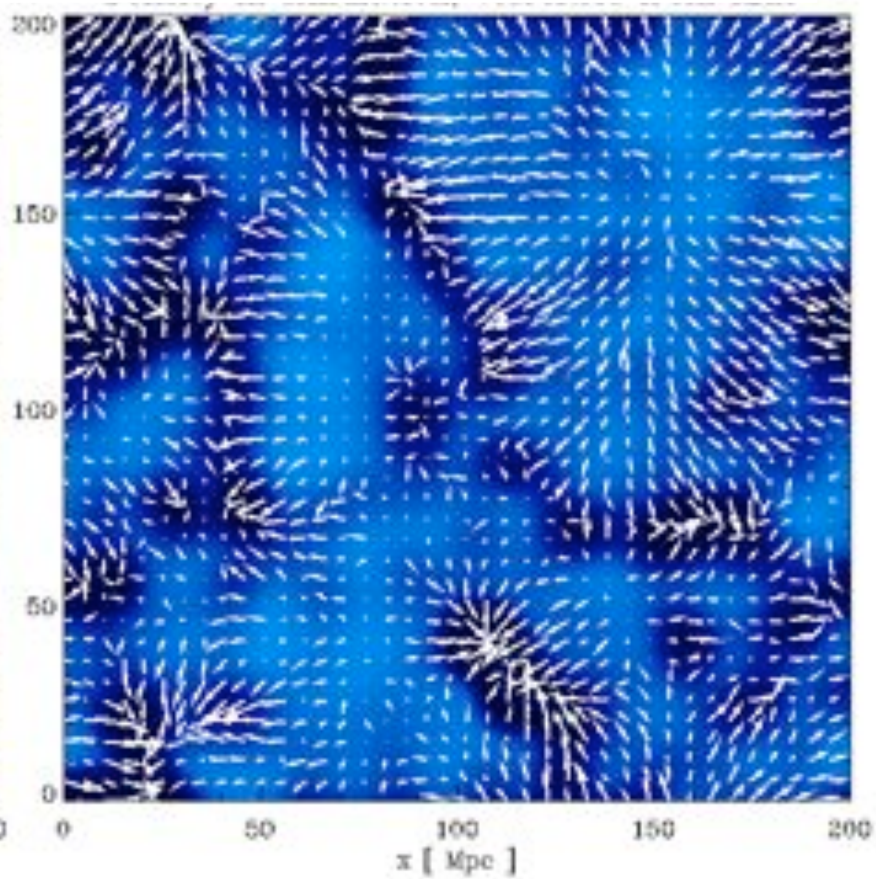
Brenier conjecture: Nlog N (true in 1d)

Test of Monge-Ampère-Kantorovich with numerical reconstruction
($N=128^3$, L=200 Mpc, 1pc= 200,000 Earth-Sun distance)

