

A STUDY OF TURBULENCE AND CONVECTION
USING FOURIER AND NUMERICAL ANALYSIS

by

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Preface

The work described in this dissertation was carried out in Cambridge between October 1961 and August 1964 and in Colchester between the latter date and February 1966. No part of it has been submitted in any form for a degree, diploma or other qualification of any other university, and it is believed to be original except where stated in the text.

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CHAPTER 1

TWO DIMENSIONAL TURBULENCE

1.1 INTRODUCTION

The study of turbulence to the present date has largely been confined to the study of three, as opposed to two, dimensional motions. There appear to be two main reasons for this. The first arises from the practical difficulty in setting up a two dimensional flow: no one has as yet produced such a flow with any refinement, in sharp contrast to the abundance of the three dimensional flows both inside and out of the laboratory, and consequently the study of two dimensional turbulence has never received any impetus from the experimentalist. Indeed there is a suspicion, which we shall discuss more fully in §2.5, that strictly two dimensional turbulence is not generally a realizable flow since any such motion, if it were to be produced, would be 'unstable' to three dimensional disturbances and would quickly degenerate into three dimensional turbulence. Thus there is even a positive discouragement from the experimental viewpoint to try to establish a two dimensional turbulence theory.

The second reason for the lack of study results from the realization that the dynamics of two and three dimensional turbulence must be radically different. In two dimensional motions there is no internal mechanism which can generate vorticity, such as the 'stretching of vortex lines' in the three dimensional situation, so the structure of the governing equations of motion and, as a result, the constraints on the vorticity and energy dissipation are necessarily not the same in the two situations. This fact, combined with the disincentive of the first reason, has led to an almost complete neglect of the subject which is discussed in this dissertation.

The original motivation for attempting this study of two-dimensional turbulence arose from the need to adopt as simple an analytic situation as possible in developing the numerical technique described in chapter 3. Whilst that motive still remains, it is to be noted that this study may now also be justified in the fact that there appear to be striking parallel situations in two and three dimensional turbulence so that the testing of a three dimensional turbulence theory in its two dimensional analogue may prove to give a valuable added insight in future.

1.2 STATEMENT OF THE PROBLEM

We shall study the motion of a fluid which satisfies the following conditions:

- (i) The fluid is unbounded.
- (ii) The velocity of the fluid is everywhere parallel to a fixed plane and is independent of the co-ordinate normal to that plane.
- (iii) The velocity field satisfies the continuity equation for an incompressible fluid, so that if the velocity

$$\mathbf{u} = (u(x, y, t), v(x, y, t), 0)$$

then

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (1.2.1)$$

- (iv) The velocity field satisfies the Navier-Stokes equation

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = \mathbf{F} - \frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u} \quad (1.2.2)$$

where $\mathbf{F}$ is the body force per unit mass exerted on the fluid, ρ is the density, and ν is the kinematic viscosity of the fluid.

- (v) The turbulence is homogeneous. Thus, given any n kinematic quantities $f_1(x, t), \dots, f_n(x, t)$

where n is an arbitrary integer and where any of the f 's may be components of a vector, then the quantity

$\overline{f_1(x, t) \cdot f_2(x+r_2, t+t_2) \dots f_n(x+r_n, t+t_n)}$,
where the overbar denotes averaging, exists and is independent of $\underline{x}$.

The averaging procedure we shall adopt will be either an averaging over space at one instant in time of a particular flow or an averaging over a set of different realizations of flows at one point in space and at one instant in time. We shall assume that there is some applicable Ergodic Theorem which implies that the ensemble average of a well-defined set of realizations is equivalent to the spatial average over almost any of the realizations in the set, so we need not distinguish which averaging procedure we have used in the above formulation of homogeneity or in many of the cases of averaging we shall describe below.

Our aim in this dissertation is to try to describe the behaviour of some of the simplest statistical quantities of two dimensional turbulence. It has become the practice in turbulence theory to ascribe some importance to the velocity and vorticity correlation functions, or alternatively to the energy and mean-square vorticity spectrum functions, and we shall concentrate on attempting to specify these in particular. To do this we shall either construct some initial value problem and try to obtain the detailed structure of these quantities or, failing the first method, try to specify their general functional form for some assumed wide class of turbulent flows. Both of these approaches are widely used in turbulence theory and need no justification here.

A large part of our analysis will be concerned with the (stochastic) Fourier-Stieltjes transform of various quantities, so we will briefly outline the definition of this operator before proceeding with the theory proper. If, say, $\omega(\underline{x}, t)$ is the vorticity field we set up the definition of the transform $dY(\underline{k}, t)$ of the vorticity as follows (see Batchelor (1953, p.30)):

Suppose P is a fixed point in space and we construct a Cartesian reference system with P as origin. Let

$$\begin{aligned}\tilde{\omega}_P(\underline{x}, t, X) &= \omega(\underline{x}, t) \quad \text{for } |\underline{x}| < X \\ &= 0 \quad \text{otherwise,}\end{aligned}$$

then the Fourier integral

$$A(\underline{k}, t, X) = (2\pi)^{-2} \int \tilde{\omega}_P(\underline{x}, t, X) e^{-i\underline{k} \cdot \underline{x}} d\underline{x}$$

exists and we can construct the Fourier-Stieltjes transform of the vorticity as

$$dY(\underline{k}, t) = \lim_{X \rightarrow \infty} \int_{\underline{k}_1}^{\underline{k}_1 + d\underline{k}_1} \int_{\underline{k}_2}^{\underline{k}_2 + d\underline{k}_2} A(\underline{k}, t, X) d\underline{k}_2 d\underline{k}_1. \quad (1.2.3)$$

We note that this definition is dependent on the position of P so that we can construct an averaging procedure either by averaging over all possible positions of P for some particular flow or by keeping P fixed and averaging over an ensemble of realizations.

As an approximation to $dY(\underline{k}, t)$ for some particular value of $d\underline{k}$, which we shall assume is such that

$$|d\underline{k}| \ll |\underline{k}|,$$

we shall take

$$dY(\underline{k}, t) \doteq \int_{\underline{k}_1}^{\underline{k}_1 + d\underline{k}_1} \int_{\underline{k}_2}^{\underline{k}_2 + d\underline{k}_2} A(\underline{k}, t, X') d\underline{k}_2 d\underline{k}_1 \quad (1.2.4).$$

provided $X' \gg d\underline{k}_1^{-1}, d\underline{k}_2^{-1}$. There is no proof that this approximation is reasonable and in fact we can construct particular vorticity fields $\omega(\underline{x})$, e.g.

$$\omega(x) = 0 \quad \text{for } |x| < X'$$

$$= \sin kx \quad \text{otherwise,}$$

for which (1.2.4) is not reasonable at all. But we shall assume that the errors involved in evaluating averaged quantities by this approximation will in general be negligible.

(In the particular contexts, §§2.2 and 2.3, in which we shall need this approximation we will require the errors to be negligible in the evaluation of, e.g., the vorticity spectrum function

$$\Omega(k) = \frac{1}{2} \int \frac{dY(\bar{k}, t) dY^*(\bar{k}, t)}{d\bar{k}_1 d\bar{k}_2} d\ell(\bar{k})$$

where the integration is around a circle, centre the origin, radius $\bar{k}$, in wavenumber space.)

1.3 SURVEY OF THE LITERATURE ON TWO DIMENSIONAL UNSTEADY FLOWS

The amount of work published on two dimensional turbulence is very sparse indeed compared with that for three dimensional theory. In fact there has been no concentrated effort to develop a theory for the unsteady two-dimensional motions of a viscous fluid at all, and the main contributions to the subject have only succeeded in stressing the conclusion that two and three dimensional motions must be radically different.

When we turn to classical inviscid theory on the other hand there is a wealth of literature. Most of this work has little relevance to the real (viscous) turbulence situation since the flows are based on particular, and usually simple, patterns of vortex singularities, but there are a few significant general results which could possibly have an interpretation for the

turbulent flow of a fluid at high Reynolds number.

In the discussion below we group the published work into three sections dealing with inviscid theory, the Fourier analysis of an unsteady viscous flow, and turbulence theory.

(i) Inviscid theory.

In an unbounded inviscid fluid it has long been known (see Lamb, pp. 202-249) that the velocity induced by a finite system of vortices has an explicit functional form. Moreover the motion of the vortices themselves can be predicted. Thus if the position of the i th vortex, of strength K_i , is (x_i, y_i) then it can be shown that

$$K_i \frac{dx_i}{dt} = - \frac{\partial W}{\partial y_i}, \quad K_i \frac{dy_i}{dt} = \frac{\partial W}{\partial x_i} \quad (1.3.1)$$

where W is a function of the (x_i, y_i) and of the K_i . In a paper by Lin (1943), which also contains a summary of the work on this subject, this result is extended to a fluid occupying a multiply connected, bounded region.

Now for an unbounded fluid the functional form of W allows three integrals of the equations of motion (see Lamb, p.229), namely

$$\sum_i K_i x_i = \text{constant}, \quad \sum_i K_i y_i = \text{constant} \quad (1.3.2)$$

and $\sum_i K_i (x_i^2 + y_i^2) = \text{constant}$

provided the fluid is at rest at infinity. It is this set of constraints which could have some relevance to the description of a turbulent motion.

The constraints have analogues for a fluid which has a continuous distribution, w , of vorticity. Thus

$$\iint u w \, dx \, dy = 0, \quad \iint v w \, dx \, dy = 0$$

and $\iint (xu + yv) w \, dx \, dy = 0$

if the velocity field decreases sufficiently fast at large distances. These results suggest that perhaps $\overline{u\omega}$, $\overline{v\omega}$ and $\overline{(xu+yv)\omega}$ are all zero in a turbulent fluid.

Now if ψ is the stream function, i.e. if

$$u = -\frac{\partial\psi}{\partial y}, \quad v = \frac{\partial\psi}{\partial x},$$

it may be shown that

$$\overline{u\omega} = -\frac{\partial}{\partial x_k} \left(\frac{\partial\psi}{\partial y} \frac{\partial\psi}{\partial x_k} \right) + \frac{1}{2} \frac{\partial}{\partial y} \left(\frac{\partial\psi}{\partial x_k} \right)^2 = 0$$

for a homogeneous motion, and similarly for $\overline{v\omega}$, but

$$\begin{aligned} \overline{(xu+yv)\omega} = & \frac{\partial}{\partial x_k} \left(y \frac{\partial\psi}{\partial x} \frac{\partial\psi}{\partial x_k} - x \frac{\partial\psi}{\partial y} \frac{\partial\psi}{\partial x_k} \right) - \frac{1}{2} \frac{\partial}{\partial x} \left(y \left(\frac{\partial\psi}{\partial x_k} \right)^2 \right) \\ & + \frac{1}{2} \frac{\partial}{\partial y} \left(x \left(\frac{\partial\psi}{\partial x_k} \right)^2 \right) \end{aligned}$$

and none of the terms on the right hand side of this equation may be zero since quantities like

$$y \frac{\partial\psi}{\partial x} \frac{\partial\psi}{\partial x_k}$$

(here we must confine ourselves to ensemble averaging since the spatial average may not exist) are not in general independent of position. Thus we can only obtain two kinematical constraints in analogy with (1.3.2):

$$\overline{u\omega} = 0 \quad \text{and} \quad \overline{v\omega} = 0. \quad (1.3.3)$$

(These are in fact particular examples of the three dimensional constraint

$$\overline{u \wedge \omega} = 0.)$$

There do exist other constraints involving derivatives of the velocity and vorticity, e.g.

$$\overline{\omega \nabla^2 u} = 0 \quad \text{and} \quad \overline{u \nabla^2 \omega} = 0,$$

but (1.3.3) appear to be the simplest and most important results. We note that none of these constraints seem to restrict in any way the form of the energy and mean-square vorticity spectra.

To return to the published work on discrete systems, Onsager (1943) and Welander (1955) developed a novel approach to the theory of a finite system of vortices.

These authors used ideas based on statistical mechanics in which they treated the vortices as particles and the equations (1.3.1) as equivalent to a Hamiltonian system. The only real conclusion resulting from this work is that there is a tendency for vortices to group themselves together: vortices of unlike sign grouping together for a flow with 'excess energy' and vortices of like sign grouping together for a flow with 'insufficient energy'. This result could be interpreted as a tendency for the motion to develop small scale eddies, albeit in 'patches', and we may have here evidence of some cascade mechanism in vorticity acting within a two-dimensional turbulent motion (see §1.4). The statistical properties of small scale motions, e.g. the equilibrium range in three dimensional turbulence, are not greatly affected by large scale inhomogeneities so we may assume that this evidence, although derived for an inhomogeneous motion, will apply to homogeneous motions as well. The only doubt about the applicability of the analysis lies in the suitability of the inviscid, discrete model to represent the behaviour of the small scale variations in a viscous fluid, but if the time scale associated with this tendency is small compared with the viscous diffusion time scale a real, nearly-discrete distribution of vorticity could be said to behave according to Onsager's prediction.

(ii) The Fourier analysis of an unsteady viscous flow.

The main attempt of Leray (1934), Kampé de Fériet (1946, 1948) and Tse-Sun Chow (1954) has been to fix bounds for the rate of dissipation of energy and vorticity in an unsteady viscous fluid. Unfortunately the work of these authors (except Leray) is marred by so many errors that we may discount many of the results, and those that remain are not highly significant.

Leray showed that the energy of a bounded fluid decreases at least as fast as exponentially, and Kampé de Fériet (1948) fixed upper bounds to the energy and vorticity spectrum functions under the same conditions. The first result could easily have been obtained by stating that the rate of dissipation of energy must be greater than or equal to the rate of dissipation based on the supposition that all the energy is in the lowest ^{wave number} Fourier mode, and the other results could be deduced from the same type of argument. The bounds provided by such analysis however are so gross that they are not very useful, particularly as the results cannot be extended for an unbounded fluid.

(In the other papers, Kampé de Fériet (1946) quotes the wrong Green's function for the Laplacian in a bounded domain and Tse-Sun Chow ignores both the possibility of the existence of a cascade mechanism for vorticity (see §1.4) and the variation of the spectrum functions with time due to the non-linear term in the equation of motion.)

Another group of authors (Fjørtoft (1953), Pedlosky (1962) and Platzmann (1962)) were concerned with the detailed interactions between Fourier modes in an unsteady two dimensional flow. The pitfall of this type of analysis is that the study of the interactions of a small number of modes (three in Fjørtoft and Platzmann's analyses) cannot in general be used to predict the behaviour of a large number of modes in a turbulent motion. It is necessary to consider the effect of all interactions in such a motion unless one has reasonable grounds for neglecting some of them. Pedlosky's paper does try to deal with the question of which interactions may be neglected but his analysis is in error insomuch as he assumes that the Fourier transform of the stream function

is a continuous function of the wavenumber and also assumes that isotropy implies that the vorticity, say, is a function of the radial distance alone - an isotropic laminar, not turbulent, flow.

(iii) Turbulence theory.

The first paper published on two-dimensional turbulence theory which was at all close to the well-established three dimensional theory was by Lee (1951). In that paper the author argued that there could not be a cascade process acting as an efficient dissipation mechanism for the energy of a flow. Using the relation for two dimensional turbulence that

$$\frac{d}{dt} \int_0^\infty E(k) k^2 dk = -2\nu \int_0^\infty E(k) k^4 dk,$$

where $E(k)$ is the energy spectrum function (see §1.5 for its definition), Lee showed by Kolmogoroff arguments that the right hand side of this equation increased at least as fast as $\nu^{-3/2}$ as ν tends to zero whereas the left hand side should have a dependence of only ν^{-1} . Consequently there is a contradiction which can only be resolved by abandoning the concept of an equilibrium range as normally held in the three dimensional theory.

An interesting privately circulated report by J. von Neumann ('Recent theories of turbulence' (1950)) also suggested the same conclusion on the grounds that the two dimensional Navier-Stokes equation has both mathematical and physical implications, e.g. the constraint on the vorticity field already mentioned, which are radically different from the three dimensional equation.

More recently the result has received support from Ogura (1962) who obtained a numerical solution for two

dimensional turbulence based on the hypothesis of a zero fourth-order cumulant (the dynamical equation based on this hypothesis having been obtained by Reid (1959)). We shall delay an evaluation of Ogura's work until chapter three but it is worthwhile to note here that Ogura also concluded that the dominant energy transfer was towards the low wavenumber region of the spectrum and that there was only slight evidence of a mechanism which transferred energy to high wavenumbers (see §1.7).

The remaining published work is devoted mainly to the establishment of the appropriate equations for two dimensional isotropic turbulence. Thus Ogura (1952), using the definitions

$$\overline{u^2} f(r) = \overline{u_p(x) u_p(x+r)} , \quad \overline{u^2} g(r) = \overline{u_n(x) u_n(x+r)} \quad (1.3.4)$$

where u_p is the component of u in the direction r , and u_n is the component in a direction normal to r , obtained the relation

$$g = f + r \frac{\partial f}{\partial r}$$

which corresponds with the relation

$$g = f + \frac{r}{2} \frac{\partial f}{\partial r}$$

in three dimensions. He also obtained relations for the one dimensional and total energy spectrum functions which again are not significantly different from the three dimensional theory.

Hama (1953) obtained the analogue of the Kármán-Howarth equation relating the time rate of change of the second order velocity correlation to the third order velocity correlation, namely

$$\frac{\partial}{\partial t} (\overline{u^2} f) = (\overline{u^2})^{3/2} \left(\frac{\partial \epsilon}{\partial r} + 3 \frac{\epsilon}{r} \right) + 2 \nu \overline{u^2} \left(\frac{\partial^2 f}{\partial r^2} + \frac{3}{r} \frac{\partial f}{\partial r} \right) \quad (1.3.5)$$

where

$$(\overline{u^2})^{3/2} k(r) = \overline{u_p^2(x) u_p(x+r)} \quad (1.3.6)$$

(the Kármán-Howarth equation is

$$\frac{\partial}{\partial t} (\overline{u^2} f) = (\overline{u^2})^{3/2} \left(\frac{\partial f}{\partial r^2} + 4 \frac{f}{r} \right) + 2\nu \overline{u^2} \left(\frac{\partial^2 f}{\partial r^2} + \frac{4}{r} \frac{\partial f}{\partial r} \right)).$$

In addition he obtained the functional relation for the energy transfer

$$\frac{\partial E(k)}{\partial t} + H(k) = -2\nu k^2 E(k)$$

and concluded that there was, at this level of analysis, little difference between the two and three dimensional theories.

Ogura's (1952) paper also includes a derivation of the energy spectrum function using Kármán's energy transfer hypothesis (see Batchelor (1953, p. 127)). The author showed that there was no significant difference between the two and three dimensional predictions from this hypothesis, but, in view of Ogura's later arguments (1962) and the results of this dissertation, we may regard the use of the Kármán transfer function as having no justification.

1.4 ENERGETICS

We now examine the rate of change of the mean kinetic energy and mean-square vorticity of a two dimensional turbulent motion*.

From the Navier-Stokes equation (1.2.2) with zero body force and the continuity equation (1.2.1) we may obtain

$$\frac{1}{2} \frac{\partial}{\partial t} (u^2) + \frac{1}{2} \frac{\partial}{\partial x_j} (u_j u^2) = -\frac{1}{\rho} \frac{\partial}{\partial x_j} (p u_j) + \nu (u_j \nabla^2 u_j) \quad (1.4.1)$$

where $u = |u|$. But

$$u_j \nabla^2 u_j = -\nabla \cdot ((\nabla \wedge u) \wedge u) - (\nabla \wedge u)^2$$

so, on averaging equation (1.4.1) and using the last

* I am indebted to Professor G.K. Batchelor for the original suggestion of the basis of the argument in this section.

result we obtain

$$\frac{dE}{dt} = -2\nu\Omega$$

where E is the mean kinetic energy and Ω is half the mean-square vorticity. As Ω is positive, E must therefore always decrease in time.

If we examine the vorticity equation

$$\frac{\partial \omega}{\partial t} + \underline{u} \cdot \nabla \omega = \nu \nabla^2 \omega \quad (1.4.2)$$

we obtain

$$\frac{d\Omega}{dt} = -\nu \overline{(\nabla \omega)^2} \quad (1.4.3)$$

i.e. the mean-square vorticity also decreases in time and the rate of decrease of Ω is proportional to the mean-square of the vorticity gradient.

Consequently, for a turbulent flow with initial energy E_0 and vorticity Ω_0 , the rate of dissipation of energy

$$\begin{aligned} \frac{dE}{dt} &= -2\nu\Omega \\ &\geq -2\nu\Omega_0 \end{aligned}$$

so that the rate of decrease of kinetic energy can be made as small as we like by decreasing the viscosity.

This is in sharp contrast to the behaviour of three-dimensional turbulence where in the inviscid limit the rate of dissipation of energy becomes independent of the viscosity, the fluid motion adjusting itself so that there is a non-zero dissipation. The reason for this is that the term $\omega \cdot \nabla \underline{u}$ occurs in the vorticity equation in three dimensions and it represents a possible mechanism for the generation of vorticity: thus Ω is not then a necessarily decreasing function of time and one cannot fix an upper bound to the rate of dissipation.

If we define a new variable ζ such that

$$\xi = \left(-\frac{\partial \omega}{\partial y}, \frac{\partial \omega}{\partial x}, 0 \right)$$

then the dissipation of vorticity, see equation (1.4.3), can be written as

$$\frac{d\Omega}{dt} = -\nu \overline{\xi^2}.$$

Now the equation for ξ , obtained by differentiating the vorticity equation, is

$$\frac{\partial \xi}{\partial t} + \nabla_{\perp} \nabla_{\perp} (\underline{u}_{\perp} \cdot \underline{\omega}) = \nu \nabla^2 \xi$$

where $\underline{\omega} = (0, 0, \omega)$. It is then a simple matter to show that

$$\frac{1}{2} \frac{d\overline{\xi^2}}{dt} = - \overline{\frac{\partial \omega}{\partial x_i} \frac{\partial u_i}{\partial x_j} \frac{\partial \omega}{\partial x_j}} - \nu \overline{(\nabla_{\perp} \xi)^2}.$$

The implication of this equation is that the quantity $\overline{\xi^2}$ does not necessarily decrease in time since the term

$$- \overline{\frac{\partial \omega}{\partial x_i} \frac{\partial u_i}{\partial x_j} \frac{\partial \omega}{\partial x_j}}$$

could be positive. We have therefore arrived at a situation similar to that in the three dimensional theory: in two dimensions Ω decreases at a rate $\nu \overline{\xi^2}$ and $\overline{\xi^2}$ is a quantity which does not necessarily decrease in time; in three dimensions E decreases at a rate $2\nu\Omega$ and Ω is a quantity which does not necessarily decrease in time. In fact in three dimensions we have the observed phenomenon that, for fully-developed turbulence, as $\nu \rightarrow 0$ the rate of dissipation of energy tends to a quantity independent of ν , i.e. $\nu\Omega \rightarrow \text{constant}$. It would seem a plausible hypothesis therefore to assume that we have an analogous situation in two dimensions and that, as $\nu \rightarrow 0$, the rate of dissipation of Ω tends to a quantity independent of ν , i.e. $\nu \overline{\xi^2} \rightarrow \text{constant}$.

In effect we are saying that a turbulent motion will adjust itself to maximize, in some sense, the

magnitude of the 'lowest order kinematic quantity' which is not subject to any physical constraints. By the 'lowest order kinematic quantity' we mean the lowest order differential (or, if we were to work in higher dimensions than three, the highest order integral) of the velocity, and by 'physical constraints' we mean, in particular for the two dimensional case, the fact that vorticity cannot be generated.

We shall now proceed to investigate both the large and small scale structure of two dimensional turbulence and find the consequences, if any, of the constraint on the vorticity and the hypothesis concerning the dissipation mechanism.

1.5 THE BEHAVIOUR OF THE LARGE SCALE MOTIONS

In three dimensional turbulence it was long thought that certain constraints necessarily governed the behaviour of the large scale motions. Thus (see Batchelor (1953, p. 88)) it was argued that the spectrum tensor

$\Phi_{ij}(\underline{k})$, where

$$\Phi_{ij}(\underline{k}) = (8\pi^3)^{-1} \int \overline{u_i(\underline{x}) u_j(\underline{x} + \underline{r})} e^{-i\underline{k} \cdot \underline{r}} d\underline{r}$$

could be expanded in a Taylor series of the form

$$\Phi_{ij}(\underline{k}) = -(4\pi^2)^{-1} L_{ijmn} k_m k_n + O(k^3), \quad (1.5.1)$$

where

$$L_{ijmn} = (4\pi)^{-1} \int r_m r_n R_{ij}(\underline{r}) d\underline{r}$$

and is invariant in time.

However Batchelor and Proudman's (1956) paper showed that $\Phi_{ij}(\underline{k})$ is in general not an analytic function near the origin of wavenumber space and that, with the correct form of the expansion (1.5.1), the coefficient of the leading term is not invariant in time. In this

section we shall adapt Batchelor and Proudman's arguments to the behaviour of the large scale motions in two dimensional turbulence and show that the situation is very similar indeed to that in three dimensions.

The form of the energy spectrum tensor for a turbulent flow at any instant must depend on the way the turbulence was generated. Thus, for instance, if the initial configuration of a turbulent motion possessed periodicity of a certain wave length this periodicity may persist throughout the history of the motion and consequently the spectrum tensor may be discrete near the wavenumber origin at all times. (This is the case with the turbulent motion considered in the numerical calculations described in chapter three.) For some flows there is therefore no expansion of the form (1.5.1) and consequently no invariance property.

Batchelor and Proudman chose for their initial motion a homogeneous motion for which all integral moments of cumulants of the velocity field, and its derivatives, converge. They also put forward a plausible argument to show that grid flow turbulence can be approximated closely by this model. But in the two dimensional problem we have no practical means of generating turbulence and so there is no such justification for using this particular initial condition. However we are seeking for analyticity and invariance in the small wavenumber structure of the spectrum function and the analysis involved in the investigation would lead one to expect that the convergence of all integral moments is the most favourable initial condition to result in these properties which we seek. The fact that it does not necessarily do so, as we shall show, can be taken as a strong indication of the non-analytic and non-invariant

small wavenumber behaviour of the spectrum function of most turbulent flows.

