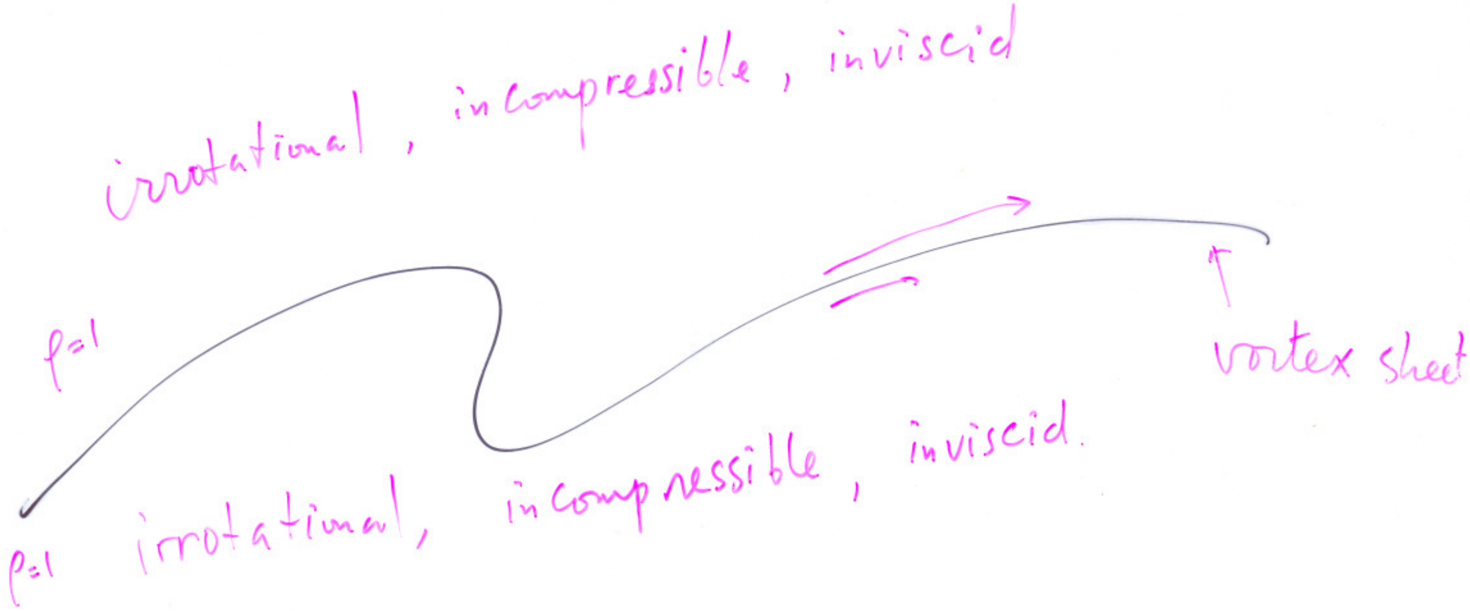
Recent Progress in Mathematical Analysis of Vortex Sheets

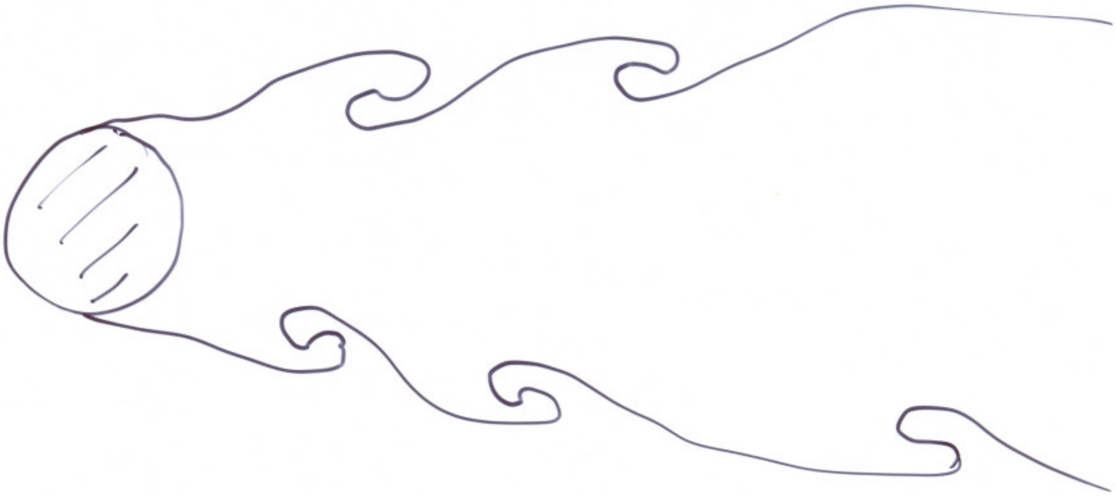
Sijue Wu
Univ. of Michigan, Ann Arbor



We consider the motion of the interface Seperating two domains of the same fluid in $\mathbb{R}^2$ that moves with different velocity along the tangential direction of the interface.