Simulated velocity field

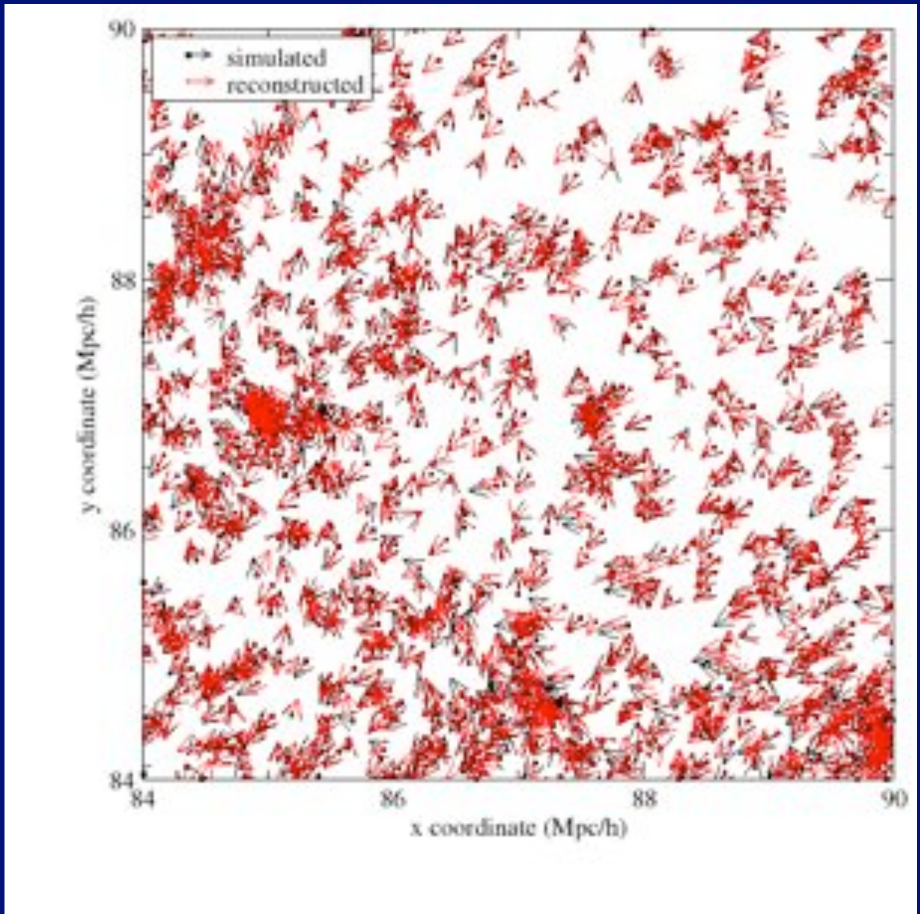


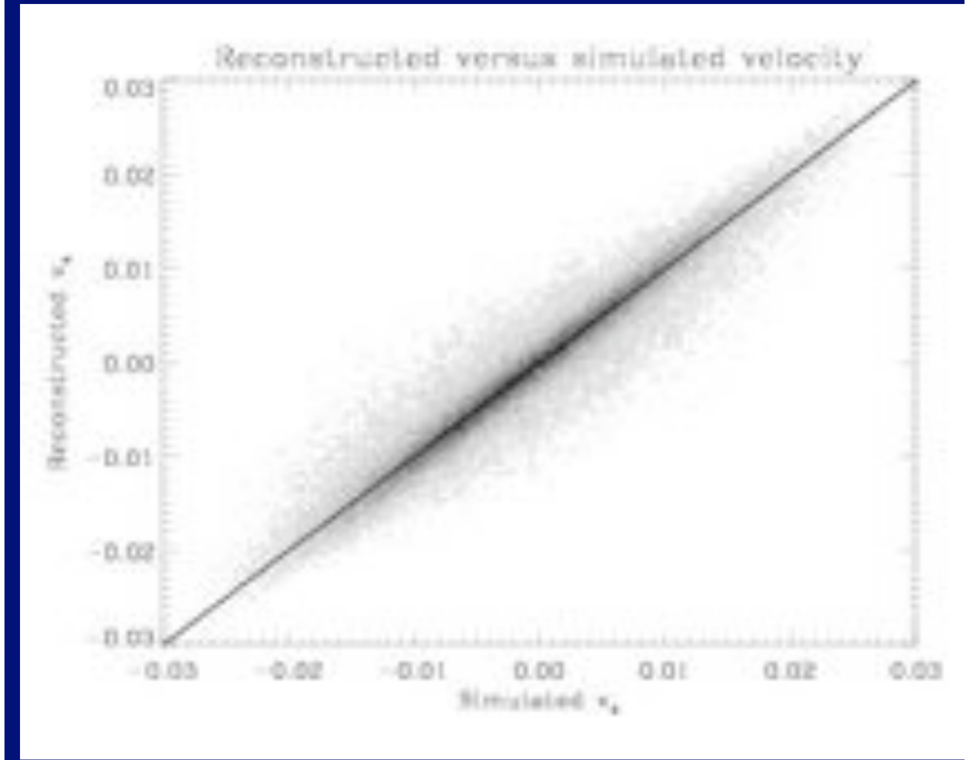
Reconstructed velocity field



Test of reconstruction with N-body simulation (2,000,000 particles)