The basic method of Batchelor and Proudman is to assume that all cumulant functions of the velocity field and its spatial derivatives can be predicted at all later times after the initial instant by a Taylor series expansion. Thus the velocity correlation

$$R_{ij}(\mathbf{r}, t) = \overline{u_i(\mathbf{x}) u_j(\mathbf{x} + \mathbf{r})}$$

is given by the series

$$R_{ij}(\mathbf{r}, t) = R_{ij}(\mathbf{r}, 0) + \left[\frac{\partial R_{ij}(\mathbf{r}, t)}{\partial t} \right]_{t=0} \cdot t + \frac{1}{2!} \left[\frac{\partial^2 R_{ij}(\mathbf{r}, t)}{\partial t^2} \right]_{t=0} \cdot t^2 + \dots$$

The time derivatives at the initial instant are obtained from the Navier-Stokes equation, e.g. in the case of the first derivative of the velocity correlation we have (see Batchelor (1953, p. 79))

$$\begin{aligned} \frac{\partial R_{ij}(\mathbf{r}, t)}{\partial t} = & \frac{\partial}{\partial r_k} \left(\overline{u_i u_k u_j'} - \overline{u_i' u_k u_j} \right) + \frac{1}{\rho} \left(\frac{\partial \overline{p u_j}}{\partial r_i} - \frac{\partial \overline{p' u_i}}{\partial r_j} \right) \\ & + 2\nu \nabla^2 R_{ij}(\mathbf{r}, t), \end{aligned} \quad (1.5.2)$$

where the dash denotes evaluation at the point $(\mathbf{x} + \mathbf{r})$. Batchelor and Proudman then derive the asymptotic form of these time derivatives at the initial instant for large t and show that the time derivatives (and hence the value of the function at a time later than the initial instant) have the behaviour $O(t^{-n})$ where n is an integer. This then leads, for the velocity correlation, to the asymptotic form of the energy spectrum tensor for small wavenumbers. Finally by determining the asymptotic expressions for all the terms in equation (1.5.2), or its Fourier transform, Batchelor and Proudman obtain the behaviour in time of the asymptotic expression for the velocity correlation.

For the two dimensional case we can repeat the form of this argument. The equivalent equation to (1.5.2) is

$$\frac{\partial R_{ij}}{\partial t} = \frac{\partial}{\partial r_k} (\overline{u_i u_k u_j} - \overline{u_i u_k} \overline{u_j}) + \frac{1}{\rho} \left(\frac{\partial p u_j}{\partial r_i} - \frac{\partial p u_i}{\partial r_j} \right) + 2\nu \nabla^2 R_{ij}(\mathbf{r}, t) \quad (1.5.3)$$

and all the terms on the right hand side, except those associated with the pressure, are exponentially small for large t at the initial instant, since we assume that all integral moments of these terms converge.

The divergence of equation (1.5.3) with respect to the i suffix is

$$\frac{1}{\rho} \nabla^2 \overline{p u_j} = - \frac{\partial^2 \overline{u_i u_k u_j}}{\partial r_i \partial r_k}, \quad (1.5.4)$$

all other terms being zero by the continuity condition, so that the appropriate solution for $\overline{p u_j}$ for an unbounded fluid is

$$\frac{1}{\rho} \overline{p u_j} = - \frac{1}{2\pi} \int \frac{\partial^2 \overline{u_i u_k u_j''}}{\partial r_i' \partial r_k'} \log |\mathbf{r}' - \mathbf{r}| d\mathbf{r}' \quad (1.5.5)$$

where $u_j'' \equiv u_j(\mathbf{x} + \mathbf{r}')$.

Now the quantity

$$- \frac{1}{2\pi} \frac{\partial^2 \overline{u_i u_k u_j''}}{\partial r_i' \partial r_k'} = A_j(\mathbf{r}'),$$

say, is exponentially small at large r' so that we may expand the logarithmic term in (1.5.5) as a Taylor series for $r' \ll r$, and as $r \rightarrow \infty$ we have

$$\overline{p u_j} \sim (\log r) \int A_j d\mathbf{r}' - \frac{\partial}{\partial r_p} (\log r) \int r_p' A_j d\mathbf{r}' + \frac{1}{2!} \frac{\partial^2}{\partial r_p \partial r_q} (\log r) \int r_p' r_q' A_j d\mathbf{r}' + \dots \quad (1.5.6)$$

The first three terms are zero since A_j can be formulated as a third derivative quantity and the integrals may be transformed into closed line integrals (by Stokes'

Theorem) which in the limit of infinite radius are zero. By similarly transforming the integral in the fourth term of (1.5.6) we finally obtain

$$\overline{pu_j'} \sim \frac{1}{2\pi} \frac{\partial^3 (\log r)}{\partial r_p \partial r_f \partial r_r} \int r_r' \overline{u_p u_f u_j''} d\mathbf{x}''.$$

Thus the terms associated with the pressure in equation (1.5.3) are $O(r^{-4})$ for large r , and we conclude that the first derivative of $\mathcal{R}_{ij}(r, \tau)$ at the initial instant is $O(r^{-4})$.

A comparison of the argument so far with Batchelor and Proudman's paper will show that the working is identical apart from the different solution of the Poisson equation, equation (1.5.4), giving rise to 'log' instead of ' $1/r$ ', and apart from certain numerical factors. This is in fact the only difference between the two and three dimensional theories. The basic physical mechanism of the production of asymptotic forms of cumulants at large separation distances is the same and is a result of the action of pressure forces which have values determined instantaneously by the whole velocity field.

Indeed the only radical difference that could arise could only come from the vorticity equation (1.4.2) since in two dimensions the 'stretching of vortex lines' term is identically zero. But this equation still retains a non-linearity involving the velocity so that the build-up of asymptotic forms of cumulants involving the vorticity will depend on time derivatives at the initial instant of cumulants involving the velocity which will then lead to power law forms, just as in the three-dimensional theory.

We will therefore adapt Batchelor and Proudman's results directly and state that all the time derivatives of the velocity correlation $\overline{u_i u_j}$ are in general $O(t^{-4})$ and that all the derivatives of the vorticity correlation $\overline{\omega_i \omega_j}$ are at most $O(t^{-7})$ at the initial instant for large t . Hence we can conclude that

$$R_{ij}(t, t) = \overline{u_i u_j} = O(t^{-4}) \quad (1.5.7)$$

$$\text{and } V(t, t) = \overline{\omega_i \omega_i} = O(t^{-7}) \quad (1.5.8)$$

at any instant. We shall also need the following results, obtained by the same form of argument;

$$S_{ijk} = \overline{u_i u_k u_j} = O(t^{-3}) \quad (1.5.9)$$

$$\text{and } \overline{u_i u_k \omega_j} = O(t^{-5}). \quad (1.5.10)$$

We now turn to the behaviour of the energy spectrum tensor $\Phi_{ij}(\underline{k})$ at small values of $\underline{k}$. From (1.5.7) we conclude that both

$$\int R_{ij}(t) dt \quad \text{and} \quad \int r_k R_{ij}(t) dt$$

converge absolutely. It therefore follows that $\Phi_{ij}(\underline{k})$ and $\partial \Phi_{ij}(\underline{k}) / \partial k_k$ are continuous functions of $\underline{k}$ for all small $\underline{k}$.

However the integral

$$\int r_k r_l R_{ij}(t) dt$$

only converges conditionally. (It does in fact converge: if t^* is a value of t sufficiently large to permit the replacement of the integrand by its asymptotic form, then

$$\int_{t > t^*} r_k r_l R_{ij}(t) dt$$

can be converted into convergent line integrals since it can be argued that the asymptotic form of R_{ij} must be a sum of terms all of the form of at least a fourth

derivative of $\log r$. The net sum of these line integrals then vanishes.)

Thus

$$-(4\pi^2)^{-1} \int r_k r_l R_{ij}(r) e^{-i\mathbf{k} \cdot \mathbf{r}} d\mathbf{r}$$

exists at $\mathbf{k}=0$ and ensures that the second derivative of Φ_{ij} is uniformly bounded at all points near the origin of wavenumber space other than the origin itself. We may therefore expand Φ_{ij} as a Taylor series with remainder, i.e.

$$\Phi_{ij}(\mathbf{k}) = \Phi_{ij}(0) + k_k \left[\frac{\partial \Phi_{ij}}{\partial k_k}(\mathbf{k}) \right]_{\mathbf{k}=0} + \frac{1}{2!} k_k k_l \left[\frac{\partial^2 \Phi_{ij}}{\partial k_k \partial k_l}(\mathbf{k}) \right]_{\mathbf{k}=0} \quad (1.5.11)$$

where $0 < |\mathbf{k}| \leq 1$.

Since the compressibility condition requires

$$k_i \Phi_{ij}(\mathbf{k}) = 0$$

for all $\mathbf{k}$ and Cramér's theorem requires $\Phi_{ij}(\mathbf{k})$ to be a tensor associated with a positive indefinite form the first two terms of (1.5.11) are zero, so that

$$\Phi_{ij}(\mathbf{k}) = O(k^2) \quad (1.5.12)$$

and the spectrum tensor will in general have a non-analytic asymptotic structure at small wavenumbers. The exact form of this structure may be obtained from the vorticity spectrum function $\Omega(\mathbf{k})$, defined by the equation

$$\Omega(\mathbf{k}) = (4\pi^2)^{-1} \int V(\mathbf{r}) e^{-i\mathbf{k} \cdot \mathbf{r}} d\mathbf{r}, \quad (1.5.13)$$

It can be shown that

$$V(\mathbf{r}) = -\nabla^2 R_{kk}(\mathbf{r})$$

so that

$$\Omega(\mathbf{k}) = k^2 \Phi_{kk}(\mathbf{k}) \quad (1.5.14)$$

and $\Omega(\mathbf{k})$ will be $O(k^4)$ for small k .

Since $V(r)$ is $O(r^{-7})$ at large r we may thus expand $\Omega(k)$ using (1.5.13) by a Taylor series in k about $k=0$ up to the fourth derivative of $\Omega(k)$. Moreover, as we shall prove, there is then a remainder term of order $k^5 \log k$. Consequently we can obtain the form of the leading term for $\Omega(k)$ and hence the form of the leading term of $\Phi_{ij}(k)$ at small k in terms of an integral moment.

The fourth integral moment of $V(r)$, namely

$$C_{lmnp} = \int r_l r_m r_n r_p V(r) dr,$$

is absolutely convergent so that the fourth derivative of $\Omega(k)$, obtained from equation (1.5.13), is a continuous function of k at all values of k including $k=0$. Also

$$\int |r_l r_m r_n r_p r_q V(r)|^2 dr$$

converges and

$$\int |r_l r_m r_n r_p r_q V(r)| dr$$

diverges at most logarithmically so that the fifth derivative of $\Omega(k)$ is a continuous function of k for all non-zero values of k and is $O(\log k)$ for small k .

Hence

$$\Omega(k) = (4\pi^2 \cdot 4!)^{-1} C_{lmnp} k_l k_m k_n k_p + O(k^5 \log k). \quad (1.5.15)$$

We need now a further relationship so that we may determine $\Phi_{ij}(k)$ in terms of $\Omega(k)$.

The incompressibility condition in two dimensions implies that we can construct a stream function, say ψ , where

$$u_i = \varepsilon_{3pi} \frac{\partial \psi}{\partial x_p},$$

Thus

$$\begin{aligned} R_{ij}(z) &= -\varepsilon_{3pi} \varepsilon_{3qj} \frac{\partial^2 \overline{\psi \psi}}{\partial x_p \partial x_q} \\ &= -\delta_{ij} \nabla^2 \overline{\psi \psi} + \frac{\partial^2 \overline{\psi \psi}}{\partial x_i \partial x_j} \end{aligned}$$

and so

$$\mathcal{R}_{ii}(\tau) = -\nabla^2 \overline{\psi^2}.$$

(Note that in two dimensional theory $\delta_{ii} = 2$)

Consequently

$$\nabla^2 \mathcal{R}_{ij}(\tau) = -\delta_{ij} V(\tau) - \frac{\partial^2 \mathcal{R}_{kk}}{\partial \tau_i \partial \tau_j}.$$

The Fourier transform of this equation is

$$-k^2 \Phi_{ij}(k) = -\delta_{ij} \Omega(k) + k_i k_j \Phi_{kk}(k).$$

Substitution from (1.5.14), which may also be obtained from this last equation, then gives the relationship required, i.e.

$$\Phi_{ij}(k) = \frac{1}{k^2} \left(\delta_{ij} - \frac{k_i k_j}{k^2} \right) \Omega(k).$$

Hence from (1.5.15)

$$\Phi_{ij}(k) = (4\pi^2 \cdot 4!)^{-1} \frac{1}{k^2} \left(\delta_{ij} - \frac{k_i k_j}{k^2} \right) C_{emnp} k_e k_m k_n k_p + O(k^2 \log k). \quad (1.5.16)$$

The energy spectrum function $E(k)$ is defined by

$$E(k) = \frac{1}{2} \int \Phi_{ii}(k) d\Omega(k)$$

where the integration is round the circle of radius k with centre $\underline{k} = 0$ and has the property that the mean kinetic energy

$$E = \int_0^\infty E(k) dk.$$

This function therefore has the form, for small values of k ,

$$E(k) = (8\pi^2 \cdot 4!)^{-1} \frac{1}{k^2} C_{emnp} \int k_e k_m k_n k_p d\Omega(k) + O(k^4 \log k). \quad (1.5.17)$$

The integral must be an isotropic tensor of the fourth order in two dimensions, so that

$$\int k_e k_m k_n k_p d\Omega(k) = A'(k) \delta_{em} \delta_{np} + B'(k) \delta_{en} \delta_{mp} + C'(k) \delta_{ep} \delta_{mn}. \quad (1.5.18)$$

There is symmetry between all four suffices, so

$$A' = B' = C'.$$

Also, by putting $e=m$, $n=p$, we obtain

$$\int k^4 d\Omega(k) = 2\pi k^5 = 8A'. \quad (1.5.19)$$

Thus using (1.5.19) and (1.5.18) in (1.5.17) we obtain

$$E(k) = Ck^3 + O(k^4 \log k) \quad (1.5.20)$$

where

$$\epsilon = \frac{3}{32\pi \cdot 4!} C_{ijj} . \quad (1.5.21)$$

Having obtained the small wavenumber structure of the energy spectrum we may now proceed to investigate its behaviour in time. To do this we take the Fourier transform of equation (1.5.3):

$$\frac{\partial \Phi_{ij}(\underline{k})}{\partial t} = i k_k Y_{ikj}(\underline{k}) - i k_k Y_{kji}(-\underline{k}) + i k_i \pi_j(\underline{k}) - i k_j \pi_i(-\underline{k}) - 2\nu k^2 \Phi_{ij}(\underline{k}), \quad (1.5.22)$$

where

$$Y_{ikj}(\underline{k}) = (4\pi^2)^{-1} \int \overline{u_i u_k u_j} e^{-i\underline{k} \cdot \underline{r}} d\underline{r}$$

and

$$\pi_j(\underline{k}) = (4\pi^2)^{-1} \int \overline{p u_j} e^{-i\underline{k} \cdot \underline{r}} d\underline{r}.$$

But we have immediately from (1.5.4) that

$$\pi_j(\underline{k}) = - \frac{k_i k_k}{k^2} Y_{ikj}(\underline{k}). \quad (1.5.23)$$

We therefore only require the asymptotic form of $Y_{ikj}(\underline{k})$ for small $\underline{k}$. This can be obtained in a similar way to the derivation of the asymptotic form of Φ_{ij} , i.e. by considering the Taylor expansion of the quantity

$$\Lambda_{ik}(\underline{k}) = (4\pi^2)^{-1} \int \overline{u_i u_k \omega} e^{-i\underline{k} \cdot \underline{r}} d\underline{r}$$

and its relationship to $Y_{ikj}(\underline{k})$. In fact we have by simple manipulation of the definitions that

$$\Lambda_{ik}(\underline{k}) = i \varepsilon_{3mj} k_m Y_{ikj}(\underline{k})$$

and hence that

$$Y_{ikj}(\underline{k}) = i \varepsilon_{3jn} \frac{k_n}{k^2} \Lambda_{ik}(\underline{k}).$$

But the asymptotic form of $\overline{u_i u_k \omega}$ is $O(r^{-5})$ for large r . Thus using a similar argument to that used in the expansion of $\Omega(\underline{k})$ we may obtain

$$\Lambda_{ik}(\underline{k}) = - (4\pi^2 \cdot 2!)^{-1} B_{ikpq} k_p k_q + O(k^2 \log k)$$

where

$$B_{ikpq} = \int \overline{u_i u_k \omega} r_p r_q d\underline{r}.$$

(The first two terms in the expansion in powers of k vanish identically by the properties of $\overline{u_i u_k w_j}$ and $\overline{u_i u_k u_j}$.)

Thus

$$Y_{ijk}(k) = - (8\pi^2)^{-1} i \varepsilon_{3jn} B_{ikpq} \frac{k_p k_q k_n}{k^2} + O(k^2 \log k). \quad (1.5.24)$$

On substituting (1.5.23) and (1.5.24) in (1.5.22) and using (1.5.16) we finally obtain the behaviour of C_{emnp} with time, which in general is non-trivial, as

$$\left(\delta_{ga} - \frac{k_g k_a}{k^2} \right) \left(\frac{1}{12} \delta_{gi} \delta_{aj} k_k k_m k_n k_p \frac{dC_{emnp}}{dt} - \varepsilon_{3fn} B_{akpq} k_n k_p k_q k_k [\delta_{fi} \delta_{gj} + \delta_{fi} \delta_{gj}] \right) = 0. \quad (1.5.25)$$

Batchelor and Proudman have evaluated Y_{ijk} in terms of the first integral moment of $\overline{u_i u_k w_j}$ but there appears to be no gain in simplification of the equation (1.5.25) or its three dimensional equivalent by doing this. Moreover in our case there is no condition imposed on the B_{ikpq} unlike the analogous situation in three dimensions so we shall take (1.5.25) as the final result. (The condition in the three dimensional theory arose from the solenoidal property of $\overline{u_i u_k w_j}$ in the j index, but this has no counterpart in two dimensions.)

We may calculate the behaviour of $\mathcal{C}$, the leading coefficient in the expansion of $E(k)$ in equation (1.5.20), from equation (1.5.25) by equating $i=j$ and integrating round a circle of radius k and centre $k=0$. The result is

$$\frac{d\mathcal{C}}{dt} = \frac{1}{32\pi} \varepsilon_{3ap} (2B_{aqqq} + B_{apqq}) \quad (1.5.26)$$

and $\mathcal{C}$ is in general not constant in time.

In the particular case of isotropic turbulence

the leading term of order τ^{-4} in the asymptotic expansion of $R_{ij}(\tau)$ can be shown to be identically zero, as in the three dimensional theory. Thus R_{ij} is at most $O(\tau^{-5})$ at large τ (indeed, as in Batchelor and Proudman, it may well be less since no such non-trivial terms have been found in the dynamical equations for the time derivatives of R_{ij}) and consequently the term of order k^2 in the expansion of Φ_{ij} is analytic for isotropic turbulence. The form of the expansion is

$$\Phi_{ij}(k) = \frac{e}{\pi} (k^2 \delta_{ij} - k_i k_j) + O(k^3 \log k)$$

and the (non-trivial) variation of $\mathcal{C}$ with time can be derived from equation (1.5.26). (It should be noted for this evaluation that the tensor B_{ikpq} contains the vorticity and that therefore it is not a tensor which has rotational and reflectional symmetry, i.e. it is a tensor composed not only of δ_{ij} tensors but also of ϵ_{3ij} tensors.)

It is perhaps more helpful in this instance to mention the closely related equation

$$\frac{d}{dt} \left(\overline{u^2} \int_0^\infty \tau^3 f(\tau) d\tau \right) = (\overline{u^2})^{3/2} \lim_{\tau \rightarrow \infty} (\tau^3 k(\tau))$$

where $f(\tau)$ and $k(\tau)$ are the correlation coefficients for isotropic turbulence defined in the usual way (see equations (1.3.4) and (1.3.6)). This relationship is obtained directly by taking the third moment of the dynamical equation for the function $f(\tau)$, namely (see equation (1.3.5))

$$\frac{\partial}{\partial t} \left(\overline{u^2} f(\tau) \right) = (\overline{u^2})^{3/2} \left(\frac{\partial k(\tau)}{\partial \tau} + 3 \frac{k(\tau)}{\tau} \right) + 2\nu \overline{u^2} \left(\frac{\partial^2 f(\tau)}{\partial \tau^2} + \frac{3}{\tau} \frac{\partial f(\tau)}{\partial \tau} \right).$$

In general, as already mentioned, $k(\tau)$ is of order τ^{-3} at large τ , so that

$$\overline{u^2} \int_0^\infty \tau^3 f(\tau) d\tau$$

is not invariant in time.

We conclude then that the general behaviour of the large scale structure of turbulence is not affected by the constraint on vorticity in two dimensional motions and that the situation is almost identical with that for three dimensional turbulence.

1.6 THE BEHAVIOUR OF THE SMALL SCALE MOTIONS

In §1.4 we postulated an hypothesis for the dissipation of vorticity, namely that in a well-developed turbulent flow and for sufficiently large Reynolds number (which we shall define more explicitly later) the rate of dissipation of vorticity is independent of the viscosity of the fluid. It is now our intention to use this hypothesis and develop an argument for the existence and specification of an equilibrium range in two dimensional turbulence.

We shall discuss the nature of the turbulent flow in terms of the transfer of vorticity across the vorticity spectrum and its eventual dissipation by the action of viscosity at high wavenumbers. Now in terms of the Fourier-Stieltjes transform $\Delta Y(k)$ of the vorticity (see equation (1.1.3)) the vorticity equation is of the form

$$\frac{\partial \Delta Y(k)}{\partial t} = \Delta S(k) - \nu k^2 \Delta Y(k) \quad (1.6.1)$$

where $\Delta S(k)$ is an integral function dependent on all the Fourier modes of the vorticity and is quadratic in the $\Delta Y(k)$. Thus, as it can be shown (see Batchelor (1953, p. 32) and equation (1.5.13)) that

$$\Delta S(k) = \lim_{\Delta k \rightarrow 0} \frac{1}{2} \frac{\Delta Y(k) \Delta Y^*(k)}{\Delta k_1 \Delta k_2},$$

we can define a new vorticity spectrum function

$$\Omega(k) = \int \Omega(\underline{k}) d\ell(k)$$

where the integration is round a circle radius k , centre $\underline{k}=0$, and derive from equation (1.6.1) the equation for this vorticity spectrum function as

$$\frac{d\Omega(k)}{dt} = S(k) - 2\nu k^2 \Omega(k) \quad (1.6.2)$$

where $S(k)$ is derived from $dS(\underline{k})$ and may be called the vorticity transfer function. The only property of $S(k)$ that we need is that it conserves vorticity so that if we integrate (1.6.2) over all k we obtain the rate of decay of mean-square vorticity

$$\frac{d\Omega}{dt} = -2\nu \int_0^\infty k^2 \Omega(k) dk = -\varepsilon_\omega, \quad (1.6.3)$$

say.

Now if we suppose that at some initial instant we have a turbulent flow whose vorticity spectrum has one sharp peak at some wavenumber k_0 - this type of spectrum could well be produced by almost any mechanism which has a fundamental length scale in its action so we shall assume that we are considering a 'typical' initial flow for two dimensional turbulence - then

$$\varepsilon_\omega \doteq 2\nu k_0^2 \Omega$$

initially. Consequently for small enough viscosity, the vorticity dissipation will be negligible for most length scales compared with the transfer term $S(k)$. The effect of the $S(k)$ term is then to transfer vorticity to harmonic Fourier modes, to alter the spectrum shape and eventually to increase the rate of dissipation of vorticity, ε_ω , until, as we assume, the 'fully-developed' stage is reached when ε_ω is independent of the particular value of ν for the flow. But what spectral shape will be established, particularly at wavenumbers greater than k_0 , by the action of this transfer term $S(k)$?

The first possibility, which we immediately discount, is an exponential (or faster) cut-off to the vorticity spectrum. This type of cut-off seems to be associated with a linear term similar to the viscous term in the vorticity equation and not to strongly non-linear systems as represented by the term $S(k)$. We may gain support for this statement from the similar structure of the $S(k)$ term to the transfer function which occurs in three dimensional theory: both functions involve an integral of quadratic terms of $\Delta Y(k)$ (for vorticity) or $\Delta Z(k)$ (for velocity) and there is no reason to suppose that the two would differ so radically as to produce an exponential cut-off in the one instance and a power-law cut-off in the other.

We now proceed to consider only power-law forms for the vorticity spectrum at high wavenumbers.

First let us suppose that for a flow with arbitrary small viscosity

$$\Omega(k) \sim k^{-n}, \quad (1.6.4)$$

where $0 < n \leq 3$, for a large range of wavenumbers for which $k \gg k_0$. The rate of dissipation of vorticity,

ϵ_ω , is then (see equation (1.6.3)) largely determined by a range of wavenumbers much greater than k_0 , and presumably in or near this dissipation range (1.6.4) ceases to be valid. Thus, if this spectrum shape is to be maintained for a relatively long period of the decay of the turbulence, vorticity must be transferred continually across the spectrum from the vorticity - containing wavenumbers to the dissipation range. We then have what is normally termed a 'cascade process' in the mean-square vorticity spectrum and, as in three dimensional theory, we expect the rate at which vorticity

is transferred to be independent of the dissipation range so that ε_0 will be independent of ν .