this interface $\longrightarrow$ vortex sheet.

We assume the fluid occupying the two domains are incompressible, irrotational, inviscid. Also we assume the surface tension $= 0$.



Q: What is the specific nature of vortex sheet motion?

There are two approaches

1) Incompressible Euler Eq'n with vortex sheet initial data. Let $v_0(x)$ be the initial velocity, vortx sheet data if $v_0(x)$ is discontinous along
$v_0(x)$ is a a curve Σ_0 $\Leftrightarrow$ curl v_0 is a measure supported on Σ_0.

a). Delort (1991) Existence of weak sol'n for any initial velocity s.t. curl $v_0 \geq 0$, and $v_0 \in L^2_{loc}(\mathbb{R}^2)$

b)

b). Scheffer (1993), Shnirelman (1997, 2003)

Nonuniqueness of weak sol'n in $L^2((0,T)\times\mathbb{R}^2)$ ~~thus~~

weak sol'n provides little information about the specific nature of vortex sheet evolution.

for example: Is vortex sheet a curve at any later time if initially it is a curve?

Second approach.

2). Assuming further that vortex sheet is a curve in $\mathbb{R}^2$ at a later time. $t>0$.

Incompressible Euler eq'n + this assumption

$\Rightarrow$ Birkhoff - Rott Eq'n:

$$\boxed{\bar{Z}_t(\alpha,t) = \frac{1}{2\pi i}\, \text{p.v.} \int \frac{1}{Z(\alpha,t)-Z(\beta,t)}\, d\beta}$$

Here: $Z(\alpha,t) = x(\alpha,t) + i\, y(\alpha,t)$ is the (complex) position fcn of the vortex sheet curve at time t with r.t. the circulation variable α.

i.e. $\gamma = \frac{1}{|Z_\alpha|}$ is the vortex strength.

$Z = Z(\alpha,t)$ = (complex) position of the vortex sheet curve at time t.

Sulem, Sulem, Bardos, Frisch (1981) Existence and uniqueness of sol'ns in analytic class for any given analytic data.

D. W. Moore (1979) analytic evidence that vortex sheet can develop singularity from analytic data.

Krasny (1986), Meiron, Baker & Orszag (1982) numerical confirmation

Duchon & Robert (1988), Caflisch & Orellana (1989) constructed specific examples of sol'ns of the Birkhoff-Rott eq'n, where a curvature singularity develops in finite time from analytic data

$\Rightarrow$ Birkhoff-Rott eq'n is illposed in $C^{1+\rho}$, $H^s(\mathbb{R})$

G. Lebeau (2001) $\rho > 0$, $s > 3/2$.

<u>Q.</u> Are there sol'ns in a class larger than $C^{1+\rho}$ or H^s. $(\rho > 0, \ s > 3/2)$?

What is the nature of the vortex sheet at and beyond the singularity formation time?

- We look for sol'ns of the B-R Eq'n in the largest possible spaces where the eq'n makes sense.

- Experimental & numerical evidences suggest that this class should include (infinite) spirals.

- The class $C^{1+\rho}$ or H^s, $\rho > 0$. $s > 3/2$ does not contain (infinite) spiral curves.

These lead us to consider sol'ns in Chord-arc class.

- . Chord-arc class contains log spirals.
- Chord-arc class is almost the largest in L^2 sense.

Chord-arc curves : (non self intersecting)

Let $\xi = \xi(s)$ describes ~~the~~ a curve Γ in arclength s.

We say Γ is chord-arc, if $\exists\ M > 1$. s.t.

$$\underbrace{|s_1 - s_2|}_{\text{arclength}} \leq M \underbrace{|\xi(s_1) - \xi(s_2)|}_{\text{chord-length}} \quad \text{for all } s_1, s_2.$$

The infimum of all such M is called the chord-arc constant.