Velocity Field

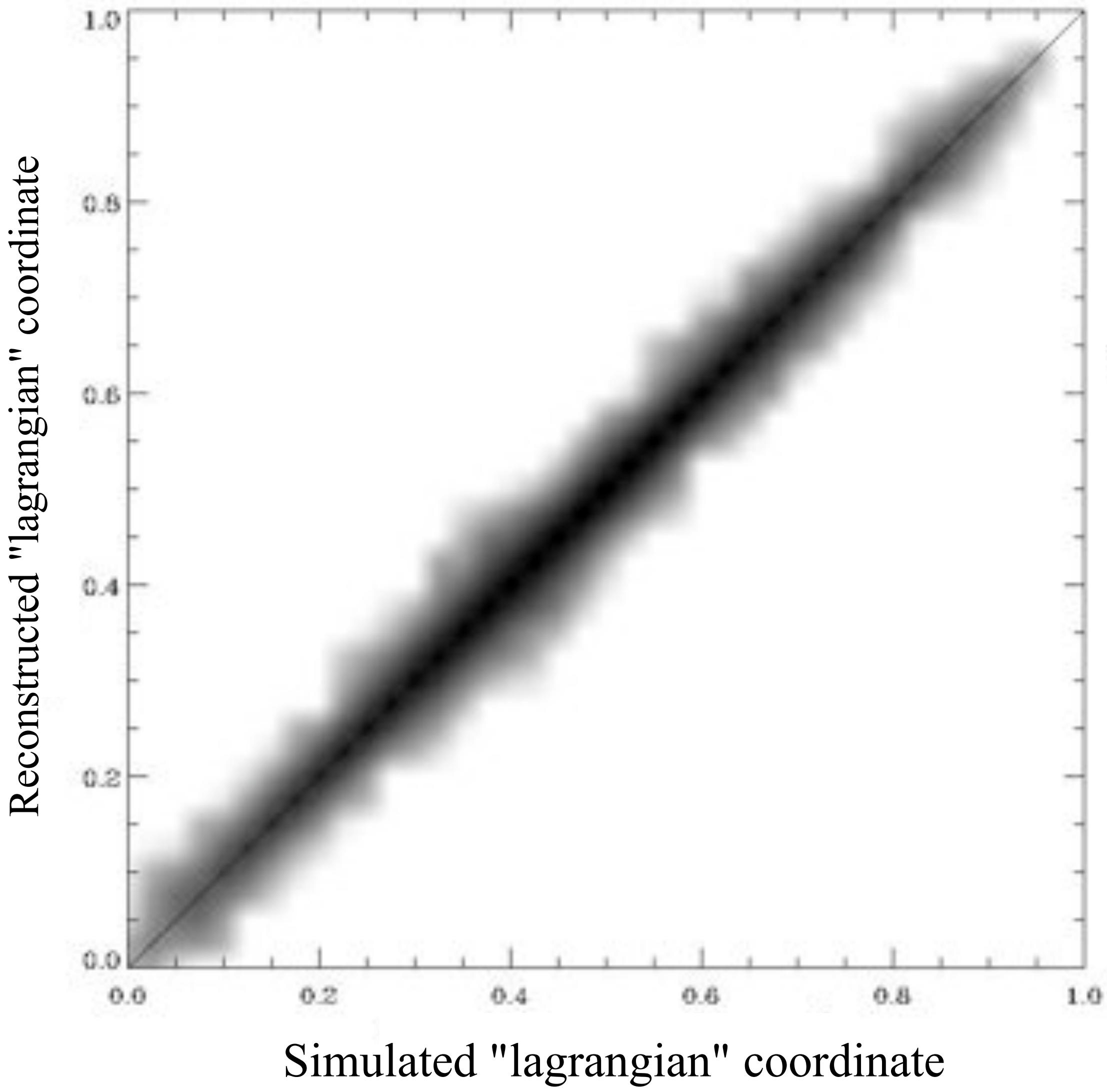


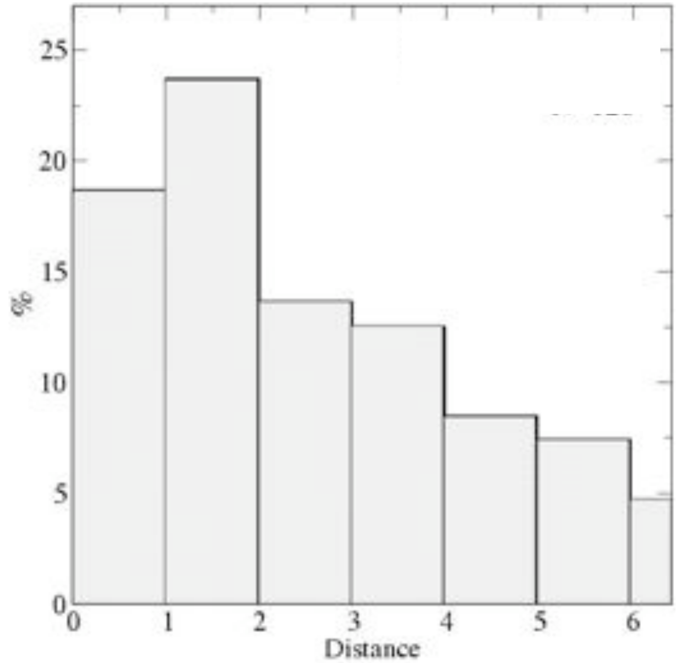


Mohayaee et al A&A 2003 ; MNRAS 2006

2,000,000 particle reconstruction