We next consider the case $3 < n$ in equation (1.6.4). Here we have that

$$k^2 \Omega(k) = o(k^{-1})$$

and consequently the range of wavenumbers which are dominant in the dissipation integral, see equation (1.6.3), are near the wavenumber k_0 . The rate of dissipation will thus remain negligible until the wavenumber k_0 marking the peak of the vorticity spectrum has increased in magnitude and has the value

$$k_0 \sim \nu^{-1/2}.$$

For this system there is no cascade process in the vorticity spectrum, although it is interesting to observe that a cascade process may exist in the spectrum of one of the derivatives of the vorticity. (E.g. if $3 < n < 5$ there may be a cascade process in the mean-square vorticity-gradient spectrum and 'mean-square vorticity-gradient' will be generated mainly in the region of k_0 and dissipated in wavenumbers greater than k_0 - the situation is in fact similar to that discussed by Moffatt (1961) for the spectrum of a magnetic field.) There appears to be no reason why this type of motion should not produce, in 'fully-developed' flow, a rate of dissipation independent of the viscosity. By the time the vorticity is transferred to the dissipative range of wavenumbers the shape of the spectrum may well have developed into a universal form independent of the initial conditions so that a reduction of the viscosity would only result in a delay in the time before the turbulence developed into a strongly dissipating system and would not affect the eventual rate of the dissipation.

However, in analogy again with three dimensional turbulence, this latter mechanism seems to contradict our present understanding of the action of strongly non-linear systems. Thus in Batchelor (1953, p. 107) we read 'that inertia forces tend to spread energy over as wide a range as is consistent with the stronger viscous damping which occurs at large wavenumbers'. The action of turbulence will tend to reduce the dominance of any one length scale in the energy spectrum in three dimensional turbulence (and, by analogy, the vorticity spectrum in two dimensional turbulence) and will try to spread the energy (or vorticity) as much as possible over all wavenumbers. It is a consequence of this view that if a power law should occur in the vorticity spectrum then α , the negative power of k , will be as small as is consistent with the transfer of energy (or vorticity) to high wavenumbers and the dissipation mechanism which is eventually dominant. In view of the strong experimental evidence in favour of this situation in three dimensional turbulence we shall carry it over to the two dimensional case: we shall thus assume that there is a cascade mechanism acting for the vorticity beyond the vorticity-containing wave-number range.

There is a further slight argument in favour of this assumption. Suppose that $3 < \alpha \leq 5$ and that there is a cascade process in the mean-square vorticity-gradient spectrum and assume that there is a Kolmogoroff-type equilibrium range of wavenumbers which also includes an 'inertial subrange'. Then the form of the vorticity-gradient spectrum in the subrange is, by the usual dimensional arguments, proportional to

$$\varepsilon^{2/3} k^{-4/3}$$

where ϵ_2 is the rate of dissipation of 'mean-square vorticity-gradient'. But this then leads to a value of $7/3$ for n and we have a contradiction.

If we suppose that $5 < n \leq 7$, or any higher valued range for n , then we may set up the same sort of argument and arrive at the same type of contradiction. We have therefore a strong indication that $n \leq 3$. However this argument cannot be said to be conclusive since it suffers from one main criticism: that a steady equilibrium range may not be feasible for the above conditions and that we should include the time as an extra parameter within the specification of the range.

It now remains to adapt the ideas of the Kolmogoroff equilibrium range of wavenumbers in three dimensional theory to the present situation in two dimensions. The arguments for postulating the hypotheses necessary for an equilibrium theory in two dimensions seem identical to those presented in, say, Batchelor (1953, §6.3) for the three dimensional flow and we will therefore omit their repetition. It may be stated, then, that when the Reynolds number is sufficiently large (which phrase remains undefined) 'there is likely to exist a range of (high) wavenumbers which is responsible for most of the viscous dissipation' of vorticity 'and for which the Fourier coefficients are statistically steady, isotropic and independent' of the Fourier coefficients of the vorticity-containing range of wavenumbers.

The external effects which determine this equilibrium range will be the transfer of vorticity into the range from the lower wavenumbers and the removal of vorticity by viscous dissipation. In a steady state

these will be equal in total magnitude and then ϵ_ω and ν will be the only relevant parameters in the description of the equilibrium range. Any statistical quantity which is in turn determined by the state of the equilibrium range will also be determined by the values of these two parameters. In particular we have by dimensional analysis that in the equilibrium range

$$\Omega(k) = (\nu \epsilon_\omega)^{1/2} \times f_n \left(\left(\frac{\nu^3}{\epsilon_\omega} \right)^{1/6} k \right). \quad (1.6.5)$$

When ν is sufficiently small we may expect a subrange to exist in the lower wavenumber end of the equilibrium range in which there is negligible viscous dissipation. In this subrange there will be no dependence on ν so that

$$\Omega(k) = C \epsilon_\omega^{2/3} k^{-1} \quad (1.6.6)$$

where C is an absolute constant.

The wavenumber which marks the lower end of the dissipation range will also be determined by the parameters ϵ_ω and ν alone and will therefore be of order

$$k_d = \left(\frac{\epsilon_\omega}{\nu^3} \right)^{1/6}.$$

Thus as equation (1.6.5) will hold only if all the dissipation takes place in the equilibrium range, a necessary condition for the existence of this range will be that

$$k_0 \ll k_d.$$

Equation (1.6.6) will hold for wavenumbers which satisfy

$$k_0 \ll k \ll k_d.$$

There is one other restriction to the range of validity of both (1.6.5) and (1.6.6) in that the time scale of the rate of dissipation of vorticity, i.e. Ω/ϵ_ω , must be much less than the time scale of the dissipation of energy, i.e. $E/2\nu\Omega$. If this condition is not satisfied then there appears to be no logical reason for expecting a cascade process in vorticity in pref-

erence to a cascade process in energy.

We note that equation (1.6.6) must be interpreted only as an asymptotic form for large wavenumbers otherwise we have possible contradictions. For example, mean-square vorticity in two dimensional flow is bounded in magnitude but if equation (1.6.6) holds exactly in the inertial subrange then in the limit of $\nu \rightarrow 0$ the vorticity in the subrange will be a divergent quantity. These difficulties are overcome if we regard equation (1.6.6) as holding asymptotically - since there exist functions of k such that asymptotically they behave as k^{-1} but whose integrals converge over an infinite wavenumber range (e.g. the function

$$\left\{ k^{1 + 2 \left(\frac{\log \log k}{\log k} \right)} \right\}^{-1}$$

has these properties).

The functional form in equation (1.6.5) is as yet undetermined beyond the inertial subrange. We propose to postulate a theory to determine the shape of the spectrum in this range which resembles that for the behaviour of convected scalar quantities in a turbulent fluid. We will therefore delay the discussion of this theory until §2.3 when we may use the results of the theory of convected scalar quantities as given in §2.2.

1.7 SIMILARITY THEORY FOR THE DECAY OF TWO-DIMENSIONAL TURBULENCE

We have so far obtained the form of the energy and vorticity spectra in the small wavenumber and, under certain conditions, in the high wavenumber regions. In particular, it was shown in §1.5 that

$$\Omega(k) = k^2 E(k) = C k^5 + O(k^6 \log k)$$

for small k where in general $\mathcal{E}$ varies with time; and in the last section it was shown that, if

$$R_\varepsilon = \frac{E \varepsilon \omega}{2 \nu \Omega^2} \gg 1 \quad (1.7.1)$$

$$\text{and } k_0 \ll k_d = \left(\frac{\varepsilon \omega}{\nu^3} \right)^{1/6}, \quad (1.7.2)$$

then an equilibrium range exists and

$$\Omega(k) = k^2 E(k) \sim \varepsilon^{2/3} k^{-1}$$

for $k_0 \ll k \ll k_d$. Here k_0 is the characteristic wavenumber of the vorticity containing eddies.

The questions that must now be asked are "how do the spectra alter in time?" and "in what stages of the development of the turbulence are the conditions (1.7.1) and (1.7.2) likely to prevail?"

Now the condition (1.7.1) is equivalent to requiring that the rate of dissipation of energy be negligible. We may therefore consider the energy to be a constant parameter for turbulence under such conditions. Moreover the viscosity ν appears to be the only other parameter that is constant throughout the development of such a flow. We therefore construct a similarity solution dependent on E and ν alone. Thus the energy spectrum will appear as

$$E(k, t) = E^{3/2} t \times f_n(E^{1/2} k t, E^{-1/2} \nu k)$$

for all k , and since there is independence of ν in the range $k \ll k_d$ we have that in this latter range

$$E(k, t) = E^{3/2} t \cdot f(E^{1/2} k t). \quad (1.7.3)$$

If most of the energy lies in this range of wavenumbers then

$$E = \int_0^\infty E^{3/2} t f(E^{1/2} k t) dk$$

which leads to the identity

$$1 = \int_0^\infty f(z) dz.$$

The vorticity spectrum is, from (1.7.3),

$$\Omega(k, t) = E^{3/2} k^2 t f(E^{1/2} k t) \quad (1.7.4)$$

for $k \ll k_d$, and if this range also contains most of the vorticity then

$$\Omega = \int_0^\infty E^{3/2} k^2 t f(E^{1/2} k t) dk$$

or

$$\Omega = \frac{1}{t^2} \int_0^\infty z^2 f(z) dz. \quad (1.7.5)$$

Thus mean-square vorticity will decrease in time as t^{-2} and the rate of dissipation of vorticity, ε_ω , will decrease as t^{-3} . (There is a virtual origin in time which can be defined by extrapolation of the vorticity/time graph back to the instant when vorticity is infinite, on the assumption that the equilibrium range always existed.) We have in fact the identity

$$4\Omega^3 = \varepsilon_\omega^2 \int_0^\infty z^2 f(z) dz.$$

Now in a well developed turbulent flow we may assume that the function $z^2 f(z)$ has one maximum which we can identify as occurring at $z_0 = E^{1/2} k_0 t$, where k_0 is the vorticity containing wavenumber. (If this is not the case then we can define k_0 as the characteristic wavenumber of the region which contributes most to the integral in equation (1.7.5).) Thus the similarity solution implies that

$$k_0 \propto t^{-1}.$$

But the dissipation wavenumber $k_d = \left(\frac{\varepsilon_\omega}{\nu^3}\right)^{1/6}$ varies as $t^{-1/2}$ so that if $k_0 \ll k_d$ at any instant in the similarity flow this condition is valid at all later instants. Similarly we may show that

$$R_\varepsilon = \frac{E \varepsilon_\omega}{2 \nu \Omega^2} \propto t$$

for this solution so that if the inequality (1.7.1) is

satisfied at any one instant, it also remains satisfied for all later instants. We therefore arrive at the surprising conclusion that the similarity solution (1.7.3) is valid for the whole 'decay' of the turbulence beyond a certain time and that an equilibrium range exists throughout this period. There is consequently never a final period of decay, in the sense of that phrase in three dimensional theory.

Of course this argument may be held as evidence for the rather special nature of the similarity solution: a similarity solution is of use only if it represents an asymptotic form to which a general class of turbulent flows converge in time and it may be the case that the solution we have constructed has not this property.

Let us again look at the situation in the three dimensional theory. It is possible there to construct a similarity solution in exactly the same way as we have done but using as parameters ε and ν (and variables k and t). It can then be shown that

$$E \propto t^{-1}, \quad K_0 \propto t^{-1/2} \quad \text{and} \quad K_d \propto t^{-1/2}$$

where K_0 and K_d denote the characteristic wavenumbers of the energy and vorticity containing ranges respectively. Consequently the Reynolds number based on E and ε (or K_0 , or K_d) remains constant in time and if $K_d \gg K_0$ at any instant it remains so for all time, i.e. the solution is self-consistent for an equilibrium range to persist throughout the development of the turbulence. In particular there would be no final period of decay.

(Note that the work by Batchelor and Proudman (1956) is crucial in the construction of this solution

since those authors showed that it was not necessary for the large scale structure of turbulence to be invariant. Most of the work on similarity (see Batchelor (1953, chapter 7) for a comprehensive survey) was published at an earlier date and consequently did not allow for this possibility.)

It has been suggested that the restriction on this similarity solution is due to the inability of the large scale motions to adjust themselves to the similarity state before they become dynamically important. A theory has therefore been constructed which allows for self-preservation of the shape of the energy spectrum only for a restricted range of (high) wavenumbers. The result of this theory is the derivation of the decay law

$$E = E_0 + \frac{A}{t - t_0}$$

where E_0 , A and t_0 are constants. This has received some degree of confirmation in some experiments by Tsuji and Hama (1953) (see also a discussion by Lin (1959))— E_0 appears to be positive in general, although it must be a relatively small quantity behind one grid in a wind tunnel (see for instance Batchelor and Townsend (1948)). We have therefore one mechanism which could lead to the eventual breakdown of the similarity, high Reynolds number flow since E_0 could be interpreted as a measure of the energy not contained in the similarity range. However this theory is not altogether satisfactory as it does not give a quantitative prediction of the breakdown of the equilibrium range. To provide this one would require a prediction for the behaviour of K_0 with time so that it could be compared with $K_d (\propto t^{-1/2})$. But the only length scale predictions from the theory are for the length λ where

$$\lambda^2 = \frac{E\nu}{\varepsilon} \propto t^2 \left(E_0 + \frac{A}{t-t_0} \right)$$

and for

$$\left(\frac{E^3}{\varepsilon^2} \right)^{1/2} \propto t^2 \left(E_0 + \frac{A}{t-t_0} \right)^{3/2}$$

and both of these quantities increase at a faster rate than that required by similarity, not slower as one would wish. Thus the theory only provides evidence for the lack of usefulness of these scales in this particular problem and does not give positive evidence for the deviation from an equilibrium state (although of course that deviation is implicit to a great deal of the underlying assumptions of the theory).

In the experiments of Batchelor and Townsend and of Stewart and Townsend (1950) upon which most of the theory of similarity was based the Reynolds number was probably not high enough for an inertial subrange to exist. (See Batchelor (1953, p. 123) for an estimate of the Reynolds number required.) Consequently it is likely that there was an appreciable amount of dissipation taking place in the wavenumbers associated with the energy containing eddies. Indeed for any finite value of the viscosity there is always some dissipation of energy taking place at all wavenumbers. This provides a second mechanism for the breakdown of the equilibrium range in three dimensional turbulence. To provide a qualitative description of this process we could use Stewart's method - see Stewart and Townsend - of scaling for turbulence at medium Reynolds number. We would then replace E , wherever it has occurred in an argument, by $E(1+\Delta)$ where Δ is a small quantity which is assumed to increase in time. Consequently we would assume that $E(1+\Delta)$ behaves according to the similarity law t^{-1} and then have a decay curve for E which corresponds roughly with the experimental results

(see Batchelor and Townsend). Moreover if we assume that K_0 is relatively unaffected by this finite viscosity mechanism and we take the similarity prediction

$$K_0 \propto t^{-1/2}$$

the dissipation range gradually coalesces with the energy containing range since

$$K_d = \left(-\frac{1}{\nu^3} \frac{dE}{dt} \right)^{1/4}$$

will decrease at a faster rate than $t^{-1/2}$.

For the two dimensional turbulence situation we first examine the consequences of self-preservation over a limited range of (high) wavenumbers. This range must necessarily include the equilibrium range so that in the latter the spectrum form will be determined by ε_ω , ν and k by Kolmogoroff arguments on the one hand and by E , ν , k , and t (since these are still the only relevant parameters by the same reasoning as before) by self-preservation requirements on the other. Thus if an inertial subrange exists these two theories will be compatible only if

$$\varepsilon_\omega = \frac{2A}{t^3}, \quad (1.7.6)$$

where A is a constant, which leads to the decay law for vorticity

$$\Omega = \Omega_0 + \frac{A}{t^2}, \quad (1.7.7)$$

where Ω_0 is a constant. If the constant is positive then Ω_0 could be interpreted as the order of magnitude of the vorticity not contained in the similarity range of wavenumbers.

Contrary to the three dimensional situation we have an actual quantitative prediction from (1.7.6) and (1.7.7) for the eventual annihilation of the equi-

librium range: a condition for the existence of that range was stated in (1.7.1), but now

$$R_\varepsilon = \frac{EA}{\nu} \varepsilon^{-3} \left(\Omega_0 + \frac{A}{\varepsilon^2} \right)^{-2}$$

and R_ε will eventually cease to be a large number, so that the rate of dissipation of energy will eventually become of importance in the decay process, if Ω_0 is positive.

If, however, Ω_0 is negative we cannot interpret it as a measure of the vorticity lying outside the similarity range. In addition the inequality (1.7.1) required for the existence of the equilibrium range would then remain valid for all time so that we could suppose that equation (1.7.7) would describe the whole decay of the turbulence. The flow would then have a finite energy but zero vorticity after only a finite time. These anomalies suggest that Ω_0 must be always either zero or positive.

It is interesting to note that the length scale formed from E and Ω , i.e.

$$(E/\Omega)^{1/2} = E^{1/2} \left(\Omega_0 + \frac{A}{\varepsilon^2} \right)^{-1/2}$$

by this prediction. So if Ω_0 is positive we have a length scale which increases at a rate slower than that required by full similarity and which eventually becomes constant. This seems to fit our intuitive ideas of how the length scales of the energy and the vorticity containing eddies should behave in these conditions and we tentatively suggest that

$$R_\lambda = (\Omega/E)^{1/2}$$

may remain of the same order as R_ε for a considerable period of the decay of two dimensional turbulence.

As regards the second possible mechanism for the breakdown of the equilibrium range, we may evaluate the effect of a finite, but small, amount of dissipation at all wavenumbers in two dimensional turbulence as follows. Suppose that the rate of change of the characteristic wavenumbers k_0 and k_d are not affected greatly by the finite viscosity so that

$$k_0 \propto t^{-1} \quad \text{and} \quad k_d \propto t^{-1/2},$$

the similarity laws, are still approximately true. Then the length scale of the vorticity containing eddies is growing much faster than the length scale which is appropriate to the vorticity dissipation. Consequently the effect of a finite viscosity on the decay of the vorticity spectrum diminishes in time. However the energy will also be affected and for this

$$\frac{dE}{dt} = -2\nu\Omega.$$

Thus if we now assume that Ω is given by the similarity law, see equation (1.7.5),

$$\Omega = \frac{A}{t^2}$$

then

$$E = E_0 - 2A\nu\left(\frac{1}{t_0} - \frac{1}{t}\right) \quad (1.7.8)$$

where $E=E_0$ when $t=t_0$. (We recall that $t=0$ was defined as the time at which Ω would be infinite, given that the similarity solution always existed.)

It seems that (1.7.8) does not necessarily imply the eventual breakdown of the equilibrium range since it is possible that

$$E_0 > 2A\nu t_0^{-1} \quad (1.7.9)$$

so that E would then be always positive and the conditions (1.7.1) and (1.7.2) could still be satisfied

at all times. But if (1.7.9) is not satisfied then R_ϵ will eventually be of order of magnitude one and the equilibrium range will then cease to exist.

(N.B. We have omitted from this discussion any effects due to inhomogeneities such as the presence of solid boundaries. These will of course strongly influence the behaviour of the large scale eddies in limiting the growth of their length scales. Consequently our discussion has been of a somewhat idealized situation.)

Both of the mechanisms we have described for the eventual breakdown of the equilibrium range have involved the Reynolds number R_ϵ . This suggests that R_ϵ could perhaps characterize the state of two dimensional turbulence. But whilst it is true that the magnitude of R_ϵ will indicate the possible presence of an equilibrium range it will not indicate the stage of development the turbulence has reached. Thus R_ϵ could be increasing in time, and then the turbulence is probably developing according to the full similarity solution, or it could be decreasing in time, when either lack of similarity and/or finite viscosity effects could be causing the annihilation of the equilibrium range.

In practice, of course, Reynolds numbers based on k_0 and Ω or k_e (the wavenumber characterizing the energy containing range) and E may prove to be more useful indicators, but we have no arguments to justify this. Indeed the only suggestion we have made concerning the quantities k_0 and k_e , namely that they are both of the order of $(\Omega/E)^{1/2}$, reduces the two Reynolds numbers to the same quantity

$$R = \frac{E}{\Omega^{1/2}}$$

and R is a monotonic increasing function of time for the self-preserving solution, (1.7.6) and (1.7.7), even right up to the extinction of the equilibrium range. Thus the choice of one of these Reynolds numbers is not certain of success.

Finally, if the equilibrium range conditions do break down, then we may assume that both the energy and the vorticity will gradually decay. The last stage of such a process will be that in which the non-linear effects will be negligible everywhere in the flow compared with the action of viscosity. This is the situation usually called the final period of decay in three dimensional turbulence. The analysis for a two dimensional flow is identical in form with that for three dimensions (see Batchelor and Proudman (1956)): the governing equation of motion is

$$\frac{\partial u}{\partial t} = \nu \nabla^2 u$$

which leads to

$$\frac{\partial \Phi_{ij}}{\partial t}(\mathbf{k}, t) = -2\nu k^2 \Phi_{ij}(\mathbf{k}, t); \quad (1.7.10)$$

so that the solution for Φ_{ij} is

$$\Phi_{ij}(\mathbf{k}, t) = \Phi_{ij}(\mathbf{k}, t_0) e^{-2\nu k^2 (t-t_0)},$$

where t_0 is a time for which the equation (1.7.10) is valid.

Now as $t-t_0 \rightarrow \infty$ this solution is dominated by the small values of k , so we may replace $\Phi_{ij}(\mathbf{k}, t_0)$ by its asymptotic form (1.5.16), i.e. as $t-t_0 \rightarrow \infty$

$$\Phi_{ij}(\mathbf{k}, t) \sim \frac{1}{96\pi^2} \frac{1}{k^2} \left(\delta_{ij} - \frac{k_i k_j}{k^2} \right) C_{lmnp}^0 k_l k_m k_n k_p e^{-2\nu k^2 (t-t_0)}$$

where C_{lmnp}^0 is the value of C_{lmnp} at $t=t_0$.

It follows that

$$E(k) \sim k^3 e^{-2\nu k^2(t-t_0)}$$

and consequently that

$$E \sim t^{-2}, \quad \Omega \sim t^{-3}$$

and

$$R \sim t^{-1/2}, \quad R_\varepsilon \doteq \frac{3}{2}.$$

CHAPTER 2

THE THEORY OF SMALL SCALE VARIATIONS OF
SCALAR AND VECTOR QUANTITIES.

2.1 INTRODUCTION

This chapter is concerned particularly with the study of the effect of a two dimensional turbulent motion on the variations of convected scalar and vector quantities such as temperature and magnetic fields. We shall assume for most of the analysis that these quantities are passive, i.e. that they do not affect the (turbulent) motion of the fluid in any way. We shall therefore rely on the results obtained in the first chapter, especially on the existence and specification of the equilibrium range as argued in §1.4 and §1.6, for the development of this theory.

Inferences can be drawn from the theory which have a bearing on the small-scale variations of the vorticity field. §2.3 and §2.5 are devoted to this aspect of the problem. In §2.3 we shall obtain a formulation for the vorticity spectrum at very high wavenumbers, and in §2.5 we shall investigate the 'stability' of the two dimensional structure of the turbulence when subjected to three dimensional perturbations.

The analysis for the three dimensional theory for convected quantities, which we intend to follow quite closely, has been developed particularly by Batchelor (1959) and Batchelor, Howells and Townsend (1959), for scalar quantities, and by Moffatt (1961, 1962), Novikov (1962) and Saffman (1963), for vector quantities. Batchelor's paper also contains a comprehensive account of the state of the theory up to 1959. As we shall have occasion to comment on this more recent work whilst developing the two dimensional theory, we shall not give a detailed criticism of the published work on this topic here.

2.2 THE THEORY OF SMALL-SCALE VARIATIONS OF CONVECTED SCALAR QUANTITIES

We shall assume that the equation governing the behaviour of a scalar quantity θ , such as temperature, is of the form

$$\frac{\partial \theta}{\partial t} + \underline{u} \cdot \nabla \theta = \kappa \nabla^2 \theta \quad (2.2.1)$$

where κ is a constant. As already mentioned we shall take θ to be a passive quantity. We shall also assume that the θ -field is statistically homogeneous (see §1.2) and for convenience we shall suppose that $\bar{\theta} = 0$.

If the term on the right hand side of equation (2.2.1), i.e. the 'conductivity' term, can be neglected then the quantity θ is convected with the fluid without amplification or diminution. Thus if at an initial instant θ is not uniform then the action of the turbulent motion through the term $\underline{u} \cdot \nabla \theta$ is to make the θ -field random by the disordered movements of the fluid. As the analysis of Batchelor (1952) on the behaviour of material lines and surfaces in turbulent fluid holds qualitatively for two dimensional motions, we can then say that, in general, lines of constant θ will increase in length and be drawn closer together. Consequently the θ -field is likely to develop large spatial gradients.

But

the But the action of the conductivity term is seen from the equation

$$\frac{\partial \overline{\theta^2}}{\partial t} = -2\kappa \overline{(\nabla \theta)^2} = -\chi,$$

say, to be responsible for the destruction of ' θ^2 -stuff', the destruction being most efficient when there exist large gradients of θ . Thus the high gradients of θ

generated by the turbulence convection lead to the destruction of θ variations by conduction.

If there is a steady external supply of $\bar{\theta}^2$ -stuff to the fluid on a 'large' length scale then we can make the reasonable postulate that a situation of dynamical equilibrium will eventually be set up in the θ -field since we would then have a source, a transfer mechanism and a sink for the $\bar{\theta}^2$ -stuff. In other words we would have a cascade mechanism set up in the $\bar{\theta}^2$ -spectrum.

If there is no such external supply of $\bar{\theta}^2$ -stuff we shall postulate that, for sufficiently small κ , the large wavenumber shape beyond the maximum in the $\bar{\theta}^2$ -spectrum will develop into a form independent of the large-scale structure. But as we can adapt all the arguments given in §1.6 for the existence of a vorticity cascade to the present context of the θ -field—especially since equation (2.2.1) has the same structure as the vorticity equation—this assumption leads to the same conclusion that there is a cascade mechanism for the $\bar{\theta}^2$ -stuff.