Def: $m_Q(f) = \frac{1}{|Q|}\int_Q f(x)\,dx$

Let $I \subset \mathbb{R}$ be an interval, We say a fcn $f \in L^1_{loc}(\mathbb{R})$ is in $BMO(I)$ if

$$\|f\|_{BMO(I)} = \sup_{Q\subset I,\ Q \text{ interval}} \frac{1}{|Q|}\int_Q |f(x) - m_Q f|\,dx < \infty.$$

Thm: (G. David) 1) If $b \in BMO(\mathbb{R})$, and $\|b\|_{BMO} < 1$, then $\xi(s) = \xi_0 + \int_0^s e^{ib(s')}ds'$ defines a chord-arc curve, with chord-arc constant $\leq \frac{1}{1-\|b\|_{BMO}}$.

2) If $\xi = \xi(s)$ defines a chord-arc curve, s-arclength then $\xi'(s)$ exist a.e. and $\exists\ b \in BMO$. s.t. $\xi'(s) = e^{ib(s)}$ Moreover, if the chord-arc constant $M \approx 1$, then $\|b\|_{BMO} \approx 0$.

Example: take $b = \varepsilon \ln|s| \in BMO$, $\varepsilon \ll 1$.

Let $\xi(s) = \int_0^s e^{i\varepsilon \ln|s'|}ds' = \int_0^s |s'|^{i\varepsilon}ds' = \frac{1}{1+i\varepsilon} s^{1+i\varepsilon}$ $(s>0)$
$\begin{cases} \frac{1}{1+i\varepsilon} \\ \ldots \end{cases}$

this is a log spiral.

<u>Thm</u>: The Birkhoff-Rott Eq'n is of "elliptic type" in the chord-arc class.

A typical elliptic eq'n: $\Delta u = u_{tt} + u_{xx} = 0$

Initial value problem: $(*)\begin{cases} u_{tt} + u_{xx} = 0 \\ u(x,0) = u_0(x) \\ u_t(x,0) = u_1(x) \end{cases}$ $\Big\}$ given

$\hat{u}(\xi, t)$ = Fourier transform of $u(x,t)$ in variable x

Sol'n $\hat{u}(\xi,t) = \frac{1}{2}\left(\hat{u}_0(\xi) + \frac{1}{|\xi|}\hat{u}_1(\xi)\right)e^{|\xi|t} + \frac{1}{2}\left(\hat{u}_0(\xi) - \frac{1}{|\xi|}\hat{u}_1(\xi)\right)e^{-|\xi|t}$

$\Rightarrow$ Conclusions:

1) $(*)$ can be solved in $L^2(\mathbb{R})$ for $0 \le t \le T$, if and only if $\hat{u}_0(\xi) + \frac{1}{|\xi|}\hat{u}_1(\xi)$ is real analytic

i.e. $\exists\, a > 0$. s.t. $\left(\hat{u}_0(\xi) + \frac{1}{|\xi|}\hat{u}_1(\xi)\right)e^{a|\xi|} \in L^2(\mathbb{R})$

2) If for $0 \le t \le T$, we have $u(\cdot, t) \in L^2(\mathbb{R})$

then $\left(\hat{u}_0(\xi) + \frac{1}{|\xi|}\hat{u}_1(\xi)\right)e^{T|\xi|} \in L^2(\mathbb{R}) \Rightarrow$

$\left(\hat{u}_0(\xi) + \frac{1}{|\xi|}\hat{u}_1(\xi)\right)e^{t|\xi|} = \left(\hat{u}_0(\xi) + \frac{1}{|\xi|}\hat{u}_1(\xi)\right)e^{T|\xi|}\, e^{(t-T)|\xi|}$ is analytic for $0 < t < T$.

$\Rightarrow u(\cdot, t)$ real analytic for $0 < t < T$.

Thm 1. Assume that $Z \in H^1([0,T], L^2_{loc}(\mathbb{R})) \cap L^2(0,T; H^1_{loc}(\mathbb{R}))$ is a sol'n of the Birkhoff-Rott eq'n for $0 \le t \le T$, satisfying that

1. $\exists$ $m, M > 0$, indep't of t, s.t.