Scale ~ 1 Mpc

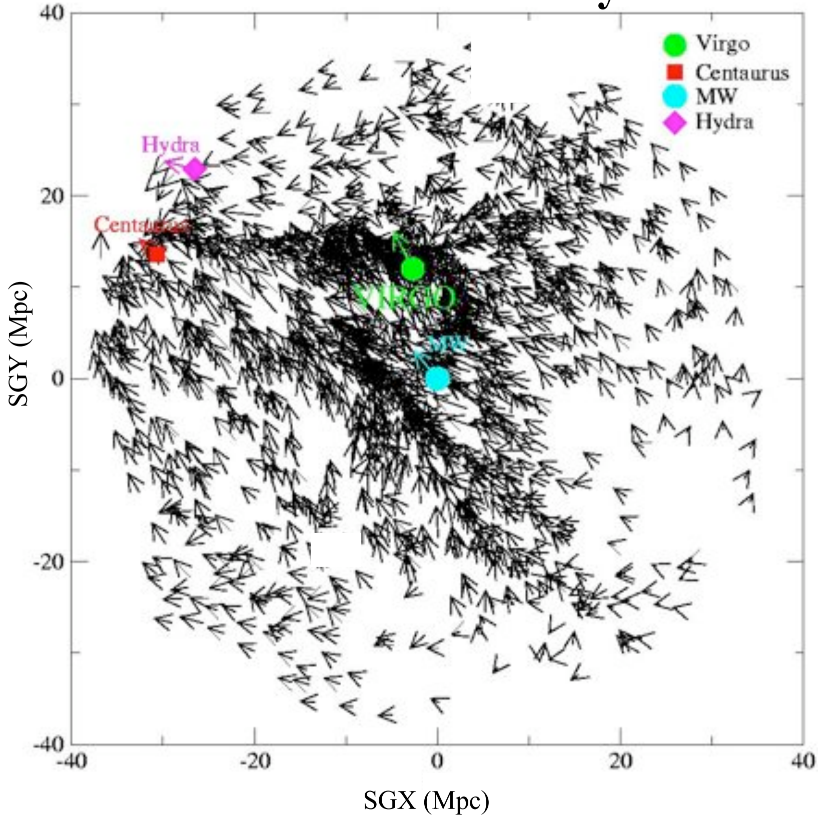




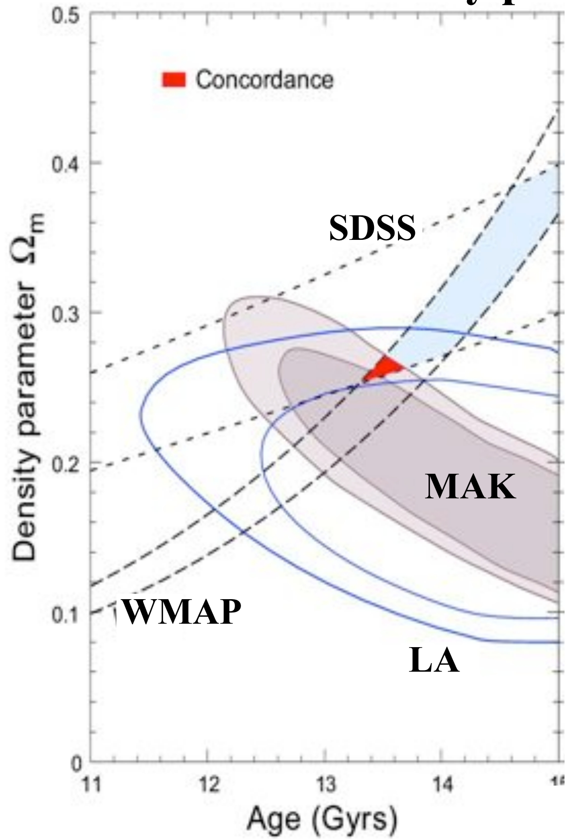
Determination of local velocity field and mass density of the Universe