In determining the shape of the $\bar{\theta}^2$ -spectrum at high wavenumbers we shall assume that k_0 , the wavenumber associated with most of the θ -variations, does not lie in the equilibrium range, i.e. $k_0 \leq k_0$. For the case when there is an external supply of $\bar{\theta}^2$ -stuff there is then the implication that the characteristic length scale of the mechanism generating $\bar{\theta}^2$ -stuff must be of order k_0^{-1} or more, which is not unreasonable. When there is no external source we can justify the assumption by a contradiction argument: suppose that

$k_0 \gg k_0$, then k_0 will lie within a region where we would expect the transfer of $\bar{\theta}^2$ -stuff to higher wavenumbers to be very efficient and so k_0 will increase in time; consequently k_0 will eventually be within the conduction range where most of the $\bar{\theta}^2$ -stuff would be destroyed (leaving a residue in wavenumbers of order k_0 , or less); but this is not the type of cascade mechanism we have postulated, so $k_0 \leq k_0$.

As in Batchelor's (1959) paper we shall now assume that the hypotheses of Kolmogoroff's equilibrium theory apply to the θ -variations at wavenumbers much greater than k_0 (and k_0): so that these θ -variations are statistically independent of the θ - and u -variations of wavenumber magnitude k_0 , and that they are statistically steady and isotropic.

In the range $k \gg k_0$, the $\bar{\theta}^2$ -spectrum will then be determined by the parameters $\epsilon_\omega, \nu, \chi$ and κ . (We have assumed that both the viscous dissipation and conduction ranges of wavenumbers are much greater than k_0 so that the rates of transfer of vorticity and $\bar{\theta}^2$ -stuff into the equilibrium range, which are both relevant parameters in the description of the $\bar{\theta}^2$ -spectrum, are equal to ϵ_ω and χ respectively.) Hence the $\bar{\theta}^2$ -spectrum function $T(k)$ is given, from dimensional arguments, by

$$T(k) = \chi \left(\frac{\kappa}{\epsilon_\omega} \right)^{1/2} \times f_n \left(\left(\frac{\kappa^3}{\epsilon_\omega} \right)^{1/6} k, \frac{\nu}{\kappa} \right) \quad (2.2.2)$$

where

$$\bar{\theta}^2 = \int_0^\infty T(k) dk.$$

If there exist wavenumbers k such that

$$k_0 \ll k \ll k_d \quad (2.2.3)$$

then, from §1.6, we may conclude that there is no dependence on ν in this range, so that

$$\Gamma(k) = \chi \left(\frac{\kappa}{\varepsilon \omega} \right)^{1/2} \times f_n \left(\left(\frac{\kappa^3}{\varepsilon \omega} \right)^{1/6} k \right). \quad (2.2.4)$$

Consequently if the range (2.2.3) includes part of the conduction range then the wavenumber marking the conduction cut-off of the $\overline{\theta^2}$ -spectrum must, from (2.2.4), be given in order of magnitude by

$$k_D = \left(\frac{\varepsilon \omega}{\kappa^3} \right)^{1/6} \quad (2.2.5)$$

and if there exist wavenumbers such that

$$k_0 \ll k \ll k_D \quad (2.2.6)$$

then in this range $\Gamma(k)$ is also independent of κ and consequently

$$\Gamma(k) \propto \chi \varepsilon \omega^{-1/3} k^{-1}. \quad (2.2.7)$$

(Again, as with the vorticity spectrum, we must interpret this as an asymptotic result and assume that the integral evaluating $\overline{\theta^2}$ for $\kappa=0$ will be convergent with the main contribution to the integral being provided by wavenumbers in the region of k_0 .)

But if the inertial subrange (2.2.3) does not contain the conduction cut-off then the $\overline{\theta^2}$ -spectrum in the whole of the subrange must be independent of κ and be given by (2.2.7). Thus (2.2.7) holds for

$$\begin{aligned} \text{the} \quad k_0 \ll k \ll k_D &= \left(\frac{\varepsilon \omega}{\kappa^3} \right)^{1/6} && \text{provided} \quad \nu \ll \kappa \\ \text{and for} \quad k_0 \ll k \ll k_d &= \left(\frac{\varepsilon \omega}{\nu^3} \right)^{1/6} && \text{provided} \quad \nu \sim \kappa \\ \text{the} \quad &&& \text{or } \nu \gg \kappa. \end{aligned} \quad (2.2.8)$$

We now proceed to determine the spectral form for $\overline{\theta^2}$ beyond these ranges.

- (i) The case of $\nu \sim \kappa$ or $\nu \gg \kappa$ in the range of wavenumbers $k > k_d$.

The range of wavenumbers $k \geq k_A$ (where this relation is intended in an order of magnitude sense only) is that in which there is an extremely small amount of vorticity. Necessarily most of the mean-square of vorticity, or of any first derivative of the velocity, is contained in wavenumbers much less than k_A since we have the vorticity spectrum prediction (1.6.6) and also the fact that there is a marked cut-off in the vorticity spectrum at k_A due to viscosity. The problem of the behaviour of the θ -variations on a length scale k_A^{-1} or less is therefore one for which the local spatial rate of change of fluid velocity is approximately uniform. The equation governing the θ -variations, equation (2.2.1), may then be replaced locally by

$$\frac{\partial \theta}{\partial t} + \underline{U} \cdot \nabla \theta = \kappa \nabla^2 \theta \quad (2.2.9)$$

where $\underline{U}$ is the approximate local velocity field in the form of a linear function of position.

This type of postulate was first used by Batchelor (1959) for the behaviour of θ -variations in three dimensional turbulence in an analogous situation. We note however that the evidence in the three dimensional theory for the uniformity of the spatial derivative of the velocity field over regions in physical space smaller than k_A^{-1} (i.e. $(\varepsilon/\nu^3)^{1/4}$) cannot be used in the present situation. Batchelor derives from Kolmogoroff's theory and an empirical value for the skewness factor that

$$\frac{(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_i}{\partial x_j})^2}{(\frac{\partial u_i}{\partial x_j})^2} \doteq 0.03 + 2 (\nu^3/\varepsilon)^{-1/2}.$$

But this relationship depends critically on the presence of the non-zero $\underline{\omega} \cdot \nabla \underline{u}$ term in the vorticity equation and on the fact that the equilibrium range includes the dominant contributions to the vorticity. Neither

of these conditions is valid in two dimensional turbulence and we therefore cannot use this equation to support our hypothesis.

Now in the equation

$$\frac{\partial \overline{\omega \omega'}}{\partial t} = - \overline{\omega' (\underline{u} \cdot \nabla) \omega} - \overline{\omega (\underline{u}' \cdot \nabla') \omega'} + 2\nu \nabla^2 \overline{\omega \omega'},$$

where dashes denote evaluation at points a distance r from the undashed quantities, we can say that for values of r lying in the equilibrium range the time derivative may be neglected and that, as r tends to zero, the first two terms on the right hand side also tend to zero. We may therefore argue, on assuming isotropy, that

$$\overline{(\omega' - \omega)^2} = o(r^2)$$

or that

$$\overline{\left(\frac{\partial u_i'}{\partial x_j} - \frac{\partial u_i}{\partial x_j} \right)^2} = o(r^2)$$

for small r . Consequently for sufficiently small r the velocity is approximately a linear function of r : but we cannot give any precise definition to the smallness of r that is required. That r must be of order

k_d^{-1} or less seems reasonable, and is all we need in the present context, but we should note that this condition could be too restrictive and that the analysis may be valid for variations on a physical scale greater than k_d^{-1} .

If the velocity is a linear function of position it may be composed of a uniform translation, a rotation and a uniform strain. Consequently if we take axes which move with the local translational velocity and which lie instantaneously along the principal axes of the straining motion then

$$\underline{U} = (\alpha x - \beta' y, -\alpha y + \beta' x, 0) \quad (2.2.10)$$

for a two dimensional motion in the (x, y) plane, where α is a measure of the strain (and is assumed without loss of generality to be positive) and p' is the local rotation. The velocity $\underline{U}$ must be a function of time so that the translational velocity, the magnitudes of α and p' , and the directions of the principal axes all vary with time.

From the form of (2.2.9) and (2.2.10) it can be seen that the variation of the translational velocity has no effect on the analysis: the reason for this being that the θ -field is always convected with the fluid even if the fluid is accelerated. We may similarly take account, at least to some extent, of the variation of the directions of the principal axes with time. For, as equation (2.2.9) is invariant in form under rotations, we may refer the local motion to axes which rotate with the principal axes. We then have that

$$\underline{U} = (\alpha x - py, -\alpha y + px, 0) \quad (2.2.11)$$

where p is now the local rotation of the fluid relative to the (rotating) principal axes. Thus we need only consider the behaviour of α and p with time in the following analysis.

In the three dimensional theory Batchelor argued that the strain is constant over long periods of time compared with the predicted time scale of the θ -variations. He also took the equivalent to p to be effectively zero — indeed the exact assumption was stronger in that the directions of the principal axes were taken to be steady as well. Saffman (1963) on the other hand did include rotation effects and he assumed that the equivalents of α and p vary with the same time scale so

that, with the addition of Batchelor's arguments for the time-scale of α , he concludes that both the strain and rotation terms can be assumed constant for the analysis.

Broadly speaking, we have not been able to find such strong evidence in the two dimensional theory for the constancy of α and p as has been obtained for the three dimensional problem. The steadiness of the principal axes can be argued convincingly, but the values of α and p' may fluctuate to an appreciable extent.

The evidence for the constancy of the directions of the principal axes is in fact the same as that for the three dimensional situation, for Batchelor cites some observations made by Welander (1955) on the drawing out of a marked fluid element in a two dimensional turbulent motion. In the experiments, long thin streaks were formed which did not contain any small-scale wriggles which would be an indication of the rotation of the axes. As we shall show below, extension of material lines of fluid can only take place if $\alpha^2 > p^2$. Consequently we have here direct evidence that the variations of the directions of the principal axes must be small only if the above condition is satisfied. However there appears to be no reason why the local rotation p' should be correlated with the rotation of the principal axes. We therefore argue that Welander's evidence may be interpreted as showing that the difference between p and p' may be ignored in most circumstances.

We may easily give an estimate of the time scale of the variation of p' , i.e. of the local vorticity,

once we have recalled that this vorticity can only be changed by the action of the viscous term. Thus if we assume that the uniformity of the vorticity extends over a region of length scale $k\lambda'$ we have, from the vorticity equation, that the time scale of variation is $(\nu k\lambda'^2)^{-1}$, or $\varepsilon_\omega^{-1/3}$. If the uniformity extends over a region greater than this, as seems to be possible, then, of course, the time scale of variation will be larger than $\varepsilon_\omega^{-1/3}$. Now in the analysis below we shall show that the time-scale of the θ -variations is of order $\varepsilon_\omega^{-1/3}$ as well. Thus if the time scale of the variation of p' is larger than that for the θ -variations it can only be in an ordinary numerical sense independent of any Reynolds number (unlike the three dimensional situation).

As the equation governing the time variation of α depends on third order correlations we have no direct theoretical estimate of the time scale associated with this variation.

We must therefore rely on Welander's evidence for the justification that α and p' may be assumed constant for periods of time long enough to produce long streaks of marked fluid particles — or for the θ -variations to behave as we shall predict.

In the following work we shall first find the consequences of assuming that α and p are constant and then try to describe qualitatively the behaviour of the θ -variations when we relax this assumption and let α and p vary with time, albeit slowly.

The equation (2.2.9) with (2.2.11) admits a solution

of the form

$$\theta = A(t) \sin(\bar{k}_1(t)x + \bar{k}_2(t)y) \quad (2.2.12)$$

where

$$\frac{dA}{dt} = -\kappa \bar{k}^2 A, \quad (2.2.13)$$

$$\frac{d\bar{k}_1}{dt} = -\alpha \bar{k}_1 - \beta \bar{k}_2, \quad (2.2.14)$$

$$\frac{d\bar{k}_2}{dt} = \beta \bar{k}_1 + \alpha \bar{k}_2. \quad (2.2.15)$$

The equations (2.2.14) and (2.2.15) then have a 'normal mode' solution of the form

$$\begin{aligned} \bar{k}_1 &= [\sqrt{(\alpha^2 - \beta^2)} + \alpha] K_1 \exp\{-\sqrt{(\alpha^2 - \beta^2)}t\} - \beta K_2 \exp\{\sqrt{(\alpha^2 - \beta^2)}t\}, \\ \bar{k}_2 &= -\beta K_1 \exp\{-\sqrt{(\alpha^2 - \beta^2)}t\} + [\sqrt{(\alpha^2 - \beta^2)} + \alpha] K_2 \exp\{\sqrt{(\alpha^2 - \beta^2)}t\}, \end{aligned} \quad (2.2.16a)$$

where K_1 and K_2 are constants, except when $\alpha = \pm \beta$ and then

$$\bar{k}_1 = K_1 - \alpha(K_1 \pm K_2)t, \quad \bar{k}_2 = K_2 + \alpha(K_2 \pm K_1)t. \quad (2.2.16b)$$

Equation (2.2.9) is intended to hold for θ -variations over a region in physical space of length scale $\bar{k}_d^{-1}$. If we choose a wavenumber $\bar{k}$ and a wavenumber interval $d\bar{k}$ such that $\bar{k}$ is much larger than $d\bar{k}_1$ or $d\bar{k}_2$, which are in turn much larger than $\bar{k}_d$, we can then assume that the Fourier-Stieltjes transform, $d\Theta(\bar{k})$, of $\theta(x)$ will in general be determined by the θ -variations mainly over a region for which (2.2.9) is valid. (See the description of the Fourier-Stieltjes transform in §1.2, particularly equation (1.2.4).) Thus the solution (2.2.12) can be thought of as being a formalized representation of the transform function $d\Theta(\bar{k})$ provided the inequalities between $\bar{k}$, $d\bar{k}_1$, $d\bar{k}_2$ and $\bar{k}_d$ are valid. As these inequalities are independent of κ we can still think of $\bar{k}$ being of order $\bar{k}_d$ as κ tends to zero: thus

(2.2.12) will be a formalized representation of $\Delta\Theta(\underline{k})$ if $k \geq k_A$ and if we interpret the sign $>$ as in the title to this section.

In the succeeding work we note that we shall consequently be seeking a solution of a mixed boundary and initial value problem. We shall assume that at some instant in time a Fourier analysis of the local Θ -variations is made and then we shall try to determine the behaviour of $A(\underline{k})$, or $\Delta\Theta(\underline{k})$, at all later instants in the range $k \geq k_A$, with the conditions at $k = k_A$ being given at all times.

Now the wavenumber of the Fourier mode $A(\underline{k})$ is seen from (2.2.16) to describe a locus in wavenumber space, the locus being either a circle, an ellipse, a pair of straight lines, a general hyperbola or a rectangular hyperbola according as $\alpha = 0$ ($\beta \neq 0$), $\alpha^2 < \beta^2$, $\alpha^2 = \beta^2$, $\alpha^2 > \beta^2$ or $\beta = 0$ ($\alpha \neq 0$). Thus if $\alpha^2 > \beta^2$ it will be seen that the wavenumber $\underline{k}$ will always eventually travel away from the origin: it is this that leads to the degradation of Θ -variations to smaller and smaller length scales until their final destruction by conduction. But the case $\alpha^2 > \beta^2$ includes loci where $\underline{k}$ travels initially towards the origin and which eventually lie on $k = k_A$ and we therefore seem to have a mechanism for the transfer of $\bar{\Theta}^2$ -stuff to smaller wavenumbers as well.

If $\alpha^2 < \beta^2$ then it can be shown that the transfer of $\bar{\Theta}^2$ -stuff between wavenumbers of different magnitude takes place slowly (relative to the case $\alpha^2 > \beta^2$) if at all, and the main action is to redistribute $\bar{\Theta}^2$ -stuff in wavenumber direction. (The reason for this is that

for transfer to be possible between wavenumbers of different magnitude the elliptical locus must be highly eccentric, i.e. $\alpha^2 \neq p^2$. But the transfer rate is dependent on $\sqrt{(p^2 - \alpha^2)}$ which is then a small quantity.) It will in fact be necessary in our analysis to ignore any transfer due to this case because the structure of the spectrum function set up by this mechanism depends on the detailed nature of $d\Theta(k)$ on $k=k_A$ and not on the general behaviour of the spectrum function $T(k)$ at $k=k_A$ as will be shown to be the case for $\alpha^2 > p^2$.

We shall now consider the initial value problem with $d\Theta(k)=0$ for all $k > k_A$ at the initial instant.

If $\alpha^2 < p^2$ then there is effectively no mechanism for the local transfer of $\bar{\Theta}^2$ -stuff to higher wavenumbers and so $d\Theta(k)$ will remain zero for all time in the range $k > k_A$.

But if $\alpha^2 > p^2$ then $\bar{\Theta}^2$ -stuff is transferred to higher wavenumbers along, approximately, an asymptote of the hyperbola locus. From (2.2.16a) the approximate equation for k is then

$$\frac{dk}{dt} = \sqrt{(\alpha^2 - p^2)} k. \quad (2.2.17)$$

Thus

$$k = k_A \exp \{ \sqrt{(\alpha^2 - p^2)} (t - t_A) \} \quad (2.2.18)$$

where t_A is the instant at which $k=k_A$, and the solution for $d\Theta(k, t)$ is, from (2.2.13) and (2.2.18),

$$d\Theta(k, t) = d\Theta(k_A, t_A) \exp \left\{ -\frac{\kappa k_A^2}{2\sqrt{(\alpha^2 - p^2)}} \left(\exp[2\sqrt{(\alpha^2 - p^2)}(t - t_A)] - 1 \right) \right\}. \quad (2.2.19)$$

The spectrum function $\Gamma(\underline{k}, t)$ for $\bar{\theta}^2$ is given approximately by

$$\Gamma(\underline{k}, t) = \int_{\underline{k}} \frac{d\Theta(\underline{k}) d\Theta^*(\underline{k})}{d\underline{k}_1 d\underline{k}_2} d\ell(\underline{k})$$

(see Batchelor (1953, p. 32)), where the star denotes conjugacy, and the integration is over a circle, centre the origin and radius $\underline{k}$. Thus the contribution to this averaged spectrum function from the local motion is

$$\hat{\Gamma}(\underline{k}, t) = \int_{\underline{k}} \frac{d\Theta(\underline{k}) d\Theta^*(\underline{k})}{d\underline{k}_1 d\underline{k}_2} d\ell(\underline{k}),$$

$$\text{i.e. } \hat{\Gamma}(\underline{k}, t) = \frac{1}{d\underline{k}} \int_{\underline{k}} d\Theta(\underline{k}) d\Theta^*(\underline{k}), \quad (2.2.20)$$

where $d\underline{k}$ is the wavenumber interval in the radial direction which corresponds to the transform $d\Theta(\underline{k})$. But this interval must, from (2.2.18), have emanated from a wavenumber interval $d\underline{k}_d$ at wavenumber $\underline{k}_d$ and time t_d , where

$$d\underline{k} = d\underline{k}_d \exp\left\{\sqrt{(\alpha^2 - p^2)}(t - t_d)\right\}, \quad (2.2.21)$$

so on substituting (2.2.19) and (2.2.21) in (2.2.20) and using (2.2.18) we obtain

$$\hat{\Gamma}(\underline{k}, t) = \left[\frac{1}{d\underline{k}_d} \left(\int_{\underline{k}_d} d\Theta(\underline{k}_d, t_d) d\Theta^*(\underline{k}_d, t_d) \right) \underline{k}_d \exp\left\{\frac{\kappa \underline{k}_d^2}{\sqrt{(\alpha^2 - p^2)}}\right\} \right] \frac{1}{\underline{k}} \exp\left\{-\frac{\kappa \underline{k}^2}{\sqrt{(\alpha^2 - p^2)}}\right\}.$$

Finally, as we have

$$\hat{\Gamma}(\underline{k}_d, t_d) = \frac{1}{d\underline{k}_d} \int_{\underline{k}_d} d\Theta(\underline{k}_d, t_d) d\Theta^*(\underline{k}_d, t_d)$$

then

$$\hat{\Gamma}(\underline{k}, t) = \left[\hat{\Gamma}(\underline{k}_d, t_d) \underline{k}_d \exp\left\{\frac{\kappa \underline{k}_d^2}{\sqrt{(\alpha^2 - p^2)}}\right\} \right] \frac{1}{\underline{k}} \exp\left\{-\frac{\kappa \underline{k}^2}{\sqrt{(\alpha^2 - p^2)}}\right\}. \quad (2.2.22)$$

There are two particular forms of this solution which are of interest. If the conditions at $\underline{k} = \underline{k}_d$ are steady in time and if we ignore the initial transient motion then the contribution to the spectrum function is

$$\hat{\Gamma}(\underline{k}, t) \propto \frac{1}{\underline{k}} \exp\left\{-\frac{\kappa \underline{k}^2}{\sqrt{(\alpha^2 - p^2)}}\right\}. \quad (2.2.23)$$

Secondly the unsteady solution in terms of k , t and k_d alone, which is obtained by using the formula, derived from (2.2.18)

$$t_d = t - \frac{1}{\sqrt{(\alpha^2 - p^2)}} \log(k/k_d), \quad (2.2.24)$$

is

$$\hat{T}(k, t) = \left[\hat{T}(k_d, t - \frac{1}{\sqrt{(\alpha^2 - p^2)}} \log(k/k_d)) k_d \exp\left\{\frac{\kappa k_d^2}{\sqrt{(\alpha^2 - p^2)}}\right\} \right] \frac{1}{k} \exp\left\{-\frac{\kappa k^2}{\sqrt{(\alpha^2 - p^2)}}\right\} \quad (2.2.25)$$

where, to satisfy the initial conditions at $t=0$, we put

$$\hat{T}(k_d, t_d) = 0$$

for all $t_d < 0$. Thus the effect of these initial conditions is simply to cause an abrupt cut-off of the spectrum at the wavenumber k given by (2.2.18) with $t_d = 0$.

It is interesting to note that if $\hat{T}(k_d, t_d)$ is oscillatory in t_d , the function $\hat{T}(k, t)$ will appear as a 'wave' travelling in wavenumber space with phase velocity

$$\sqrt{(\alpha^2 - p^2)} k.$$

Moreover the unsteady solution is probably more relevant than the steady solution for most situations since, even if the overall statistical quantities at $k=k_d$ are steady in time (and now there seems to be some doubt about this in view of the work of Kolmogoroff (1962) and Oboukhov (1962)), it is less likely that the local supply of $\bar{\theta}^2$ -stuff at these wavenumbers is actually steady. We shall have cause to return to this type of solution, and in particular to the 'wave' solution, in the next chapter when we discuss the results of the computational model.

Before constructing the spectrum function $T(k, t)$

we must note that the above analysis apparently violates one of the necessary conditions for its validity, for one component of $d\mathbf{k}$ is not always greater than k_a . The radial component of $d\mathbf{k}$, i.e. the symbol dk defined above, is an increasing function of time (see equation (2.2.21)) and will always satisfy this condition, but the azimuthal component, dk_a say, behaves as

$$dk_a \propto \exp \{ - \sqrt{(d^2 - p^2)} t \}$$

and consequently will become smaller than k_a after a finite time. We obviate this difficulty in the following manner.

If $d\Theta(k, t, dk, dk_a)$ represents the Fourier-Stieltjes transform with wavenumber interval (dk, dk_a) at time t and wavenumber k then the above analysis predicts, at a time δt later, the value of

$$d\Theta(k + \delta k, t + \delta t, dk + \delta(dk), dk_a - \delta(dk_a)) \quad (2.2.26)$$

when k lies on the outward travelling asymptote. But at this time and wavenumber we can then construct a new $d\Theta$, with a different wavenumber interval, based on the prediction (2.2.26). For, say,

$$d\Theta(k + \delta k, t + \delta t, dk + \delta(dk), dk_a) \quad (2.2.27)$$

is defined to be the integral of all θ -variations over the given interval, and this interval includes the interval given in (2.2.26). Moreover θ -variations with wavenumbers included in (2.2.27) but not in (2.2.26) must all have zero amplitude since we have assumed that the only transfer of $\bar{\theta}$ -stuff to wavenumber $k + \delta k$ at time $t + \delta t$ is along the asymptote. We can then use (2.2.27) as a basis for predicting the transfer over the next time interval δt , and successive repetitions of this whole process allows us to make

predictions (which are the same as before) for all wavenumbers $k > k_d$.

If we suppose that the spectrum function $T(k, t)$ is independent of time then we may assume that the average of $\hat{T}(k, t)$, which is given by (2.2.25) for $\alpha^2 > p^2$ and zero otherwise, is of the form

$$T(k) \propto \frac{1}{k} \exp \{ -\kappa P k^2 \} \quad (2.2.28)$$

where P is a parameter which must be connected with some statistical function of $(\alpha^2 - p^2)^{-1/2}$. (In fact we shall be able to obtain its order of magnitude as $\epsilon_\omega^{-1/2}$ in §2.3 when discussing the application of (2.2.28) to the vorticity spectrum.)

We note that the steady solution (2.2.28) will only be attained after a time of the order of

$$-\frac{P}{2} \log(k_d^2 \kappa P) \quad (2.2.29)$$

which is the time taken for transfer of $\bar{\theta}^2$ -stuff from $k = k_d$ to the conduction cut-off wavenumber. Thus for very small values of κ it is hardly likely that the state (2.2.28) should ever be achieved.

The above analysis was constructed from the initial value problem with $d\Theta = 0$ for $k > k_d$. This was done in order to obviate the difficulties which arise from arbitrary initial distributions, and in particular from initial distributions of Θ -variations with wavenumbers lying on, or near to, the inward travelling asymptotes when $\alpha^2 > p^2$. In this latter case it appears to take a time of the order of (2.2.29) for such a distribution to be transferred from near the conduction cut-off wavenumber to $k = k_d$ and throughout this time it would seem that there was indeterminacy in the

analytical solution. However there are two reasons why we may ignore these arbitrary initial conditions in general (for sufficiently large κ).