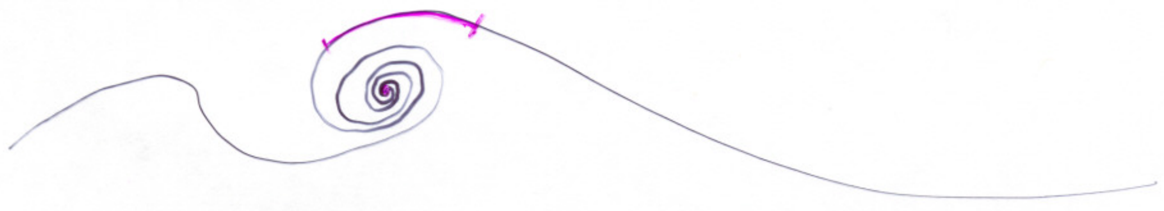
$$m|\alpha-\beta| \le |Z(\alpha,t) - Z(\beta,t)| \le M|\alpha-\beta| \quad \forall \alpha,\beta \in \mathbb{R},\ 0 \le t \le T$$

Then there is a constant $C(m,M) > 0$ as follows. If also

2. on some fixed interval (a,b), $\exists$ $\delta_0 > 0$, satisfying

$$\sup_{[0,T]} \|\ln Z_\alpha(\cdot,t)\|_{BMO(a,b),\delta_0} \le C(m,M)$$

Then in fact $Z_\alpha \in C((a,b)\times(0,T))$, and for each $t_0 \in (0,T)$ $Z_\alpha(\cdot,t_0)$ is analytic on (a,b).



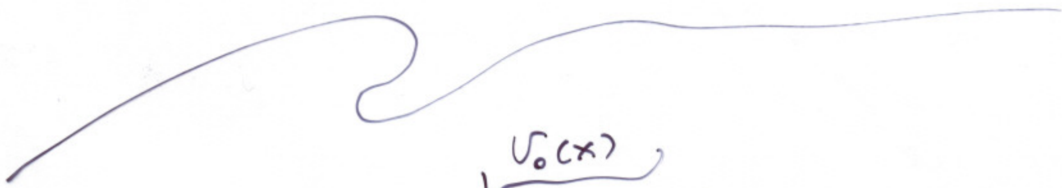
1. means: $Z = Z(\alpha,t)$ is chord-arc and the vortex strength $0 < \frac{1}{M} \le \gamma = \frac{1}{|Z_\alpha|} \le \frac{1}{m} < \infty$

2. $\Leftrightarrow$ The vortex sheet curve does not roll-up too fast on a section (a,b).

G. Lebeau (2001). Assuming $Z \in C^{1+\rho}$, $\rho > 0$, (stronger assumption)

Cor: After the singularity formation time, the vortex sheet will not be a chord-arc curve that doesn't roll-up too fast.

Initial value problem for the Birkhoff-Rott eq'n.



If the initial velocity $V_0(x)$ is given, then the position of the initial vortex sheet is given, (where the discontinuity is.) and the initial vortex strength γ is also given.

$\Rightarrow$ $Z_0 = Z(\alpha, 0)$ is completely given.

$\Rightarrow$ $\ln Z_\alpha(\alpha, 0)$ is given.

$(1+i) \ln Z_\alpha(\alpha, 0) =$ ~~[illegible]~~ Real part + i Imaginary part

Thm 2: For any real valued fcn $\omega_0 \in H^{3/2}(\mathbb{R})$, there exist a $T = T(\|\omega_0\|_{H^{3/2}}) > 0$, such that the Birkhoff-Rott eq'n has a sol'n $z = z(\alpha, t)$ for $0 \le t \le T$, satisfying $\mathrm{Im}\{(1+i)\ln z_\alpha(\alpha, 0)\} = \omega_0(\alpha)$, and $z(\cdot, t)$ is real analytic for each $0 < t < T$.

Thm 3: Assume that $z \in H^1([0,T], L^2_{loc}(\mathbb{R})) \cap L^2(0,T, H^1_{loc}(\mathbb{R}))$ is a sol'n of the Birkhoff-Rott eq'n for $0 \le t \le T$, $T > 0$, satisfying properties 1 and 2 of Thm 1. on some interval (a, b). Assume further that $\omega_0 = \mathrm{Im}\{(1+i)\ln z_\alpha(\cdot, 0)\}$ is analytic on (a, b). Then $z_\alpha \in C((a,b) \times [0, T))$, and $\mathrm{Re}\{(1+i)\ln z_\alpha(\cdot, 0)\}$ is also analytic on (a, b).