Mohayaee & Brent Tully: ApJL 635, 2005

Reconstructed velocity field



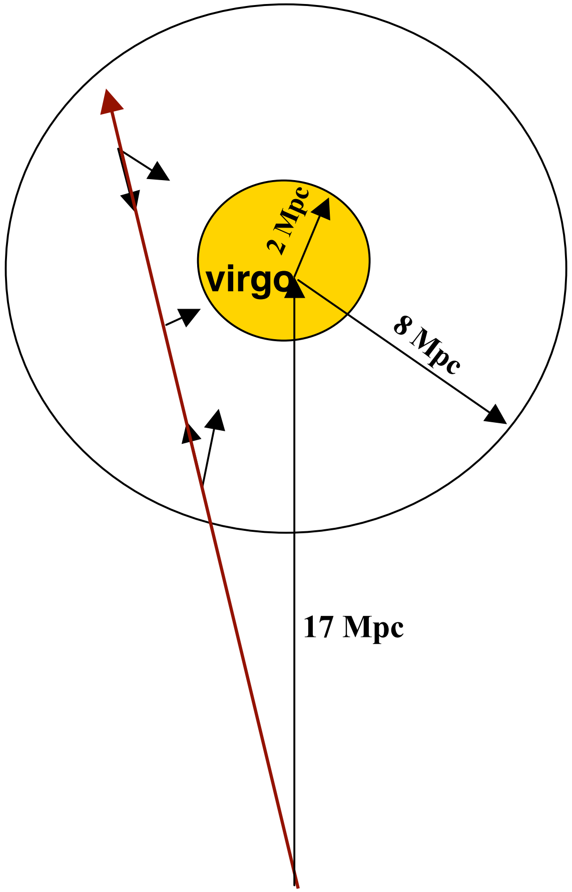
Determination of density parameter

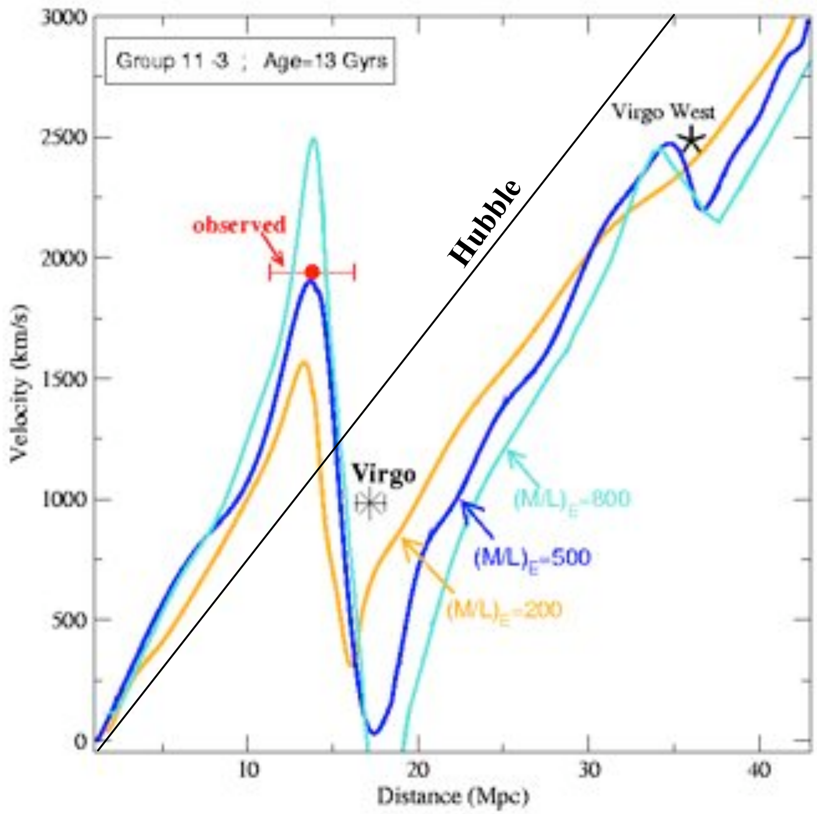


Ω_m=0.24 ± 0.03
t =13.2 ± 0.6 Gyrs

Determination of local fluctuation in the mass density

Mohayaee & Brent Tully 2005





$M/L_{virgo} = 500 \pm 70$

$M_{virgo} = (8. \pm 1.1) \times 10^{14}\ M_o$

$M/L = 280 \pm 20$

$t = 13.2 \pm 0.6$ Gyrs

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Frisch et al Nature 2002, 417, 260 ; Brenier et al MNRAS 2003,346,501

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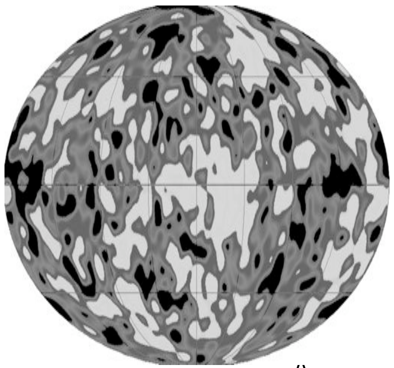