Firstly, if we consider the apparent real situation where α and β vary (slowly) with time, the only occasions when there are arbitrary distributions of $\bar{\theta}^2$ -stuff, and $\alpha^2 > \beta^2$, are the instants when $(\alpha^2 - \beta^2)$ changes sign from being negative to positive. Because whilst $\alpha^2 > \beta^2$ the θ -variations can be expected to stay close to the outward travelling asymptotes and in these circumstances the above analysis will be valid, so that it is necessary for α^2 to become less than β^2 , when transfer will follow quasi-elliptical loci, before θ -variations will be distributed throughout wavenumber space. As α and β seem to vary slowly over time periods of order P , the time scale of the period between two successive changes of sign of $(\alpha^2 - \beta^2)$ from negative to positive is likely to be much larger than this characteristic time P of transfer of the θ -variations. However it is doubtful if it is also much larger than the time scale (2.2.29) when κ is not of the same order as ν .

But the second reason indicates that we may be able to ignore all θ -variations on, or near to, the inward travelling asymptote^{altogether} so that P , and not (2.2.29), will be the only relevant time scale to consider. If $d\Theta(\underline{k}, t, dk, dk_a)$ is the transform of θ at wavenumber $\underline{k}$ on this asymptote, then at the time t we must choose a value of dk_a greater than k_a . But dk_a can be shown to increase exponentially with time so that the wavenumber interval will eventually include wavenumbers distant from the inward travelling asymptote

with loci behaving in a very different manner from the asymptote locus. Thus it is necessary to continually reduce the value of $d\bar{k}_a$. We can in fact try a method which is the exact opposite of that constructed above for wavenumbers lying on the outward travelling asymptote: hence at time $t+\delta t$ the analysis would predict

$$d^{\Theta}(\bar{k}-\delta\bar{k}, t+\delta t, d\bar{k}-\delta(d\bar{k}), d\bar{k}_a+\delta(d\bar{k}_a)), \quad (2.2.30)$$

where all the δ -quantities are positive, and we would like to construct from this the value of, say,

$$d^{\Theta}(\bar{k}-\delta\bar{k}, t+\delta t, d\bar{k}-\delta(d\bar{k}), d\bar{k}_a). \quad (2.2.31)$$

But the wavenumber interval in (2.2.31) is included within that in (2.2.30) so that it is impossible from this analysis to construct the function (2.2.31) and thus to continue a solution along an inward travelling asymptote.

The physical reason for this failure is apparent when we realize that to obtain a knowledge of (2.2.31) from (2.2.30) we must evaluate d^{Θ} by considering the Θ -variations over a region in physical space larger than k_a^{-1} . Thus the indeterminacy arises from the fact that the behaviour of Θ -variations outside the scope of our analysis actually contributes to the determination of the quantity (2.2.31).

It would of course be possible to reproduce the spectral form (2.2.28) from the Θ -variations along the inward travelling asymptote but it would then be necessary for (2.2.31) to be approximately equal to (2.2.30) whenever the wavenumber interval was changed. This seems to be an unlikely sequence of events to demand from the large scale variations simply because there is a whole range of possible values for (2.2.31)

at each of these changes. Thus we argue that the transfer of $\bar{\Theta}^2$ -stuff along an inward travelling asymptote is a highly unlikely event and we conclude that the results (2.2.22) and (2.2.28) are in fact valid.

(We note that Saffman (1963) reached this same conclusion. His argument was based on a comparison between the above type of analysis with wavenumbers lying only near the outward travelling asymptote and Pearson's analysis for a random vector field in a uniform straining motion of infinite extent. He shows that the divergence one obtains in the latter analysis, but not in the former, is wholly due to variations of the vector field which are on a large length scale, i.e. the divergence is critically dependent on the infinite extent of the straining motion. Hence the analysis which excludes this divergence is argued to be the relevant analysis for the behaviour of the small scale variations over a uniform region of small extent. However from Saffman's argument in terms of spectral functions (he also deals with the analysis in terms of covariance tensors in physical space) it is not altogether clear why one should neglect wavenumbers lying near the inward travelling asymptote since he argues that they should be neglected only because they eventually have magnitude $k < k_a$: this is an argument in support of considering only wavenumbers $k > k_a$ with boundary conditions at $k = k_a$ and we have seen that this does not exclude an indeterminacy in the resulting spectrum function. (See §2.4 for details of the possible magnitude of this indeterminacy for vector quantities.) The argument for neglecting the awkward wavenumbers is critically dependent, as we have shown, on an argument which involves the wavenumber interval Δk associated

with the Fourier-Stieltjes transform $d\Theta$ and not merely on the magnitude of the wavenumber k .)

If we tentatively extend the range of validity of this analysis to include wavenumbers less than k_d then we may combine the results (2.2.7) and (2.2.28) to cover the whole of the range $k \geq k_0$. We then have

$$T(k) = B \chi \varepsilon_0^{-1/3} k^{-1} \exp \{-\kappa P k^2\}$$

where B is a constant. But the rate of destruction of $\bar{\Theta}^2$ -stuff, χ , is largely determined by wavenumbers $k \gg k_0$, so that

$$\begin{aligned} \chi &= 2\kappa \int_0^\infty k^2 T(k) dk \\ &\doteq 2B \chi \kappa \varepsilon_0^{-1/3} \int_0^\infty k \exp \{-\kappa P k^2\} dk \end{aligned}$$

and on substituting $P = C \varepsilon_0^{-1/3}$, where C is a constant, we obtain $C = B$. Thus

$$T(k) = C \chi \varepsilon_0^{-1/3} k^{-1} \exp \{-C \kappa \varepsilon_0^{-1/3} k^2\} \quad (2.2.32)$$

for $\nu \sim \kappa$ and $\nu \gg \kappa$ in the range $k \gg k_0$.

Finally, we note that the anomalous behaviour of the Θ -variations when $\alpha^2 < \beta^2$ is peculiar to the two-dimensionality of the problem. Although a similar configuration of the strain and rotation can be obtained in the three dimensional problem, a slight variation of either the strain or rotation is sufficient to destroy the closed form of the loci so that transfer of $\bar{\Theta}^2$ -stuff to different wavenumber magnitudes can always be assumed to be present.

(ii) The case of $\nu \ll \kappa$ in the range of wavenumbers $k \gg k_D$

In the case when $\nu \ll \kappa$ we have shown that the $\bar{\Theta}^2$ -spectrum is independent of the conductivity κ for wavenumbers $k \ll k_D = (\varepsilon_0/\kappa^3)^{1/6}$. But when k is of order

k_D or greater we must expect conduction to play an important role in governing the behaviour of the θ -variations so that there will be a sharp cut-off of the $\bar{\theta}^2$ -spectrum. In the range $k_D \ll k \ll k_d$ we then have that $\Gamma(k)$ falls off more rapidly than k^{-1} and that $\Omega(k)$ has the form k^{-1} . The conditions of this situation are in fact closely analogous with the three dimensional theory covered in the paper by Batchelor, et al. (1959) - so much so in this instance that we can follow through those authors' arguments exactly, merely replacing ε by ε_ω , $(\varepsilon/\kappa^2)^{1/4}$ by $(\varepsilon_\omega/\kappa^2)^{1/6}$, $(\varepsilon/\nu^3)^{1/4}$ by $(\varepsilon_\omega/\nu^3)^{1/6}$ and substituting all the algebraic details which result from these changes. We shall not, therefore, describe in full all the steps of the argument here but give only a brief synopsis of the method.

First, the integral term in the Fourier transform of the θ -equation

$$\frac{\partial}{\partial t} \{ \Delta \Theta(k) \} + i \int k'_1 \Delta Z_1(k-k') \Delta \Theta(k') = -\kappa k^2 \Delta \Theta(k),$$

where $\Delta Z(k)$ is the Fourier-Stieltjes transform of the velocity, can be assumed to be dominated by those values of k' much smaller than k when $k \gg k_D$ since the magnitude of $\Delta \Theta(k')$ near $k'=k$ decreases very rapidly with increasing k' . (This step can be shown to be consistent with the result we shall obtain on this basis.) Secondly we can postulate that the resulting integral term is approximately steady in time (being in the nature now of a source term with a characteristic time that associated with the vorticity spectrum in the same wavenumber range, i.e. $\varepsilon_\omega^{-1/3}$, as compared with a time scale $(\kappa k^2)^{-1}$ associated with the conduction term) so that time derivatives may be neglected. The basic equation can therefore be reduced

to

$$\kappa^2 \kappa^4 \overline{d\Theta(\underline{k}) d\Theta^*(\underline{k})} = \overline{dZ_i(\underline{k}) dZ_j^*(\underline{k})} \cdot \overline{\frac{\partial \Theta}{\partial x_i} \frac{\partial \Theta}{\partial x_j}}$$

(see Batchelor, et al.).

We can now obtain the relation between the $\overline{\Theta^2}$ and energy spectra at the wavenumber $\underline{k}$ by dividing by $dk_1 dk_2$ and taking the limit $d\underline{k} \rightarrow 0$. But in the range of wavenumbers we are considering, the energy spectrum is isotropic and determined by the single parameter ϵ_ω . Moreover $\overline{\frac{\partial \Theta}{\partial x_i} \frac{\partial \Theta}{\partial x_j}}$ must also be isotropic, since it is largely determined by variations of length scale κ_D^{-1} . Thus

$$\kappa^2 \kappa^4 \Gamma(\underline{k}) = E(\underline{k}) \cdot \overline{(\nabla \Theta)^2}.$$

If we now integrate over the circumference of a circle, centre the origin, radius $\underline{k}$, and substitute $\overline{(\nabla \Theta)^2} = \chi/2\kappa$ we obtain the spectral form for the Θ -variations in the wavenumber range $\kappa_D \ll \underline{k} \ll \kappa_d$ as

$$\Gamma(\underline{k}) \propto \frac{\chi \epsilon_\omega^{2/3}}{\kappa^3} \underline{k}^{-7}. \quad (2.2.33)$$

We note that this result is consistent with the prediction (2.2.7) for the range $\kappa_0 \ll \underline{k} \ll \kappa_D$ in that both give the same order of magnitude, namely $\chi \left(\frac{\kappa}{\epsilon_\omega}\right)^{1/2}$ for the function $\Gamma(\underline{k})$ if they are extrapolated to cover wavenumbers of magnitude κ_D .

Finally, in the range $\underline{k} \gg \kappa_d$ the conditions for the uniform strain approximation of §2.2(i) are valid and consequently the $\overline{\Theta^2}$ -spectrum will be given by equation (2.2.28). However there will be but a minute

quantity of $\overline{\theta^2}$ -stuff in this range and this prediction can hardly have any practical value.

A summary of the results obtained for convected scalar quantities in the high wavenumber range is shown in figure 1.

2.3 THE VORTICITY SPECTRUM FOR WAVENUMBERS $k \gg k_d$

As the vorticity equation has the same form as the θ -equation (2.2.1) we may tentatively construct an argument to predict the vorticity spectrum in the range $k \gg k_d$ based on the uniform strain approximation given in §2.2 (i). Novikov (1962) and Saffman (1963) have attempted a similar task for three dimensional motions where the vorticity equation has the same structure as the equation for a vector quantity such as a magnetic field.

The main difficulty of such an argument lies in the fact that the vorticity in the high wavenumber region is not a dynamically passive quantity but actually contributes to the local straining field. Thus any change in the small scale variations of the vorticity will affect the local conditions. However if we consider only those parts of the vorticity spectrum which make a negligible total contribution to the mean-square vorticity, say the vorticity in the wavenumber range

$k \gg k_d$, we may hope that the local straining motion is not affected greatly by any variations in this range and that the local strain approximation is then a reasonable basis on which to work.

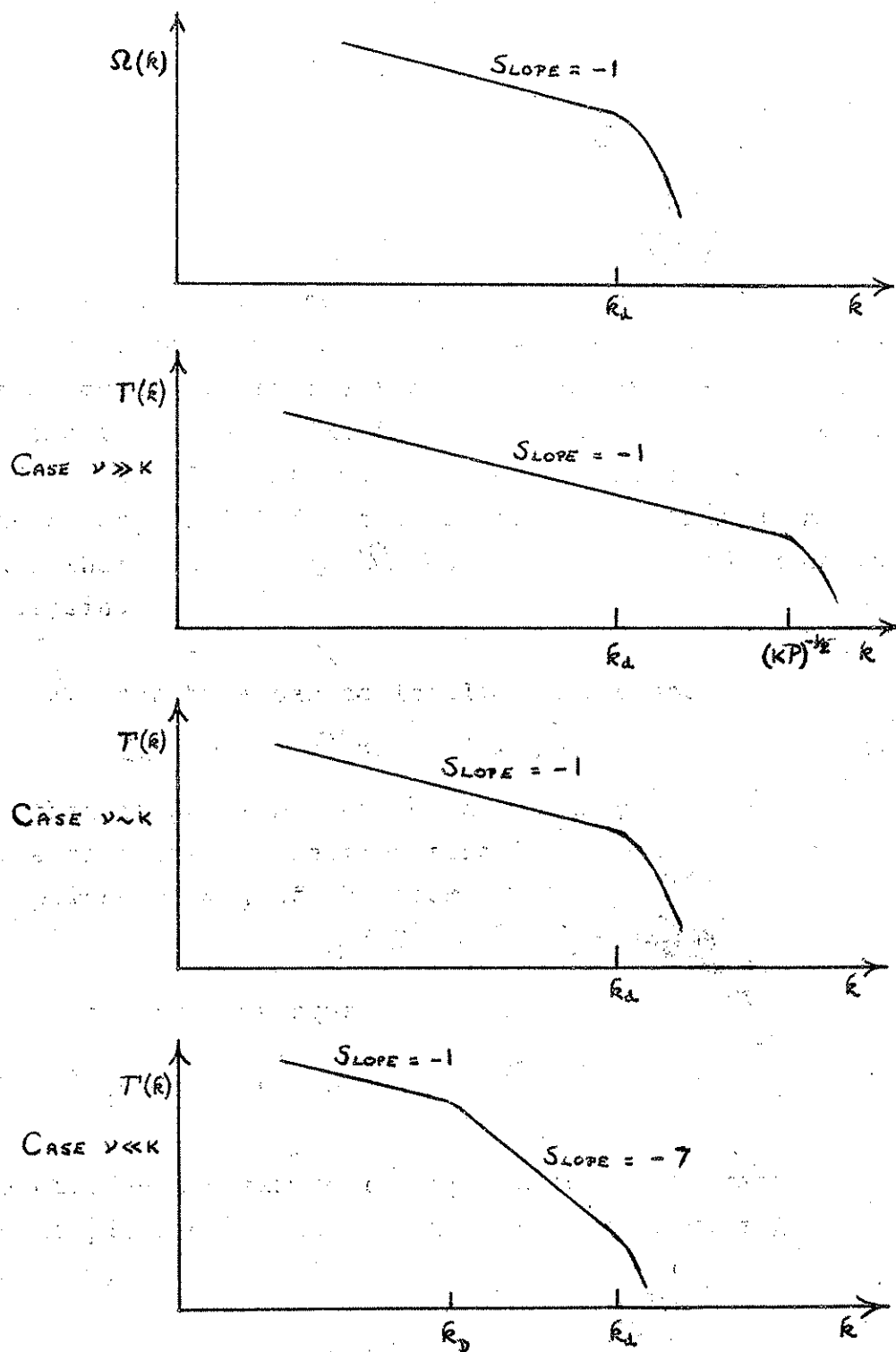


Fig. 1: LOGARITHMIC GRAPHS OF THE SPECTRA OF VORTICITY AND $\overline{\theta^2}$ - STUFF AT LARGE WAVENUMBERS.

The uniform strain analysis reduces the interaction term $\underline{u} \cdot \nabla \omega$ into a linear form for the small scale variations of the vorticity. Now the full linearization of the interaction term is of the form

$$\underline{U} \cdot \nabla \omega + \underline{u} \cdot \nabla \hat{\Omega}$$

where $\underline{u}$ and ω will be the velocity and vorticity of the small-scale variations and $\underline{U}$ and $\hat{\Omega}$ are the local velocity and vorticity fields represented approximately by the first few terms in a Taylor series expansion as a function of position. If then we are to use the theory of §2.2 (i) for the vorticity spectrum we must show that the term $\underline{u} \cdot \nabla \hat{\Omega}$ has negligible effect on the analysis.

$\hat{\Omega}$ can be expanded locally in the form

$$\hat{\Omega} = \Omega^0 + \Omega_{,j}^1 x_j + \Omega_{,jk}^2 x_j x_k + \dots$$

so that if we represent U_i tensorially by $\alpha_{ij} x_j$ we have that the interaction term is, to the first order in powers of x , of the form

$$\alpha_{ij} x_j \frac{\partial \omega}{\partial x_i} + u_i (\Omega_i^1 + \Omega_{ij}^2 x_j + \Omega_{ji}^2 x_j)$$

and the vorticity equation is then

$$\frac{\partial \omega}{\partial t} + 2 \frac{\partial \tilde{\omega}}{\partial t} + \alpha_{ij} x_j \frac{\partial \omega}{\partial x_i} + u_i (\Omega_i^1 + \Omega_{ij}^2 x_j + \Omega_{ji}^2 x_j) = \nu \nabla^2 \omega. \quad (2.3.1)$$

We have here referred the motion to the (rotating) principal axes of the α_{ij} tensor which have angular velocity $\tilde{\omega}$.

We note immediately that we must assume that $\frac{\partial \tilde{\omega}}{\partial t}$ is negligible - but the evidence given in §2.2 (i) suggests that this is a reasonable step to make.

Now if we are to represent the local vorticity variations by the Fourier-Stieltjes transform $dY(\underline{k})$ then the arguments presented in §2.2 (i) show that we must use the form for $dY(\underline{k})$ with finite wavenumber interval $d\underline{k}$ where

$$k > dk_1, dk_2 > k_d.$$

But equation (2.3.1) also involves the local velocity variations, so by the same argument we must select a transform $d\underline{Z}(\underline{k})$ for the velocity with finite wavenumber interval (which we may take to be the same $d\underline{k}$). The relation between $d\underline{Z}$ and dY is then (only approximately)

$$i(k_1 dZ_2 - k_2 dZ_1) = dY$$

or

$$d\underline{Z} = \left(\frac{ik_2}{k^2}, -\frac{ik_1}{k^2}, 0 \right) dY.$$

We now discuss the formalized representation of $dY(\underline{k})$ for the solution of (2.3.1), i.e.

$$\omega = \omega(t) \sin(\underline{k}(t) \cdot \underline{x} + \varepsilon(t)) \quad (2.3.2)$$

$$\text{with } \underline{u} = \left(\frac{k_2}{k^2}, -\frac{k_1}{k^2}, 0 \right) \omega(t) \cos(\underline{k}(t) \cdot \underline{x} + \varepsilon(t)).$$

This is a solution if

$$\frac{d\omega(t)}{dt} + \nu k^2 \omega(t) = 0 \quad (2.3.3)$$

$$\frac{d\varepsilon(t)}{dt} + \frac{1}{k^2} (k_2 \Omega'_1 - k_1 \Omega'_2) = 0 \quad (2.3.4)$$

$$\frac{dk_j}{dt} + d_{ij} k_i + \frac{k_2}{k^2} (\Omega_{ij}^2 + \Omega_{ji}^2) - \frac{k_1}{k^2} (\Omega_{2j}^2 + \Omega_{j2}^2) = 0. \quad (2.3.5)$$

The effect of the Ω'_i term is thus, from (2.3.2) and (2.3.4), to cause the vorticity waveform to travel in physical space with phase velocity

$$\frac{1}{k^2} (k_2 \Omega'_1 - k_1 \Omega'_2)$$

and group velocity of the same order of magnitude.

If this phase velocity ^{with $\underline{k} = \underline{k}_m$} is much smaller than $(k_m P)^{-1}$,

where P is the characteristic time associated with the uniform strain analysis and k_m is the smallest wavenumber for which we wish to apply the analysis, then a waveform will remain under the same local conditions for a sufficiently long period for the uniform strain analysis to be valid and consequently we may then ignore the Ω_i^1 term altogether. We therefore require

$$P |\Omega_i^1| \ll k_m. \quad (2.3.6)$$

The third and fourth terms in equation (2.3.5) will be negligible compared with the second term if

$$|\Omega_{ij}^2| \ll k_m^2 |d_{\alpha\beta}| \quad (2.3.7)$$

for any i, j, k and l ($||$ denotes modulus).

Both inequalities (2.3.6) and (2.3.7) will obviously be satisfied for sufficiently large k_m , so the uniform strain analysis will always be valid for some large enough high wavenumber range. We can in fact be more precise than this.

$|\Omega_i^1|$ and $|\Omega_{ij}^2|$ are the moduli of a gradient of vorticity and of a second gradient of vorticity respectively, they therefore have orders of magnitude which can be determined by Kolmogoroff arguments since both quantities are largely determined by the equilibrium range of wavenumbers. Thus

$$|\Omega_i^1| \sim \left(\frac{\epsilon_\omega}{\nu}\right)^{1/2} \quad \text{and} \quad |\Omega_{ij}^2| \sim \frac{\epsilon_\omega^{2/3}}{\nu}.$$

If, further, we assume that

$$P \sim \epsilon_\omega^{-1/3}$$

(on the basis that P will be a relevant quantity for

the description of the vorticity spectrum in the equilibrium range) and that

$$|d_{ij}| \sim \Omega^{1/2}$$

then the inequalities (2.3.6) and (2.3.7) reduce to

$$k_m^2 \gg \left(\frac{\varepsilon_\omega}{\nu^3}\right)^{1/6} = k_d$$

and

$$k_m^2 \gg \left(\frac{\varepsilon_\omega}{\nu^3}\right)^{1/3} \cdot \left(\frac{\varepsilon_\omega^2}{\Omega^3}\right)^{1/6}.$$

Thus if we take $(\varepsilon_\omega^2/\Omega^3)$ to be of order one we have that the condition for the uniform strain analysis of §2.2 (i) to be valid for the vorticity spectrum is

$$k_m \gg k_d$$

where k_d is the viscous cut-off wavenumber. From (2.2.28) we then have the vorticity spectrum form

$$\Omega(k) \propto \frac{1}{k} \exp\{-\nu P k^2\}$$

in the range $k \gg k_d$. The proportionality and the value of P can be determined by Kolmogoroff arguments so that

$$\Omega(k) = D \varepsilon_\omega^{2/3} k^{-1} \exp\{-C \nu \varepsilon_\omega^{-1/3} k^2\} \quad (2.3.8)$$

where C and D are constants.

In view of the spectral form in the range $k_0 \ll k \ll k_d$ (see equation (1.6.6)) it is tempting to assume that equation (2.3.8) can be taken to be the form of the spectrum function throughout the range $k \gg k_0$. If then we evaluate ε_ω from

$$\varepsilon_\omega = 2\nu \int_0^\infty k^2 \Omega(k) dk$$

by using equation (2.3.8) alone we may show finally that $D = C$.

It is to be noted that the same type of discussion may be constructed for the application of the uniform strain analysis to vorticity in three dimensions. The appropriate basic equation is then

$$\frac{\partial \underline{\omega}}{\partial t} + \underline{U} \cdot \nabla \underline{\omega} - \underline{\omega} \cdot \nabla \underline{U} - \nu \nabla^2 \underline{\omega} = -2 \frac{\partial \underline{\tilde{\omega}}}{\partial t} - \underline{u} \cdot \nabla \underline{\hat{\Omega}} + (\underline{\hat{\Omega}} + 2 \underline{\tilde{\omega}}) \cdot \nabla \underline{u}$$

using an obvious adapted notation. Saffman and Novikov considered only the terms on the left hand side of this equation with $\underline{U}$ as a linear function of position. We shall therefore examine the magnitudes and effects of the additional terms.

As before, the terms involving $\underline{\tilde{\omega}}$, the angular velocity of rotation of the principal axes may be discounted immediately on the basis of Batchelor's (1959) original discussion of the uniform strain approximation.

The expressions $\underline{\omega} \cdot \nabla \underline{U}$, $\underline{u} \cdot \nabla \underline{\hat{\Omega}}$ and $\underline{\hat{\Omega}} \cdot \nabla \underline{u}$ could all include additional terms of the expansions of $\underline{U}$ and $\underline{\hat{\Omega}}$ as Taylor series not considered before but which could result in a factor of at most λ in the forms of the expressions. But all of these possibilities, except one, can be argued to be negligible by reasoning as before, if we restrict the range of validity of the analysis to

$$k \gg k_d = \left(\frac{\varepsilon}{\nu^2} \right)^{1/4}.$$

The exceptional case is the term

$$\underline{\Omega}^{\circ} \cdot \nabla \underline{u},$$

where $\underline{\Omega}^{\circ}$ is the local vorticity. If one includes this term in the analysis then one can still have a solution of the form

$$\underline{\omega} = \underline{\omega}(t) \sin(\underline{k}(t) \cdot \underline{x})$$

and the equation governing the behaviour of $\underline{k}(t)$ is unaffected. But the equation for $\omega(t)$ is

$$\frac{\partial \omega_i(t)}{\partial t} = (\alpha_{ij} + \varepsilon_{ijl} \Omega_k^o \frac{k_l k_j}{k^2}) \omega_j(t) - \nu k^2 \omega_i(t) \quad (2.3.9)$$

where α_{ij} is the local strain (and rotation) tensor.

As the equation for $\underline{k}$ is unaltered we can still assume that $\underline{k}$ always lies near an outward travelling asymptote and under such an assumption the tensor associated with Ω_k^o in (2.3.9) can be taken to be independent of time. The tensor expression in brackets in (2.3.9) then has constant eigenvalues and, since the trace of the tensor is zero, there must be at least one eigenvalue with positive real part. This tensor in fact has the same characteristics as the α_{ij} tensor alone. For this reason, Saffman's and Novikov's analyses give the correct form of result and it is only the definition of their symbols that need correction. In Saffman's paper in particular, α should not be defined as the greatest of the positive real parts of the eigenvalues of the α_{ij} tensor but of the expanded tensor in (2.3.9), and σ , which is defined in terms of α , must also be changed. Saffman's form of the energy spectrum, namely equation (A3) in his paper:

$$E(k) = \frac{\varepsilon}{\nu} \left(\frac{\nu \sigma}{\alpha} \right)^\sigma \frac{k^{2\sigma-3}}{(\sigma-1)!} \exp \left\{ -\nu \frac{\sigma}{\alpha} k^2 \right\}$$

then remains valid.

We note also that the comparison between the theoretical prediction and experiment discussed by Saffman is not altered in any way by these changes since the empirical evaluations of α and σ do not rely on their original definitions but rather on their positions within the final differential equation which leads to the above spectral form.

Lastly, it is interesting to notice that our comments on the three-dimensional analysis clearly show that such an analysis can hold only for wave-numbers $k \gg k_A$. Novikov's attempt to evaluate ϵ by extrapolating the prediction to cover the inertial sub-range can therefore have no justification at all.

2.4 THE THEORY OF CONVECTED VECTOR QUANTITIES

For the sake of completeness we now extend the theory of convected quantities to vector fields. We shall assume that the basic equations for the vector field $\underline{H}$ are

$$\frac{\partial \underline{H}}{\partial t} + \underline{u} \cdot \nabla \underline{H} = \underline{H} \cdot \nabla \underline{u} + \lambda \nabla^2 \underline{H} \quad (2.4.1)$$

$$\nabla \cdot \underline{H} = 0 \quad (2.4.2)$$

and we shall, for simplicity, once again assume that the convected field does not affect the turbulence in any way. Thus $\underline{H}$ could represent the magnetic field in a conducting fluid with the condition that the Lorentz force can be neglected.

If the $\underline{H}$ -field is parallel to the vorticity everywhere then equation (2.4.1) reduces to the form (2.2.1) for a convected scalar quantity and all the results obtained in §2.2 apply.

If the $\underline{H}$ -field is everywhere normal to the vorticity then the term $\underline{H} \cdot \nabla \underline{u}$ in equation (2.4.1) will be non-zero in general and it is possible for a weak $\underline{H}$ -field to be amplified by the turbulence interaction. The analogous three dimensional situation has been discussed fully by Moffatt (1961, 1962) and Saffman (1963)

using methods similar to those already used for the scalar field.

Now the equation (2.4.2) implies in this case the existence of a scalar A such that

$$H_1 = -\frac{\partial A}{\partial y}, \quad H_2 = \frac{\partial A}{\partial x}.$$

Moreover we may integrate equation (2.4.1) to give an equation for the behaviour of A with time, i.e.

$$\frac{\partial A}{\partial t} + \underline{u} \cdot \nabla A = \lambda \nabla^2 A.$$

This is the same equation as (2.2.1) and we may therefore use the theory of §2.2 to predict the $\overline{A^2}$ -spectrum. To obtain the magnetic energy spectrum we use the relation

$$\Gamma_m(k) = k^2 \Gamma_a(k)$$

where Γ_m and Γ_a are the $\overline{H^2}$ - and $\overline{A^2}$ -spectra. Thus if $\nu \ll \lambda$ we have from (2.2.7) and (2.2.8) that

$$\Gamma_m(k) = \frac{4}{3} \lambda \overline{H^2} \varepsilon_\omega^{-1/3} k \quad (2.4.3)$$

for $k_0 \ll k \ll k_m = (\varepsilon_\omega / \lambda^3)^{1/6}$, and from (2.2.33) that

$$\Gamma_m(k) = \frac{4}{3} \lambda^{-2} \overline{H^2} \varepsilon_\omega^{2/3} k^{-5} \quad (2.4.4)$$

for $k_m \ll k \ll k_d$. (We have chosen the constants of proportionality in (2.4.3) and (2.4.4) so that the two predictions agree when they are extrapolated to $k = k_m$ and so that the total energy is $\overline{H^2}$.)

If $\nu \sim \lambda$ or $\nu \gg \lambda$ then from (2.2.32) we have

$$\Gamma_m(k) = 2C\lambda \overline{H^2} \varepsilon_\omega^{-1/3} k \exp\{-\lambda C \varepsilon_\omega^{-1/3} k^2\} \quad (2.4.5)$$

for all $k \gg k_0$. (Again the constant of proportionality has been adjusted to give a total energy of $\overline{H^2}$.)

Obviously such spectra need be maintained by only

a weak applied field H_0 at wavenumbers of order k_0 . We may take as a measure of this 'seed' field

$$\overline{H_0^2} \sim \int_0^{k_0} T_m(k) dk$$

which, from (2.4.3) and (2.4.5), we assume to be of order $k_0 T_m(k_0)$. Thus on substituting back in these two equations we obtain

$$\overline{H^2} \sim R_m \overline{H_0^2} \quad (2.4.6)$$

for all (small) λ , where

$$R_m = \frac{\varepsilon \omega^3}{k_0^2 \lambda}$$

R_m can be interpreted as a magnetic Reynolds number.

The rate of dissipation of magnetic energy, D , is given by

$$D = \int_0^\infty k^2 T_m(k) dk$$

and may be shown to be of order $R_m^2 D_0$, where D_0 is the rate of dissipation of the seed field alone.

The only significant difference between these results and those of the three dimensional theory is that the relations for $\overline{H^2}$ and D are independent of the ratio of ν to λ . This is a consequence of the fact that, for wavenumbers less than the conduction cut-off, the functional form of the spectrum is proportional to k whether the wavenumber lies in the inertial subrange or not.

If H represents a magnetic field we cannot neglect the Lorentz force in all circumstances. For if we let λ tend to zero then equation (2.4.6) indicates that the magnetic energy will increase without limit so that eventually the Lorentz force will become appreciable.

How large λ must be (for given H_0) for the validity of the analysis presented here remains an open question.

There is a third possible configuration of the convected $\underline{H}$ -field which we could consider: that where $\underline{H}$ is a general three dimensional field. Now if $\underline{H}$ does represent a magnetic field then the Lorentz force would introduce three dimensional perturbations in the Navier-Stokes equation and might cause the turbulence to degenerate into a three dimensional motion. For this reason the behaviour of a passive $\underline{H}$ -field in this configuration is only of theoretical importance. However there are certain anomalies with this problem which make it interesting.

First, if we transform the $\underline{u}$ - and $\underline{H}$ -fields then, as equation (2.4.1) is linear in $\underline{H}$ and as there is no dependence of the $\underline{u}$ -field on the z -direction, there will be no interaction between Fourier modes of the $\underline{H}$ -field with different wavenumber components, k_3 , in the z -direction. The only useful method of constructing spectral forms is then to consider the two dimensional (i.e. in k_1 and k_2) spectra of the Fourier modes of the $\underline{H}$ -field for particular values of k_3 .

Secondly the equation for H_3 is again the same as that for a convected scalar quantity (with the three dimensional Laplacian). The spectrum for the H_3 component alone with k_3 suitably small can therefore be very little different from that obtained in §2.2. Moreover as $|k_3|$ is increased we would expect the only effect to be a lowering of the conduction cut-off wavenumber.

The remaining two components of $\underline{H}$ present a more

complex problem however. The two dimensional divergence of $\underline{H}$ is not now zero (although presumably it is approximately zero for the small scale variations since $\frac{\partial H_3}{\partial z}$ cannot be amplified) and we cannot use the quantity A as in the above analysis. But for the range $k \ll k_d$ we could construct an analysis by ignoring the anomalous H_3 behaviour and by considering the vector potential $\underline{A}$ of $\underline{H}$ (as in Moffatt (1962)): in this way we would regain the results (2.4.3) and (2.4.4), for suitably small k_3 , if we redefine $\overline{H^2}$ to be the mean-square of the $\underline{H}$ variations with fixed wavenumber component k_3 in the z -direction.

We may also regain the result (2.4.5) since the uniform strain analysis is not dependent on being able to construct a potential function at all (see Saffman (1963)). Thus if we consider the simplified case of $p=0$, $\alpha \neq 0$ and ignore the conduction term we can obtain a formalized solution for the Fourier-Stieltjes transform of $\underline{H}$ in the form

$$\underline{H} = \underline{H}(t) \sin(\underline{k}(t) \cdot \underline{x})$$

where

$$\frac{dk_1}{dt} + \alpha k_1 = 0, \quad \frac{dk_2}{dt} - \alpha k_2 = 0, \quad \frac{dk_3}{dt} = 0$$

$$\frac{dH_1(t)}{dt} - \alpha H_1(t) = 0, \quad \frac{dH_2(t)}{dt} + \alpha H_2(t) = 0, \quad \frac{dH_3(t)}{dt} = 0$$

and
$$k_1 H_1(t) + k_2 H_2(t) + k_3 H_3(t) = 0$$

(see equations (2.2.12) and (2.2.15) for comparison).

Consequently on the outward travelling asymptote $k_1=0$ the $\underline{H}$ -vector is dominated by H_1 and

$$H_1 \propto e^{\alpha t} \propto k_2$$

and this produces a (two-dimensional) spectral form

$$T_m(k) \propto k \quad (2.4.7)$$

where $k^2 = k_1^2 + k_2^2$. This is the same form as (2.4.5) for small k .

We note that if we had not been able to discount transfer of energy along the inward travelling asymptote $k_2=0$, it would have seemed feasible to have taken the component H_1 as dominating the $\underline{H}$ -vector for this locus and that it would have behaved as

$$H_1 \propto e^{\alpha t} \propto k_1^{-1} \quad (2.4.8)$$

which would have led apparently to a spectral form

$$T_m(k) \propto k^{-3}. \quad (2.4.9)$$

Thus at small k (but with $k > k_d$) it would have seemed that (2.4.8) would be dominant over (2.4.7). This would then have led to an indeterminacy since the constant of proportionality in (2.4.9) would be unknown. However, as we have argued in §2.2 (i), it is not admissible to consider wavenumbers which lead to the prediction (2.4.9) since H_1 cannot behave as (2.4.8) except under favourable, and unlikely, conditions which depend on $\underline{H}$ -variations on a length scale larger than the region of uniformity in physical space which we have assumed for this analysis. (The predictions (2.4.8) and (2.4.9) are those for which Saffman's arguments did not give adequate reasons for ignoring.) Thus (2.4.7) and, in more detailed form, (2.4.5) are valid results for this configuration of the $\underline{H}$ -field.

2.5 THE STABILITY OF TWO DIMENSIONAL TURBULENCE

We showed in §2.4 that a convected vector field which is not aligned parallel to the vorticity of two

dimensional turbulence is amplified by the turbulence interaction, but that when the field is everywhere parallel then there is no such amplification. This suggests that if we have a vector field in the latter configuration and introduce a three dimensional perturbation then this perturbation will probably be amplified and the vector field will degenerate into a three dimensional configuration. Thus the vector field everywhere parallel to the vorticity may be said to be 'unstable'.

Now the equations (2.4.1) and (2.4.2) which we used in §2.4 are of the same form as the equations for a three dimensional vorticity field. Consequently there appears to be the possibility that the two dimensional configuration of the turbulence may prove to be unstable in the sense that a three dimensional perturbation may be amplified and lead to a fully three dimensional motion.

We have already (see §2.3) argued the validity of using the uniform strain approximation for the vorticity field and we therefore intend to use this theory as a model in which to investigate the stability of the small scale vorticity variations. Thus we shall consider a small scale vorticity perturbation field $\underline{\omega}$, with associated velocity field $\underline{u}$, which satisfies the equations

$$\frac{\partial \underline{\omega}}{\partial t} + \underline{U} \cdot \nabla \underline{\omega} = \underline{\omega} \cdot \nabla \underline{U} + \underline{\hat{\Omega}} \cdot \nabla \underline{u} + \nu \nabla^2 \underline{\omega} \quad (2.5.1)$$

$$\nabla \cdot \underline{\omega} = 0 \quad (2.5.2)$$

where $\underline{U}$ is the local velocity field

$$\underline{U} = (\alpha x - \beta y, -\alpha y + \beta x, 0),$$

referred to the principal axes of the strain, and $\hat{\Omega}$ is the local (absolute) vorticity

$$\hat{\Omega} = (0, 0, 2(p + \tilde{\omega}))$$

where $\tilde{\omega}$ is the angular velocity of the principal axes. We intend in this analysis to show that the configuration is unstable and we shall also investigate the possibility of stabilizing the motion by superimposing a strong uniform rotation. The quantity $\tilde{\omega}$ is therefore introduced at this stage to save a repetition of the analysis: when there is no superimposed rotation we shall assume that $\tilde{\omega}$ is zero. α , p and $\tilde{\omega}$ will all be taken to be constant.

As we have already discussed the effect of the Fourier-Stieltjes transform in this type of analysis we shall use the simpler formalized representation of this transform in terms of the amplitude of one sinusoidal wave. Thus equations (2.5.1) and (2.5.2) have a solution of the form

$$\underline{w} = \underline{w}'(t) \sin(\underline{k}(t) \cdot \underline{x})$$

which has an associated velocity field

$$\underline{u} = \frac{\underline{k} \wedge \underline{w}'(t)}{k^2} \cos(\underline{k}(t) \cdot \underline{x})$$

if

$$\frac{dk_1}{dt} + \alpha k_1 + p k_2 = 0, \quad \frac{dk_2}{dt} - p k_1 - \alpha k_2 = 0, \quad \frac{dk_3}{dt} = 0 \quad (2.5.3)$$

$$\frac{d\omega'_i}{dt} = (\alpha \omega'_1 - p \omega'_2, p \omega'_1 - \alpha \omega'_2, 0)_i + \frac{2(\tilde{\omega} + p)}{k^2} k_3 (\underline{w}' \wedge \underline{k})_i - \nu k^2 \omega'_i \quad (2.5.4)$$

$$\omega'_1 k_1 + \omega'_2 k_2 + \omega'_3 k_3 = 0. \quad (2.5.5)$$

It will be noticed that the equations for k_1 and k_2 in (2.5.3) are the same as those in §2.2 (i) so that we already know the behaviour of the wavenumber

loci. Moreover in the case $\alpha^2 > \rho^2$ we also know that the only relevant transfer of $\bar{\omega}^2$ takes place along the outward travelling loci of the hyperbolic asymptotes.

Now if we substitute $\omega_1 = \omega_2 = k_3 = 0$ in the above equations we obtain the equation for the vorticity in two dimensional turbulence and this has a solution

$$\omega_3' \propto \exp \left\{ -\nu \int^t (k_1^2 + k_2^2) dt \right\}. \quad (2.5.6)$$

Thus if this configuration is unstable we require that the three dimensional perturbation must grow relative to (2.5.6), i.e. $(\omega_1'^2 + \omega_2'^2 + \omega_3'^2)^{1/2}$ as predicted by equations (2.5.4) must decrease at a less rapid rate than ω_3' as given in (2.5.6). (As in §2.4 we are working in terms of two-dimensional spectra, even for the three dimensional vorticity field, so that we are comparing the spectrum for a particular value of k_3 with the above spectrum for zero k_3 .)

We may eliminate the viscous terms in (2.5.4) by substituting

$$\underline{\omega}' = \underline{\omega}^+ \exp \left\{ -\int^t \nu k^2 dt \right\}.$$

The equations for $\underline{\omega}^+$ are then the same as (2.5.4) but with the viscous terms removed. This set of equations does not give a closed form of solution in general so we shall now discuss only particular approximate solutions.

If $k_3^2 \gg k_1^2 + k_2^2$ then, on using equation (2.5.5) we may show that the equations for $\underline{\omega}^+$ become

$$\begin{aligned} \left(\frac{d}{dt} - \alpha - 2(\bar{\omega} + \rho) \frac{k_1 k_2}{k_3^2} \right) \omega_1^+ &= (2\bar{\omega} + \rho) \omega_2^+ \\ \left(\frac{d}{dt} + \alpha + 2(\bar{\omega} + \rho) \frac{k_1 k_2}{k_3^2} \right) \omega_2^+ &= -(2\bar{\omega} + \rho) \omega_1^+. \end{aligned}$$

From these we may eliminate either ω_1^+ or ω_2^+ and then neglect the terms containing factors $k_1 k_2 / k_3$ to obtain

$$\left(\frac{d^2}{dt^2} - \alpha^2 + (2\tilde{\omega} + \beta)^2 \right) (\omega_1^+, \omega_2^+) = 0.$$

When $\tilde{\omega} = 0$ and $\alpha^2 < \beta^2$ the solutions for ω_1^+ and ω_2^+ oscillate with constant amplitude so that, as the viscous damping factor would be much larger in this case than in (2.5.6), the configuration would be stable. But when $\tilde{\omega} = 0$ and $\alpha^2 > \beta^2$ then asymptotically

$$\omega_1, \omega_2 \propto \exp \left\{ \left(\sqrt{\alpha^2 - \beta^2} - \nu k_3^2 \right) t \right\} \quad (2.5.7)$$

and it may be possible for this to grow relative to (2.5.6). However the approximation $k_3^2 \gg k_1^2 + k_2^2$ would eventually break down in this case so that there is no conclusive evidence of instability. Moreover if a rotation $\tilde{\omega}$ is superimposed so that

$$(2\tilde{\omega} + \beta)^2 > \alpha^2,$$

ω_1^+ and ω_2^+ would necessarily be oscillatory so that the configuration can be stabilized. As both α^2 and β^2 have the value $\frac{1}{2}\Omega$ we could thus ensure stability for this case if $\tilde{\omega}^2 \gg \Omega$.

It is interesting to note that in the case $\alpha = 0$, $\tilde{\omega} + \beta \neq 0$ (and irrespective of the relative magnitudes of k_1 , k_2 and k_3 , provided $k_3 \neq 0$) the solution for ω^+ is the inertial wave solution: the configuration would then necessarily be stable. We shall take this to be an indication of the general behaviour of ω^+ whenever $\alpha^2 < \beta^2$. (We have not been able to obtain a solution for ω^+ when $\alpha^2 < \beta^2$, except when $\alpha = 0$ or when $k_3^2 \gg k_1^2 + k_2^2$, which is valid for a complete circuit of the elliptical loci but one would presume that

such a solution would necessarily be oscillatory—
although, of course, of unknown amplitude.)

When $k_3^2 \ll k_1^2 + k_2^2$, $k_3 \neq 0$ and $\alpha^2 > p^2$ we can obtain a solution by transforming the reference axes so that one of them, k_1' say, coincides with the outward travelling asymptotes. If in this transformation k and ω^+ go into k_1' and ω^* respectively then the equations for these variables are

$$\frac{dk_1'}{dk} - \sqrt{(\alpha^2 - p^2)} k_1' = 0, \quad k_2' = 0 \quad (2.5.8)$$

$$\left. \begin{aligned} \frac{d\omega_1^*}{dk} &= -\sqrt{(\alpha^2 - p^2)} \omega_1^* + 2(\tilde{\omega} + p) \left(\frac{k_3}{k_1'}\right)^2 \omega_2^* \\ \frac{d\omega_2^*}{dk} &= -2\tilde{\omega} \omega_1^* + \sqrt{(\alpha^2 - p^2)} \omega_2^* \\ \frac{d\omega_3}{dk} &= -2(\tilde{\omega} + p) \left(\frac{k_3}{k_1'}\right) \omega_2^* \end{aligned} \right\} \quad (2.5.9)$$

where we have confined our attention to wavenumbers lying on the asymptote itself. Thus we may eliminate ω_1^* from equations (2.5.9) and obtain

$$\left(\frac{d^2}{dk^2} - (\alpha^2 - p^2) + 4\tilde{\omega}(\tilde{\omega} + p) \left(\frac{k_3}{k_1'}\right)^2 \right) \omega_2^* = 0. \quad (2.5.10)$$

Equation (2.5.10) with (2.5.8) is in general of Bessel type and the solutions may be shown to be

$$\omega_2^* = P_1 J_1(z) + Q_1 Y_1(z) \quad \text{when } \tilde{\omega}(\tilde{\omega} + p) > 0$$

$$\omega_2^* = P_2 \exp\{\sqrt{(\alpha^2 - p^2)}z\} + Q_2 \exp\{-\sqrt{(\alpha^2 - p^2)}z\} \quad \text{when } \tilde{\omega}(\tilde{\omega} + p) = 0$$

$$\omega_2^* = P_3 I_1(z) + Q_3 K_1(z) \quad \text{when } \tilde{\omega}(\tilde{\omega} + p) < 0$$

where the P 's and Q 's are constants; J_1 , Y_1 and I_1 , K_1 are the Bessel and modified Bessel functions respectively; and

$$z = \left| \frac{16\tilde{\omega}(\tilde{\omega} + p)}{(\alpha^2 - p^2)} \left(\frac{k_3}{k_1'}\right)^2 \right|^{1/2}$$

Now when $\tilde{\omega} = 0$ we have that the configuration must be unstable since ω_2^* increases exponentially with time and also since the viscous damping term is comparable to that in (2.5.6). Thus the small scale variations of two dimensional turbulence are unstable to three dimensional perturbations.

When we superimpose an angular velocity $\tilde{\omega}$ then, as $k_1'^2$ tends to infinity, z tends to zero and

$$Y_1(z) \sim z^{-1}, \quad K_1(z) \sim z^{-1}$$

i.e. it is not possible to stabilize the configuration since, for example, when $\tilde{\omega}(\tilde{\omega} + \rho) > 0$

$$\omega_2^{*2} \sim \frac{\alpha^2 - \rho^2}{16\tilde{\omega}(\tilde{\omega} + \rho)} \left(\frac{k_1'}{k_3} \right)^2 \quad (2.5.11)$$

as $k_1'^2$ tends to infinity and the vorticity is clearly going to grow relative to (2.5.6). The only effect which an increase in the value of $\tilde{\omega}$ would have is to decrease the rate of amplification of the instability.

If $k_3 = 0$ and $\alpha^2 > \rho^2$ then the solution will be given by (2.5.9) if we ignore the terms associated with $\tilde{\omega}$. Thus there will always be an exponentially increasing component of vorticity ω and the exponential factor will be independent of the rotation $\tilde{\omega}$ for this case.

Of course our analysis is strictly limited to wavenumbers $k \gg k_A$ so that, if we restricted the perturbations to be small at $k \sim k_A$ relative to the two dimensional field, then the three dimensionality would only dominate the spectrum at wavenumbers well beyond the viscous cut-off — and for $k_3 \neq 0$ the wavenumber marking the beginning of the three dimensionality range

would be associated with a certain magnitude achieved by ω_z^* in (2.5.11). However if an instability mechanism is predicted for wavenumbers covered by the uniform strain approximation then it is suggestive that wavenumbers smaller than this may also be unstable so that appreciable perturbations may actually occur at $k \sim k_d$. Thus if we define the parameter

$$N^2 = \frac{\bar{\omega}^2}{\bar{\Omega}} \left(\frac{k_z}{k} \right)^2,$$

where we have assumed that $(\alpha^2 - \beta^2)$ is of order $\bar{\Omega}$, we postulate that a condition for three dimensional perturbations to be negligible at a particular wavenumber $\underline{k}$ is that N^2 should be sufficiently large.

Consequently if the turbulence were bounded between two parallel planes a distance L apart, we could take as an estimate of the smallest value of k_z the value L^{-1} and three dimensional perturbations would be small so long as

$$N^2 = \frac{\bar{\omega}^2}{\bar{\Omega}} (kL)^{-2}$$

is sufficiently large.

An alternative way of trying to stabilize the two dimensional configuration is to impose a strong uniform magnetic field $\underline{H}$ parallel to the vorticity. The vorticity and magnetic field equations are then coupled so that we must consider perturbation fields $\underline{\omega}$ and $\underline{k}$ of both. Thus

$$\frac{\partial \omega_i}{\partial t} + \underline{U} \cdot \nabla \omega_i - \underline{\omega} \cdot \nabla U_i - \nu \nabla^2 \omega_i = \bar{\Omega} \cdot \nabla \omega_i + \frac{\mu}{4\pi c} \underline{H} \cdot \nabla (\nabla \cdot \underline{k})_i$$

$$\frac{\partial k_i}{\partial t} + \underline{U} \cdot \nabla k_i - \underline{k} \cdot \nabla U_i - \lambda \nabla^2 k_i = \underline{H} \cdot \nabla \omega_i$$

where μ is the magnetic permeability and $\underline{H} = (0, 0, H)$. We shall also take the simplified case of $\lambda = \nu$. Then these equations have a (formal) solution

$$\underline{\omega} = \underline{\omega}(t) \exp \left\{ -\int^t \gamma \underline{k}^2 dt \right\} \sin \left(\underline{k}(t) \cdot \underline{x} \right) \quad (2.5.12)$$

$$\underline{h} = \sqrt{\frac{4\pi e}{\mu}} \underline{k}_3^{-1} \underline{h}(t) \exp \left\{ -\int^t \gamma \underline{k}^2 dt \right\} \sin \left(\underline{k}(t) \cdot \underline{x} \right)$$

with $\mathcal{H} = \sqrt{\frac{\mu}{4\pi e}} H$

if the equations for $\underline{k}$ are (2.5.3) and

$$\frac{d\omega_i}{dt} = (\alpha \omega_1 - p \omega_2, p \omega_1 - \alpha \omega_2, 0)_i + \frac{2p \underline{k}_3}{\underline{k}^2} (\underline{\omega} \wedge \underline{k})_i + \mathcal{H} (\underline{k} \wedge \underline{k})_i$$

$$\frac{dh_i}{dt} = (\alpha h_1 - p h_2, p h_1 - \alpha h_2, 0)_i + \mathcal{H} \left(\frac{\underline{k}_3}{\underline{k}} \right)^2 (\underline{\omega} \wedge \underline{k})_i$$

(we have now assumed that $\tilde{\omega} = 0$ and that the principal axes are stationary).