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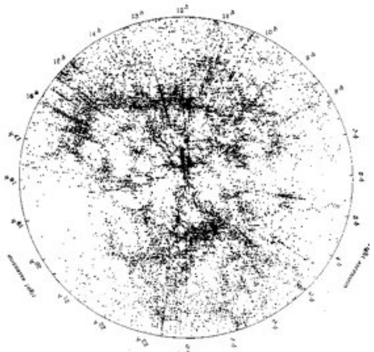

present (x)

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Legendre transform : $\Theta(x)=\mathrm{Max}_q\,(x.q-\Phi(q))$ hence $q=\nabla_x\Theta(x)$

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Solution: Variational approach à la Kantorovich
(Y. Brenier 1984)

Discrete case | N galaxies

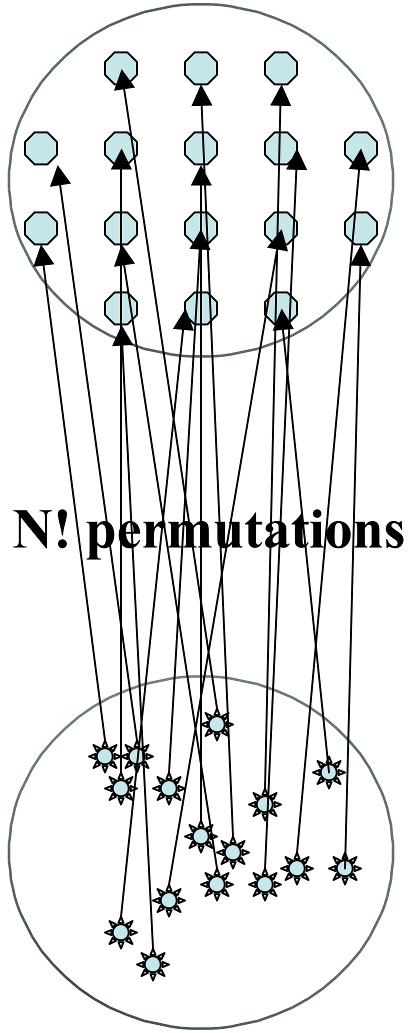
Assignement problem in optimization theory

$$\min_{m(n)}\sum_n (x_n-q_{m(n)})^2$$

discretize

$$I\sim\int d\tau\int\tau^{3/2}\left(\rho|v|^2+(3/2)|\nabla_x\varphi|^2\right)$$

initial (q)



final (x)