As in the analysis of the effect of a strong rotation the most critical configuration occurs when $\underline{k}_3^2 \ll \underline{k}_1^2 + \underline{k}_2^2$, $\underline{k}_3 \neq 0$, and $\alpha^2 > p^2$. We shall therefore deal only with this case here. Again we transform axes so that $\underline{k}$, $\underline{\omega}$ and $\underline{h}$ go into $\underline{k}'$, $\underline{\omega}'$ and $\underline{h}'$ where $\underline{k}'_2 = 0$ is the outward travelling asymptote. The relevant equations when $\underline{k}'_2 = 0$ are then

$$\frac{d\underline{k}'_1}{dt} - \sqrt{(\alpha^2 - p^2)} \underline{k}'_1 = 0 \quad (2.5.13)$$

$$\left(\frac{d}{dt} + \sqrt{(\alpha^2 - p^2)} \right) \omega'_1 = (\mathcal{H} \underline{k}_3) \underline{k}'_2 + 2p \left(\frac{\underline{k}_3}{\underline{k}'_1} \right)^2 \omega'_2 \quad (2.5.14)$$

$$\left(\frac{d}{dt} - \sqrt{(\alpha^2 - p^2)} \right) \omega'_2 = -(\mathcal{H} \underline{k}_3) \left(\frac{\underline{k}'_1}{\underline{k}_3} \right)^2 \underline{h}'_1 \quad (2.5.15)$$

$$\left(\frac{d}{dt} + \sqrt{(\alpha^2 - p^2)} \right) \underline{h}'_1 = (\mathcal{H} \underline{k}_3) \left(\frac{\underline{k}_3}{\underline{k}'_1} \right)^2 \omega'_2 \quad (2.5.16)$$

$$\left(\frac{d}{dt} - \sqrt{(\alpha^2 - p^2)} \right) \underline{h}'_2 = 2p \underline{h}'_1 - (\mathcal{H} \underline{k}_3) \omega'_1. \quad (2.5.17)$$

Equations (2.5.15) and (2.5.16) with (2.5.13) can be shown to have solutions

$$\omega'_2 \propto \exp \left\{ \left(\sqrt{(\alpha^2 - p^2)} \pm \mathcal{H} \underline{k}_3 i \right) t \right\} \quad (2.5.18)$$

$$\underline{h}'_1 \propto \exp \left\{ \left(-\sqrt{(\alpha^2 - p^2)} \pm \mathcal{H} \underline{k}_3 i \right) t \right\}$$

and on substituting these solutions into (2.5.14) and (2.5.17) we may obtain the solutions of ω'_1 and $\underline{h}'_2$ as

$$\omega'_1, \underline{h}'_2 \propto \exp \left\{ \pm \sqrt{(\alpha^2 - p^2 - \mathcal{H}^2 \underline{k}_3^2)} t \right\} + f_{\omega'_1, \underline{h}'_2}(t)$$

where f_{ω_i}, f_{k_i} are particular integrals of the differential equations and are exponentially decreasing functions of time.

We can obviously 'stabilize' the ω_i and k_i components by requiring

$$H^2 k_3^2 > \alpha^2 - p^2$$

and the k_i component is always stable. But the vorticity component ω_i always contains an exponentially increasing function of time whatever the value of H , or H . However we cannot immediately conclude from this that the two dimensional configuration cannot be stabilized by a strong magnetic field, for this dominant vorticity component is, from (2.5.12) and (2.5.18),

$$\omega_2 \propto \exp \left\{ \sqrt{(\alpha^2 - p^2)} t - \int^t \nu k^2 dt \right\} \sin (k_1 x + k_3 z \pm H k_3 t),$$

i.e. it is a travelling wave. The phase velocity associated with this wave is $\pm \sqrt{\frac{\mu}{4\pi\rho}} H \frac{k_3}{k}$ and the group velocity is $\pm \sqrt{\frac{\mu}{4\pi\rho}} H$. There are now two possibilities when H is large:

(i) the phase velocity may be large enough to render this analysis invalid since the wave would not stay long enough under the same local conditions for a prediction to be feasible. If k_d^{-1} is the length scale of the region for which uniformity can be assumed then, as the characteristic time scale of the uniform strain analysis is P , this analysis is invalid if

$$\sqrt{\frac{\mu}{4\pi\rho}} H \frac{k_3}{k} \gg (k_d P)^{-1}.$$

When this condition holds then the small scale variations would be propagated rapidly through fluctuating local conditions. It may then be possible to have stability since one would expect ω and k to fluctuate as well, and they may do so without the amplitude of ω growing

in time.

(ii) The fluid may be bounded by two parallel boundaries, a distance L apart, where it may be possible to dissipate the energy associated with the propagation of this wave. Stability would then result if the amplification rate was small relative to the rate at which the energy of the disturbance is propagated to (and dissipated at) the boundaries. Thus if we take the amplification rate as P^{-1} stability would result, whatever the wavenumber of the disturbance, if

$$\sqrt{\frac{\mu}{4\pi c}} H \gg L P^{-1}.$$

CHAPTER 3

A NUMERICAL CALCULATION FOR THE STUDY OF TURBULENCE

3.1 INTRODUCTION

In this chapter we shall describe a numerical calculation which we believe is the basis of a reasonable model for a high Reynolds number homogeneous turbulent flow.

The use of numerical computations in the study of unsteady viscous flow problems is a relatively modern development in fluid mechanics and only in a few recent examples has the method achieved any degree of success. Up to now the main difficulty in setting up a successful computational model of this sort has lain in the fact that it has been necessary to have an accurate description of the flow for all length scales right down to those where viscous forces are dominant. Consequently the higher the Reynolds number of the flow one wishes to investigate so a more detailed description of the motion is required. The limitation of the Reynolds number one can use is then determined directly by the size of the computer available. This is a fundamental difficulty in constructing a successful model for high Reynolds number turbulence and in overcoming it we believe that the model presented here is potentially the most promising mode of attack.

Before giving a detailed description of the method we used we shall briefly survey the numerical methods which have been used in the past in the study of unsteady viscous flows and try to evaluate their possible contribution to the study of turbulence. All the methods discussed have in fact been listed already by Batchelor (1962).

(i) Expansions in powers of t .

Taylor and Green (1937) have described a calculation in which they expanded the velocity as a Taylor series in time. Their initial flow was a simple sinusoidal form so that the method consisted of expanding the amplitudes of the fundamental and harmonic modes as power series and evaluating the coefficients of the series from the time derivatives of the velocity field at the initial instant. Although the original calculation was performed by hand the method is adaptable to automatic computation. (We have, in fact, done this, and so also has Jain (1960) - independently-but we were unable to develop the calculation much further than the original work.)

One advantage of this method is that the spatial average of any quantity can be obtained as a function of the modal amplitudes and its behaviour in time may then be readily investigated. But there are two serious disadvantages: first, the analysis will be of use only in the initial development of a simple flow whilst the series converges rapidly; and secondly the amount of data involved in the calculation increases enormously as the number of terms in the power series is increased. This second disadvantage may be amenable to numerical approximation but the first is characteristic of this method and renders it ineffective for a general study of turbulence.

(ii) Expansion in powers of the Reynolds number.

Goldstein (1940) adapted Taylor and Green's technique to produce an expansion of the velocity field as a power series in R . The first approximate solution for this method was obtained from the final period

equation

$$\frac{\partial u}{\partial t} = \frac{1}{R} \nabla^2 u \quad (3.1.1)$$

and successive approximations were obtained by comparing coefficients of powers of R . Goldstein, also working without an automatic computer, considered only one simple sinusoidal solution of (3.1.1).

The criticisms in (i) generally apply to the present context as well. In particular the convergence of the power series in R is probably too slow for this method to be useful except in the decay of turbulence in or near the final period. It does not therefore seem to be adaptable enough for our purposes.

As Batchelor (1962) points out, if some method could be devised to compute a solution based on a power series in R^{-1} then some useful information could be obtained even if there is a finite radius of convergence. However the main feature of such an expansion is that the first order solution would be involved with the full non-linearity of the equations of motion and this is at the heart of the turbulence problem.

(iii) Methods based on hypotheses or models of turbulence.

It has been a basic postulate of this dissertation that the development of a turbulent motion can be predicted by the Navier-Stokes equation. This has not proved in the past to be amenable to solution in itself and various authors have either added an extra hypothesis to the system or replaced this whole mode of attack by constructing new models for the non-linear mechanics of a fluid motion.

As yet no new non-linear models have been developed numerically for the study of an incompressible unsteady flow but an illustration of this kind of approach can be seen in the so-called 'particle-in-cell' method developed by Harlow (1964) and others for compressible flows. There is little information on which one can base a judgement on the utility of this approach for turbulence but it certainly remains a possible numerical method.

The addition of an extra hypothesis has so far been the basis of one set of calculations, being the work of Ogura (1962, 1963). This author uses the hypothesis of zero fourth order cumulants of the velocity at two different points in space. This hypothesis is based on some experimental observations and was known to be only approximately true. Ogura confirms this by showing that the resulting equations of motion can, with certain initial conditions, predict a range of negative values for the energy spectrum - by definition a positive definite function. So in effect this work has merely shown an inconsistency within the hypothesis and has not been able to make any definite predictions of the behaviour of turbulence.

The limitations of this approach in computational work are inherently bound up with the accuracy of the additional hypothesis which is postulated. Consequently, as there is as yet no hypothesis for turbulence which is satisfactory for a large range of length scales, it is difficult to envisage any success on this basis at the moment.

- (iv) Direct numerical integration of the governing equations.

The nature of the Navier-Stokes equations is such that if sufficient information is given of the description of an initial flow then the development of the flow can be predicted forward in time. The equations are of the first order in the time differential so that an integration forward in time of the velocity field is in principle a simple matter if the pressure field can be evaluated at every instant.

Emmons (1947) was the first to attempt such a calculation (by hand). In his case the flow was two dimensional and bounded between parallel plates. He prescribed the initial velocity of the fluid at the points of intersection of a square network in space and then evaluated the velocity at these points at later instants from a finite difference form of the Navier-Stokes equation. The technique is now widely used in many unsteady flow problems, e.g. in thermal convection (see the next chapter) and in numerical weather prediction, and a detailed account of the method is given by Fromm (1964).

This method does offer a number of advantages: the integration forward in time can be made as accurate as one wishes by reducing the time step involved in the computation; the calculation can in principle be developed for an unlimited time; spatial averages can be evaluated by a simple numerical routine; and finally the initial description of the flow can, with ingenuity, be made quite complicated (although one would need to proceed cautiously in order to conserve the continuity condition). However the method does suffer from the fundamental difficulty discussed in the introduction to this section that a computation of

this type will only be successful for a limited range of Reynolds numbers. If the Reynolds number is too high then the turbulence will generate velocity variations on a length scale of the same order as the mesh size of the network and the velocities at neighbouring points will then become uncorrelated and will violate the finite difference form of the continuity condition. Thus until some method is found to impose a suitable constraint on the variations on this length scale, other than by viscosity, it is doubtful if this approach can be fruitful in the study of turbulence.

Lastly this method also suffers from the defect that boundary conditions are involved. If rigid, or free, boundary conditions are stipulated then effects due to inhomogeneities may play an important part in the development of the flow. On the other hand if periodic boundary conditions are given, then the calculation of the pressure term appears to be a dominant feature in the length of the computation (see Fromm) and the computation time is increased by a large factor.

3.2 DIRECT NUMERICAL INTEGRATION OF THE FOURIER TRANSFORMED EQUATIONS

It is apparent from the above discussion that method (iv), the direct integration of the governing equations, seems to be the most promising approach to adopt for a numerical calculation provided some way can be devised to overcome the difficulty inherent in the discreteness of the representation of the velocity field in physical space. But if we represent the

velocity field by a finite set of discrete Fourier modes, i.e.

$$\underline{u}(\underline{x}) = \sum_{\underline{k}} \underline{A}(\underline{k}) e^{i\underline{k} \cdot \underline{x}}$$

and use the Fourier transformed form of the Navier-Stokes equations (see Batchelor (1953, p. 82) for its derivation and form in three dimensional turbulence) then we do in fact avoid this difficulty.

The representation of a velocity field by the same Fourier modes throughout the development of a motion will be that of an actual fluid motion provided the velocity field is real, i.e.

$$\underline{A}(-\underline{k}) = \underline{A}^*(\underline{k}), \quad (3.2.1)$$

where the asterisk denotes conjugacy, and provided the continuity condition

$$\underline{k} \cdot \underline{A}(\underline{k}) = 0 \quad (3.2.2)$$

is satisfied. But the Fourier transformed Navier-Stokes equations are such that the values of the time derivatives of the $\underline{A}$'s preserve both these conditions. Thus a computational model on this basis is free of the difficulty associated with the finite difference method.

The model is also well adapted to the description of a homogeneous motion since the representation is naturally that of a periodic flow. This periodicity may of course lead to 'free' boundary effects or to a restriction, not present in aperiodic flows, on the behaviour of the large scale eddies. But we may minimize both these effects by first ensuring arbitrary phase factors in the complex Fourier modes $\underline{A}(\underline{k})$ and secondly by having only a small amount of energy associated with the modes of smallest wavenumber.

By representing the fluid motion at every instant by the same Fourier modes we shall naturally not be following the actual development of a fluid motion, for such a development would always involve harmonic Fourier modes which do not occur in the description. It will therefore be necessary to discover how the development of the flow is altered by this approximation.

We shall now discuss in detail the method of calculation which we have used to model two dimensional turbulence. The method can in principle be extended to three dimensional turbulence but the duration, size and accuracy required by such a calculation are probably still beyond the reach of the modern computer.

(i) The governing equation and its integration.

The vorticity equation in terms of the stream function $\psi(x)$, where

$$u = \left(-\frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial x} \right),$$

is

$$\frac{\partial \nabla^2 \psi}{\partial t} + \psi_x \nabla^2 \psi_y - \psi_y \nabla^2 \psi_x = \nu \nabla^4 \psi. \quad (3.2.3)$$

Thus if $\phi(k)$ is defined as the Fourier transform of $\psi(x)$, i.e.

$$\psi(x) = \sum_k \phi(k) e^{i k \cdot x}, \quad (3.2.4)$$

where $k = (k_1, k_2)$, then equation (3.2.3) becomes

$$\frac{\partial \phi(k)}{\partial t} = \sum_{k' + k'' = k} [k'_1 k''_2 - k'_2 k''_1] \frac{k''^2}{k^2} \phi(k') \phi(k'') - \nu k^2 \phi(k)$$

or, writing the non-linear term in a more symmetric form,

$$\frac{\partial \phi(k)}{\partial t} = \frac{1}{2} \sum_{k' + k'' = k} \frac{k''^2 - k'^2}{k^2} (k' \wedge k'')_3 \phi(k') \phi(k'') - \nu k^2 \phi(k) \quad (3.2.5)$$

where

$$(k' \wedge k'')_3 = k'_1 k''_2 - k'_2 k''_1.$$

From (3.2.4) we have that

$$\phi(-\underline{k}) = \phi^*(\underline{k})$$

and this is the only restriction imposed on the ϕ 's since the use of the stream function guarantees the continuity condition.

In the actual calculation we chose the length scale so that the vector $\underline{k}$ had integer components. We then considered only modes having wavenumbers in the range

$$-n < k_1, k_2 < n$$

where n is a positive integer, in order to facilitate the counting procedures in the computer programme. However we made the further restriction that only the modes where

$$k < n \tag{3.2.6}$$

were allowed to be non-zero. The slight directional preferences imposed by the square network of values of $\underline{k}$ were then not a feature of the cut-off boundary in wavenumber space and the geometry of the system could be taken to be approximately axisymmetric.

No particular time scale was chosen for the calculations, but in the description of the results the time values will usually be cross-referenced to the initial value of $\Omega^{-1/2}$.

The behaviour in time of the set of modes satisfying (3.2.6) was calculated from (3.2.5) but all other modes were kept at zero amplitude throughout the computation. The numerical procedure used in the integration of (3.2.5), which is now essentially a

set of simultaneous first order differential equations, was that provided in the library of subroutines for the EDSAC 2 machine, the computer used for these calculations. This procedure is one developed by Gill (1951) from the Runge-Kutta process and consists of integrating forward the $\phi(k)$ in time from the values of $\phi(k)$ at each instant, the accuracy being equivalent to expanding the $\phi(k)$ as a Taylor series in time up to, and including, the fourth power of t . In this way the values of $\phi(k)$ as predicted by (3.2.5) were evaluated at time intervals of δt .

(ii) The numerical errors involved in the method.

Now there are two inaccuracies in this method: one is the error involved in the truncation of the Taylor series in the integration process and the other results from the evaluation of the right hand side of (3.2.5) by a machine with a limited accuracy in the storage and arithmetic of numbers. (The EDSAC 2 machine worked at an accuracy of 1 in 10^9 .)

The first error was made negligible by reducing the value of δt until the calculation produced a set of values for the development of the $\phi(k)$ which were not changed (to the order of 1 in 10^9) by yet further reduction of δt . This error could then be said to be less than or equal to that resulting from the limited accuracy of the machine.

However the second source of errors is inherent in the calculation model as set up and if the errors are not negligible then it would be necessary to increase the accuracy of the machine before continuing with the calculation. The non-linear term in equation

(3.2.5) consists of a large number of summations of quantities which may possibly be all of the same order of magnitude. But if just two numbers x and y , both having relative error ϵ say, are added together then the maximum relative error in the summation is

$$\frac{|x| + |y|}{x + y} \epsilon$$

and, if $x \neq -y$, this can be much larger than ϵ . The non-linear term must therefore be investigated to ensure that these 'cancellation errors' are not significant in the evaluation of the right hand side of (3.2.5) for any value of k .

The strongest evidence that we have that these errors were negligible for the computations we carried out is in the observation that the predicted behaviour in time of quantities like the energy and vorticity spectra was highly reproducible. Thus the calculated development of two initial flows which differed from each other by slight alterations in all the Fourier amplitudes showed only slight differences even over a period of time of the order of $5\Omega_0^{-1/2}$ (where Ω_0 is the initial value of Ω in the computation). But if the predicted development was due largely to cancellation errors one would not expect the behaviour of the spectra to be reproducible at all, since errors of this nature would be to some extent random.

Secondly the convergence of the numerical integration process is again evidence of the reproducibility of this model under slightly differing conditions. We therefore believe that all numerical errors could be ignored.

In fact we can construct estimates of the cancellation errors involved in the evaluation of the non-linear term for particular values of the wavenumber $\underline{k}$. Thus, for instance, when $\underline{k}$ takes its smallest values we can show that the errors are negligible, irrespective of the value of the cut-off wavenumber n . The non-linear term in (3.2.5) is bounded above for these $\underline{k}$ in order of magnitude by

$$\sum_{\text{all } \underline{k}'} \underline{k}'^2 |\phi(\underline{k}')|^2 \quad (3.2.7)$$

since $|\underline{k}''^2 - \underline{k}'^2| \doteq 2\underline{k} \cdot \underline{k}'$ for $\underline{k}' \gg \underline{k}$ and, we assume,

$$\phi(\underline{k}'') \sim \phi(\underline{k}')$$

when $\underline{k}'' \doteq \underline{k}'$. Thus the non-linear term is at most of order $2E$ (E the energy) whatever the value of n . Consequently if the error in ϕ is $\varepsilon\phi$ then the order of magnitude of the cancellation errors for the smallest wavenumber modes is at most $4\varepsilon E$. In the actual calculations E was of order 10 and ε of order 10^{-9} so the estimate of the cancellation errors is of the order of $4 \cdot 10^{-8}$. But fluctuations of the ϕ 's were always a feature of the development of the flows and, from occasional inspection of the behaviour of individual modes, it was observed that $\frac{\partial \phi}{\partial t}$ was usually at least of the order of 10^{-2} for modes in the first two wavebands (see below for the definition of the waveband). Thus the cancellation errors were negligible for the smallest wavenumbers.

(It is also interesting to note that the non-linear term for small $\underline{k}$ is determined mainly by interactions of the small wavenumber modes: (3.2.7) implies that the maximum contribution to the non-linear term from interactions of modes with wavenumbers of order $\underline{k}'$ is $E(\underline{k}')$. But from the first chapter we know

that

$$E(k') \sim k'^{-3}$$

in the inertial range, so the contribution of the large wavenumber modes to the non-linear term for small wavenumbers is uniformly convergent whatever the value of the viscosity. This conclusion is also confirmed by the results described below.)

The only estimate we have of the cancellation errors for wavenumbers near to $k=n$ is dependent on the value of n and indicates that, with the maximum value of 12 for n which we used, the calculations were working almost to the limit of the accuracy of the computer. Thus with $k=n$ the non-linear term in (3.2.5) is at most of the order

$$\sum_{k' < k''} k k' |\phi(k') \phi(k'')| \leq \sum_{k'} k k' |\phi(k')|^2$$

since

$$|k'' - k'| \leq k^2, \quad |(k' \wedge k'')| \leq k k'$$

and since the largest values of ϕ were contained in the smaller wavenumber modes. Now

$$\sum_{k'} k' |\phi(k')|^2$$

was of order 10 in the calculations so the maximum order of magnitude of the non-linear term was 10^2 . With a relative error of 10^{-9} in the values of the ϕ 's this would give a maximum possible error in the non-linear term of order $2 \cdot 10^{-7}$. But it was observed that $\frac{\partial \phi}{\partial t}$ was in general of order $10^{-1} \phi$ and ϕ , for large k , was of order 10^{-4} so the non-linear term was of order 10^{-5} which is not very much greater than the possible error. The evidence we have given already is sufficient to show that the cancellation errors were in fact negligible, but this estimate clearly shows that an increase in the value of n

much beyond 12 was not feasible for this particular calculation model.

Finally, a check on the net accuracy of both the integration process and the evaluation of the non-linear term was provided by a series of calculations for which the viscous terms were identically zero. For these computations the energy and mean-square vorticity should both remain constant. In practice they varied by one part in 10^8 over times of the order of $\mathcal{R}^{-1/2}$ despite marked fluctuations of the individual Fourier components. This was taken as a clear indication of the accuracy of the model.

(iii) The initial flow.

The initial values of the Fourier modes were generated within the computer in a pseudo-random fashion. A computer subroutine was used to produce a sequence of random numbers having a unit normal deviate. Each term in the sequence was then scaled by a factor, depending only on the wavenumber magnitude of the mode with which it was to correspond, so as to approximate to a spectrum of a specified shape. In particular both the energy and vorticity spectra were designed to have one maximum at prescribed wavenumbers in all initial flows.

This method was used in order to avoid selecting particular simple, and thus perhaps uncharacteristic, initial flows and also to be able to generate a number of different initial flows having roughly the same overall structure. It was also hoped to be able to produce approximately isotropic configurations by these means so that the model would have favourable

conditions in which to confirm the predictions made for two dimensional turbulence in the first two chapters of this dissertation. In practice, however, we could not achieve this isotropic distribution. The reason for this was that in all initial flows it was necessary to have the peak of the energy spectrum at a small wavenumber. Consequently most of the energy was distributed amongst a small number of modes and only a slightly uneven distribution between them resulted in a noticeable directional preference.

(iv) Evaluation of the results.

The averaging procedure we adopted for the evaluation of the results was, in the first instance, the spatial average. It would also have been possible to construct an ensemble average since the realization of an initial flow was through a pseudo-random process, but it was not feasible, in the computer time available, to calculate the development of a large number of different initial flows for a sufficiently long time.

The Fourier components of the velocity are

$$(-ik_2 \phi(\underline{k}), i k_1 \phi(\underline{k}))$$

so that the energy of the flow is

$$E = \frac{1}{2} \overline{u^2} = -\frac{1}{2} \sum_{\underline{k}} \sum_{\underline{k}'} (k_1 k_1' + k_2 k_2') \phi(\underline{k}) \phi(\underline{k}') e^{i(\underline{k} + \underline{k}') \cdot \underline{x}},$$

i.e.

$$E = \frac{1}{2} \sum_{\underline{k}} k^2 \phi(\underline{k}) \phi^*(\underline{k}).$$

The mean-square vorticity can similarly be shown to be

$$\Omega = \frac{1}{2} \overline{\omega^2} = \frac{1}{2} \sum_{\underline{k}} k^4 \phi(\underline{k}) \phi^*(\underline{k}).$$

Now as there are only discrete values of k we could not use the spectral functions $E(k)$ and $\Omega(k)$ which are defined for continuously varying k . We therefore used the approximate spectral functions $\bar{E}(k_0)$ and $\bar{\Omega}(k_0)$ defined as the summation of

$$k^2 \phi(k) \phi^*(k) \quad \text{and} \quad k^4 \phi(k) \phi^*(k)$$

respectively for all modes with wavenumbers in the range

$$k_0 - 0.5 < k \leq k_0 + 0.5.$$

This particular bandwidth was chosen in order that there was a reasonably large number of modes in each waveband and also in order that this number should be approximately proportional to the value of k_0 .

For most of the results we restricted k_0 to take integer values only and made comparison between the spectrum functions for these k_0 under different conditions, such as varying ν and n . However in the final evaluation of the shape of the spectral functions half-integer values of k_0 were also used.

No continuous observation was made of the degree of anisotropy of a flow during a calculation. In fact occasional evaluations were sufficient to show that only approximate isotropy was ever achieved. For these evaluations we used the function

$$\overline{\psi(x) \psi(x+r)} = \sum_k \phi(k) \phi^*(k) \cos(k \cdot r) \quad (3.2.8)$$

which, for an isotropic flow, should depend on the magnitude of r alone. This function was evaluated for $r = (x, 0)$ and $r = (0, y)$ and comparison was made between the coefficients of $\cos kx$ and $\cos ky$ which occur in the right hand side of (3.2.8). Table 1 shows two typical evaluations of these coefficients. Case (a) is taken from the calculation with $n=8$ and $\nu=0$,

and case (b) from the calculation with $n=12$ and $\nu=0$ (see §3.3 (ii)). The coefficients of $\cos k_x$ and $\cos k_y$ are of the same order of magnitude for most values of k but very few show any degree of equality, and hence any degree of isotropy. However, from this cursory investigation it was noticeable that the coefficients did fluctuate in time (see also §3.4) and it seemed as if there was no recognizable fixed preferential direction of anisotropy.

Apart from (3.2.8) no correlation functions were evaluated. Such functions are mainly of use in investigating the structure of the large scale variations and in our case these would have shown a strict periodicity as a result of the finite Fourier representation.

3.3 DESCRIPTION OF RESULTS

The main aim in carrying out the series of computations described below was first to examine the effect of the absolute cut-off of possible Fourier modes at the wavenumber $k=n$ and secondly to try to obtain an estimation of the spectral functions at wavenumbers larger than the vorticity containing range. As the total number of Fourier modes was relatively small it was necessary to confine the vorticity containing range to the first few bandwidths (in fact the first three). In this way the vorticity near the cut-off wavenumber would be small and reasonably favourable conditions would prevail for minimising the effect of the cut-off. The behaviour of the vorticity containing modes was then likely to be influenced by the discreteness of the representation, but an intermediary range of wavenumbers between these modes and $k=n$ might

be influenced only slightly by either the discreteness or the cut-off. It is in this latter range that we hoped to obtain results for the spectral functions.

(i) The first calculation had the viscosity term identically zero and $\kappa=4$. There were then only 22 independent complex Fourier modes in the calculation. The values of E and Ω were 11.7 and 52.3 respectively and the integration process was accurate with the time interval 0.015 for δt , i.e. approximately a tenth of the time scale $\Omega^{-1/2}$. The calculation was continued over some sixteen time scales.

The behaviour of the $\bar{E}(k)$ with time is given in figure 2. Initially there was only a small amount of vorticity in $\bar{\Omega}(3)$ but by the time $t=0.4$, i.e. within about three time scales, $\bar{\Omega}(3)$ contained the largest proportion of the vorticity. Figure 2 indicates that this was also the time taken for the $\bar{E}(k)$ to settle to steady values. Soon after that, i.e. at about $t=0.6$, it can be seen that oscillations were set up between the $\bar{E}(k)$ with ever increasing amplitude — $\bar{E}(1)$ and $\bar{E}(3)$ being in phase and $\bar{E}(2)$ out of phase.

This calculation clearly shows the possible drastic effect of the cut-off wavenumber. The development of the flow could be interpreted (and this is a consistent interpretation for all the succeeding calculations too) as trying to adjust itself to some 'equilibrium' state, which in this case was achieved by about $t=0.4$. But for this particular calculation the state then contains a significant amount of energy and vorticity in all modes, since the cut-off prevents the development of the normal single-peaked spectra, and eventually

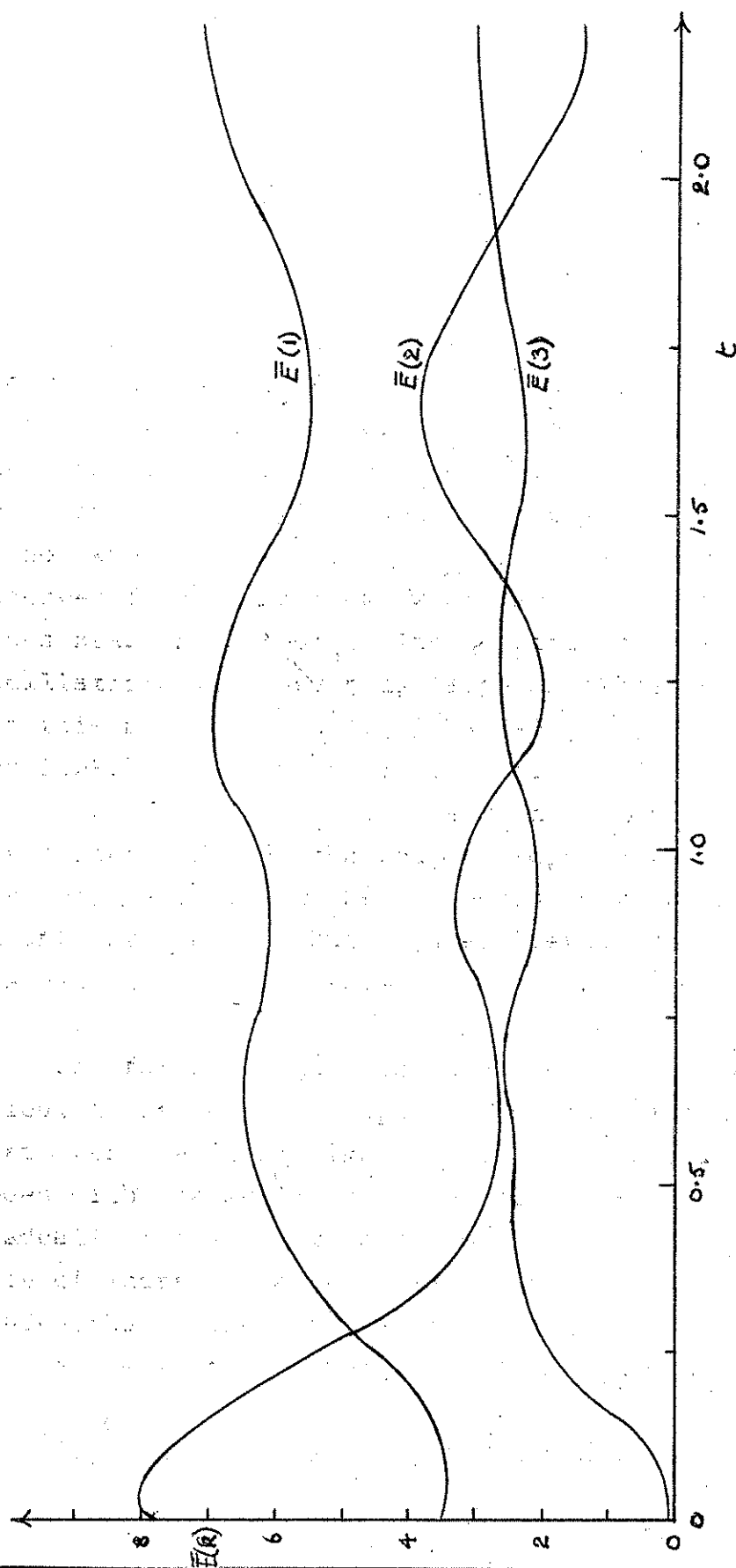


FIG. 2: THE BEHAVIOUR OF THE $\bar{E}(k)$ WITH (ACTUAL) TIME FOR $n=4$ AND $\gamma=0$.

an 'unstable' oscillation is set up.

In all the succeeding calculations the value of n was larger and the oscillatory and cut-off effects appeared to be independent. Thus after the initial period of the adjustment of the spectral functions the only effect of the cut-off was to prevent vorticity from being cascaded to wavenumbers larger than n so that there was a consequent build-up of the vorticity in the largest wavenumbers. All the computations were stopped after this build-up of vorticity became noticeable so that the eventual result of this process is not known. However the oscillatory effects were observed in all the flows whatever the state of the modes near the cut-off. The amplitudes of the oscillations were never as large as those in figure 2, but this may be a result of not allowing the flow to develop for a sufficiently long time. The most probable explanation of this phenomenon seems to lie in the discreteness of the Fourier representation so that only a finite number of modes contain significant amounts of energy: but we shall return to a fuller discussion of this in §3.4.

One further detail of this, and all succeeding, calculations was that, apart from an initial adjustment over the first time scale, the energy in the modes with the smallest wavenumber, i.e. $k=1$, gradually increased throughout the calculation. The rate of increase was approximately three or four hundredths of the total energy in one time scale. We shall attempt to explain this phenomenon in §3.5.

(ii) A second series of calculations were per-

formed mainly to establish the value of δt for accurate time integration at various values of ν and also to observe the transfer of vorticity to larger wavenumbers.

A different initial flow from (i) was used but the computation was again set up with $\nu=4$ and $\nu=0$. After a while ν was increased to 6 and then successive changes were made through the values 8, 10 and 12. The total energy and vorticity were 11.9 and 56.1 respectively and it was sufficiently accurate to take the values of δt as 0.015 for $\nu=4, 6$ and 8, and 0.01 for $\nu=10$ and 12. The introduction of many more Fourier modes into the calculation did not therefore imply a drastic reduction of δt . In fact the limiting size of δt for those calculations which included viscosity (see (iii) and (iv) below) was determined just as much by the viscous term as the non-linear term.

Each time ν was increased, vorticity was transferred into the new wavebands. The time scale of this transfer was such that the new wavebands contained a sufficient amount of vorticity to produce a k^{-1} spectrum law within a time of about $\nu^{-1/2}$ after the change of ν (although of course the transfer continued after this time as well).

After each change of ν there was a marked difference in the behaviour of the spectra only for those wavenumbers near the former value of ν . There was very little noticeable change in the other waveband spectra.

In order to confirm this last observation the two calculations for $\nu=4$ and $\nu=6$ were allowed to overlap for a while, and so also were those

for $\kappa=6$ and $\kappa=8$. In the $\kappa=4$ and 6 calculations, after a time of about $\Omega^{-1/2}$, a difference of only one part in ten was observed in the values of $\bar{\Omega}(1)$ and $\bar{\Omega}(2)$ as given by the two predictions; and in the $\kappa=6$ and 8 calculations a difference of at most one in a hundred occurred in the first four waveband spectra over a time of about $\frac{1}{2} \Omega^{-1/2}$. Thus it seemed that the model could provide reasonable estimates of the behaviour of the spectral functions in time for wavenumbers $k < \kappa-2$.

As already stated, fluctuations occurred in all of the waveband spectra, especially in the highest wavenumbers, throughout these calculations. But unlike case (i) above the fluctuations were by no means regular. Consequently as these calculations were never extended over long periods of time, in order not to allow the development of a significant amount of vorticity near $k=\kappa$, these calculations could not be used for predicting the shape of the spectra.

It is relevant to point out that the time taken by the computer to calculate one time step in the integration process was about 10 minutes for $\kappa=8$, 15 minutes for $\kappa=10$ and 30 minutes for $\kappa=12$. The EDSAC 2 computer could not be relied upon for periods in excess of these times so the calculation with $\kappa=12$ was performed at the limit of the capabilities of that machine.

(iii) It is apparent from the above results that the calculation model may give a reasonably accurate description of the development of an inviscid motion for all except the largest wavenumber modes provided

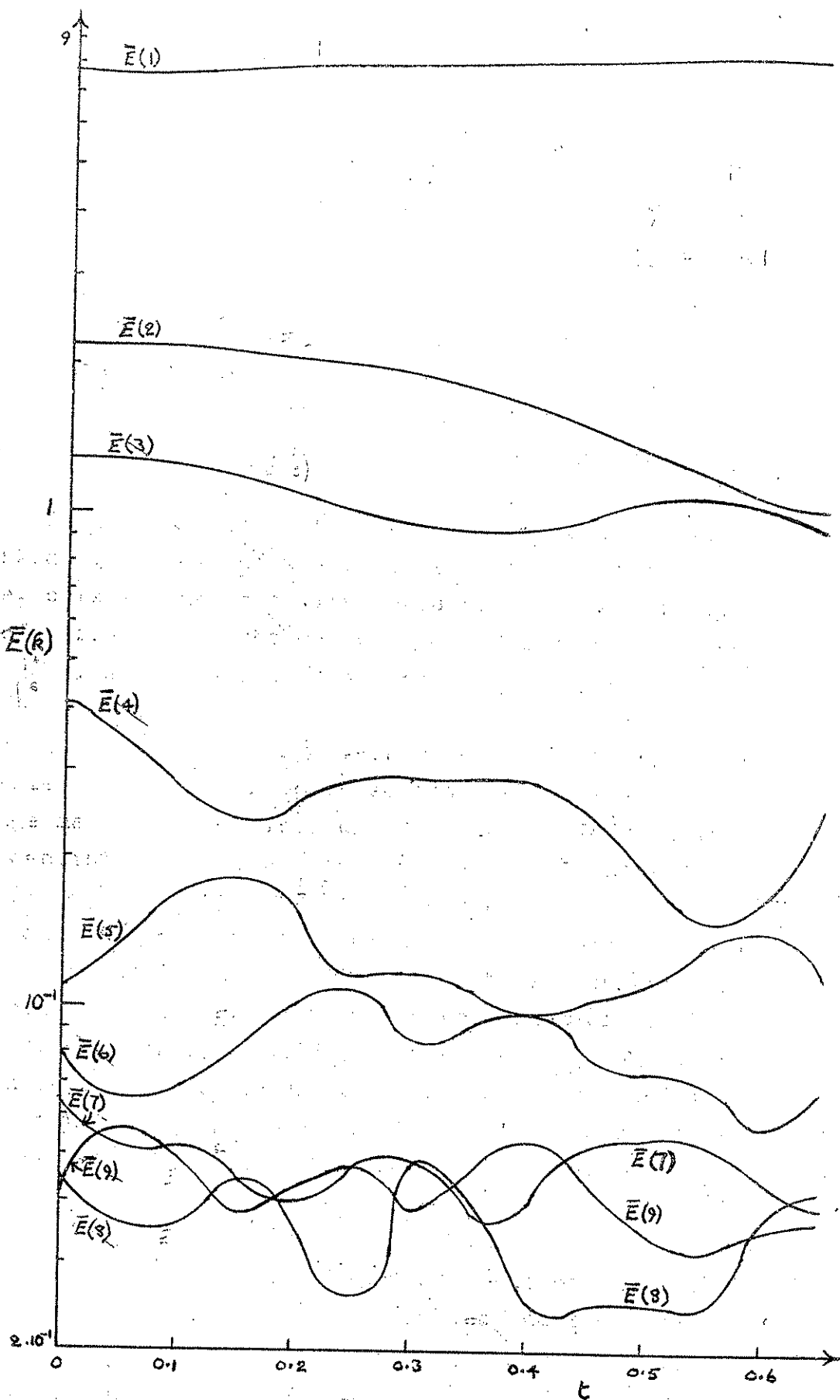
the vorticity in the latter modes is sufficiently small. But, as has been pointed out, vorticity is continually transferred to the largest wavenumbers and this would presumably wreck the calculation once a sufficient amount were allowed to accrue near $k=n$. Consequently we introduced the viscous term into the next series of calculations in order to limit this growth of vorticity by dissipation. The value of the viscosity was so chosen that viscous effects were practically negligible for all except the highest wavenumber bands.

The first calculation of this type had an initial distribution taken from one of the calculations described in (ii) above when an appreciable amount of vorticity had been transferred to the highest waveband. The value of n was 10, ν was 0.01 and δt was again 0.01. The initial values of E and Ω were 11.9 and 56.1 respectively and the calculation was continued over a time of the order of $5 \Omega_0^{-1/2}$, where Ω_0 is the initial value of Ω . With this configuration it was found that ϵ_0 was of order 27 (see §3.6), so that the viscous cut-off wavenumber k_d was about 17.3 and was therefore significantly larger than n .

The behaviour of the waveband spectra for integer values of k_0 are shown in figure 3. The fluctuations in this diagram are typical of all the calculations we carried out except in case (i) above.

In view of the rapid fluctuations we decided to average the spectral functions in time as well as over physical space when determining the shape of the spectra.

FIG. 3: THE BEHAVIOUR OF THE $\bar{E}(k)$ $\longrightarrow$
WITH (ACTUAL) TIME FOR $n=10$, $\nu=0.01$.
(LOGARITHMIC SCALE FOR THE $\bar{E}(k)$.)



This step seemed justified in that there was no marked decay or growth in any of the functions over the whole duration of the calculation so that the system could be supposed to be oscillating about some fixed state. The resulting values for the vorticity spectrum are plotted in figure 4: the centres of the vertical lines denote the average values of $\log \bar{\Omega}(k)$ for integer values of k and the crosses the values for half-integer values of k ; the height of the vertical lines is twice the standard deviation of the fluctuations of $\log \bar{\Omega}(k)$ from the average value.

Despite the fact that the fluctuations are quite marked, the range of wavenumbers from $k=4$ to $k=7$, i.e. between the vorticity containing range and the range affected by the cut-off, does seem to indicate a k^{-1} law, as predicted in the first chapter.

But in the largest wavenumbers there is an appreciable amount of vorticity so that the apparent k^{-1} range may have been influenced by this. To test the susceptibility of the spectrum shape to changes in conditions near the cut-off wavenumber we therefore repeated the calculation, with the same initial values and with $\nu=10$, $\delta t=0.01$, but with viscous terms identically zero except for a narrow band of wavenumbers satisfying $k > 8.5$ where the viscosity was 0.1 . With this configuration the accretion of vorticity near the cut-off was limited by fairly strong viscous effects but the modes away from this region were governed by an inviscid equation.

The high wavenumber spectral functions took the order of $\Omega^{-1/2}$ to adjust themselves and then they

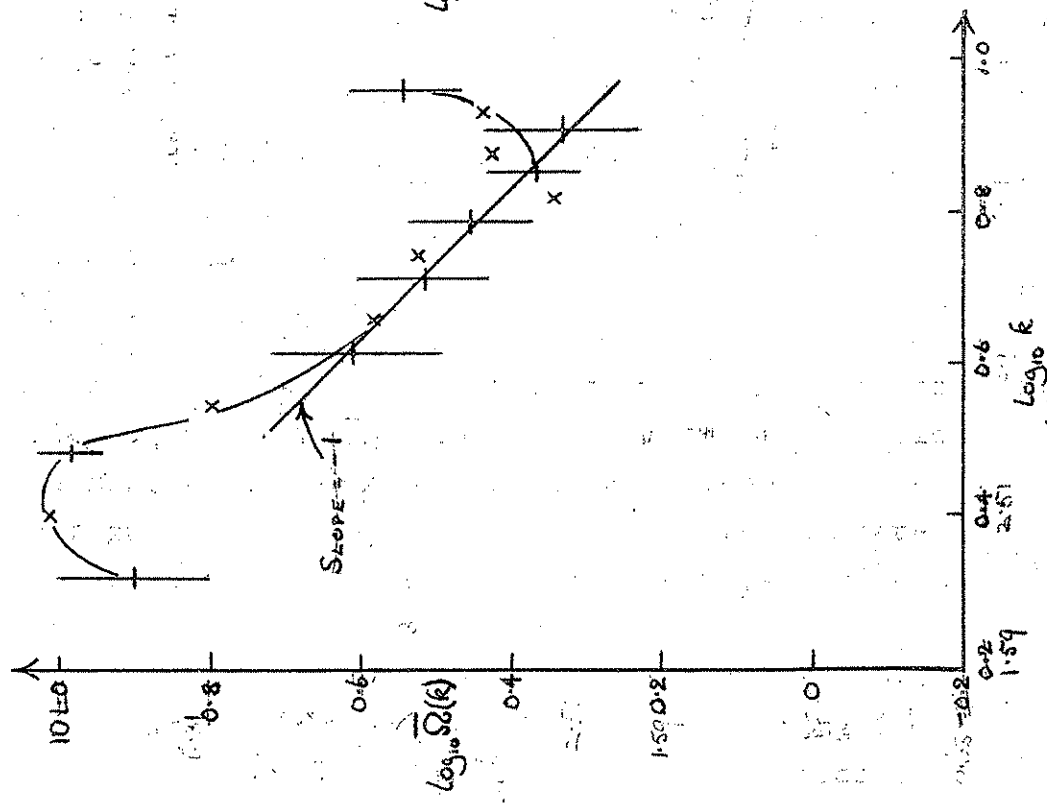


FIG. 4: $n=10$ AND $\nu=0.01$.

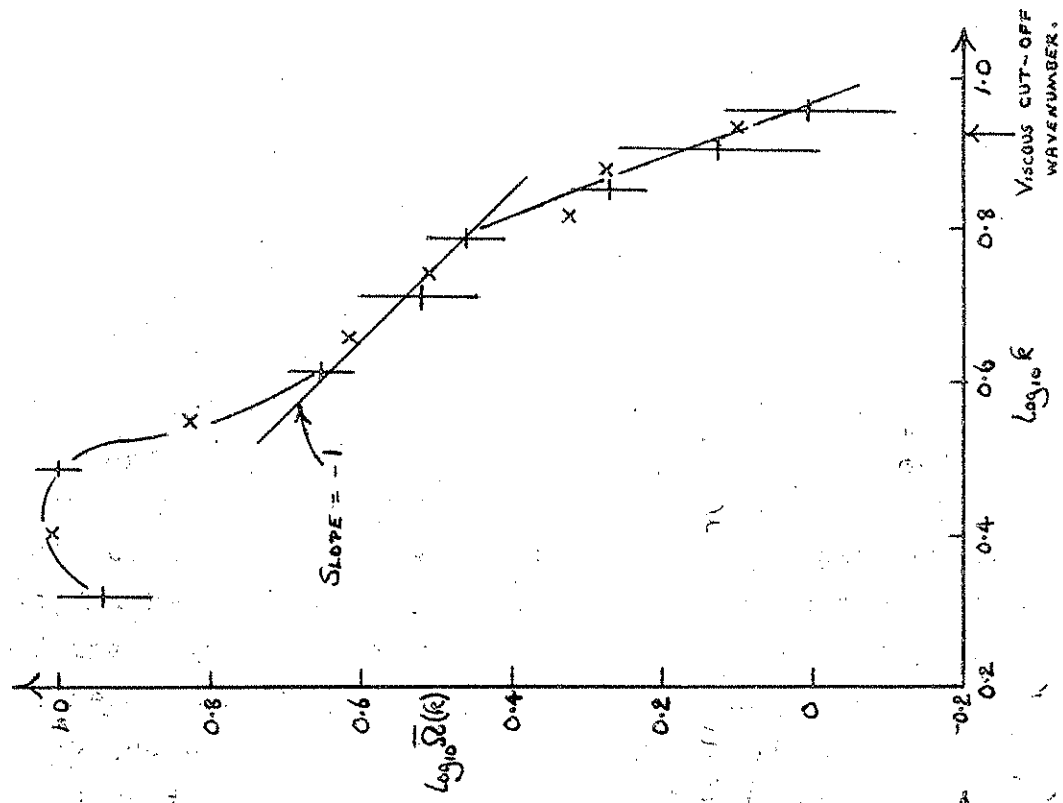


FIG. 5: $n=10$ AND $\nu=0$ FOR $k < 8.5$,
 $\nu=0.1$ FOR $k \geq 8.5$.

fluctuated as in the previous calculations. The vorticity spectrum functions, averaged over the time elapsing after the initial period of adjustment, are shown in figure 5.

This time the region of approximately k^{-1} behaviour is from $k=4$ to about $k=6$, and beyond that there is a significant fall away from a k^{-1} law right to the highest wavenumber. The values of $\bar{\Omega}(8.5)$ and $\bar{\Omega}(9)$ are necessarily as small as they are because of the strong viscous effects at these wavenumbers but this cannot be an explanation for the reduction of values of $\bar{\Omega}(k)$ between $k=6.5$ and $k=8.0$. We can only conclude that the vorticity in these particular wavenumber bands is very much dependent on the state of the highest wavenumber bands and consequently cannot be held to be reliable in predicting the spectral shape of an actual turbulent flow. However there still remains a narrow range of wavenumbers where a k^{-1} law is feasible.

By dimensional arguments we obtained, see equation (1.6.6), the form for the vorticity spectrum in the inertial subrange as

$$\bar{\Omega}(k) = C \varepsilon_w^{2/3} k^{-1}$$

where C is a universal constant. Now ε_w took the value 27 (again see §3.6) for both the above calculations and moreover, the values of $\bar{\Omega}(k)$ in the k^{-1} ranges in figures 4 and 5 are not significantly different. Thus both these calculations give the same value for C namely 1.9.

Lastly these two calculations, starting from the same initial values, had two striking similarities.

It was observed that the development of the spectra in the first five bandwidths as predicted by the two calculations differed very slightly indeed over the whole duration of this calculation, that is for approximately four times the initial value of $\Omega^{-1/2}$. Secondly the occurrences of the maxima and minima of the spectral fluctuations were identical in the two calculations, even for the spectra in the highest wavenumber bands. Both of these observations indicate the surprisingly slight effect of the cut-off wavenumber and of variations in the conditions of the large wavenumber modes.

(iv) Two further calculations were performed: with $\kappa=8$ and with viscous terms non-zero only for those wavenumbers $k > 5.5$. The first calculation had a value for ν of 0.5 and the initial distribution was the same as the initial distribution of the $\kappa=8$ calculation described in (ii) above. The initial values of E and Ω were therefore 11.9 and 56.1. This calculation was carried forward for about five times the initial value of $\Omega^{-1/2}$ after the flow had adjusted itself to the new conditions near $k=\kappa$. The second calculation had a value for ν of 0.25 and the initial distribution was taken from the first calculation after the latter's initial period of adjustment. This calculation was then carried forward for the same time as the other calculation. (The values of k_a for the first and second calculations were - see §3.6 - 2.5 and 3.5 respectively so that they were well within the range of wavenumbers used in the calculation.)

As in the second $\kappa=10$ calculation in (iii) above the high values of the viscosity led to values of the vorticity spectrum at large wavenumbers which were

well below a k^{-1} law - but this time the viscosity was so large that there was no k^{-1} range at all. However for the viscous region we might expect that (see equation (2.3.8))

$$\overline{\Omega}(k) = D \varepsilon_\omega^{2/3} k^{-1} \exp \{ - C \nu \varepsilon_\omega^{-1/3} k^2 \}. \quad (3.3.1)$$

As a test for this figure 6 shows the graph of $\log(k\overline{\Omega})$ against k^2 for the calculation with $\nu=0.5$ (the values of $\overline{\Omega}$ having been obtained by averaging over the duration of the calculation). Unfortunately the number of data points in the viscous range (and beyond the vorticity containing range, which was confined to approximately $k < 3.5$) is so small and so close to $k=n$ that the results are inconclusive.

The calculation with $\nu=0.25$ was notable only in the fact that the two similarities described in (iii) above between the two $\nu=10$ calculations were also characteristic of the two $\nu=8$ calculations.

3.4 THE NATURE OF THE FLUCTUATIONS OF THE SPECTRAL FUNCTIONS

The fluctuations of the waveband spectra with time were a notable feature in all the calculations described in §3.3. We believe that they are inherent to the calculation model as set up. For in a turbulent motion each individual Fourier mode of the velocity field is naturally a fluctuating function of time. Consequently, as each waveband contains only a finite and, moreover, small number of modes, the fluctuations of the waveband spectra are merely a reflection of the behaviour of these individual modes.

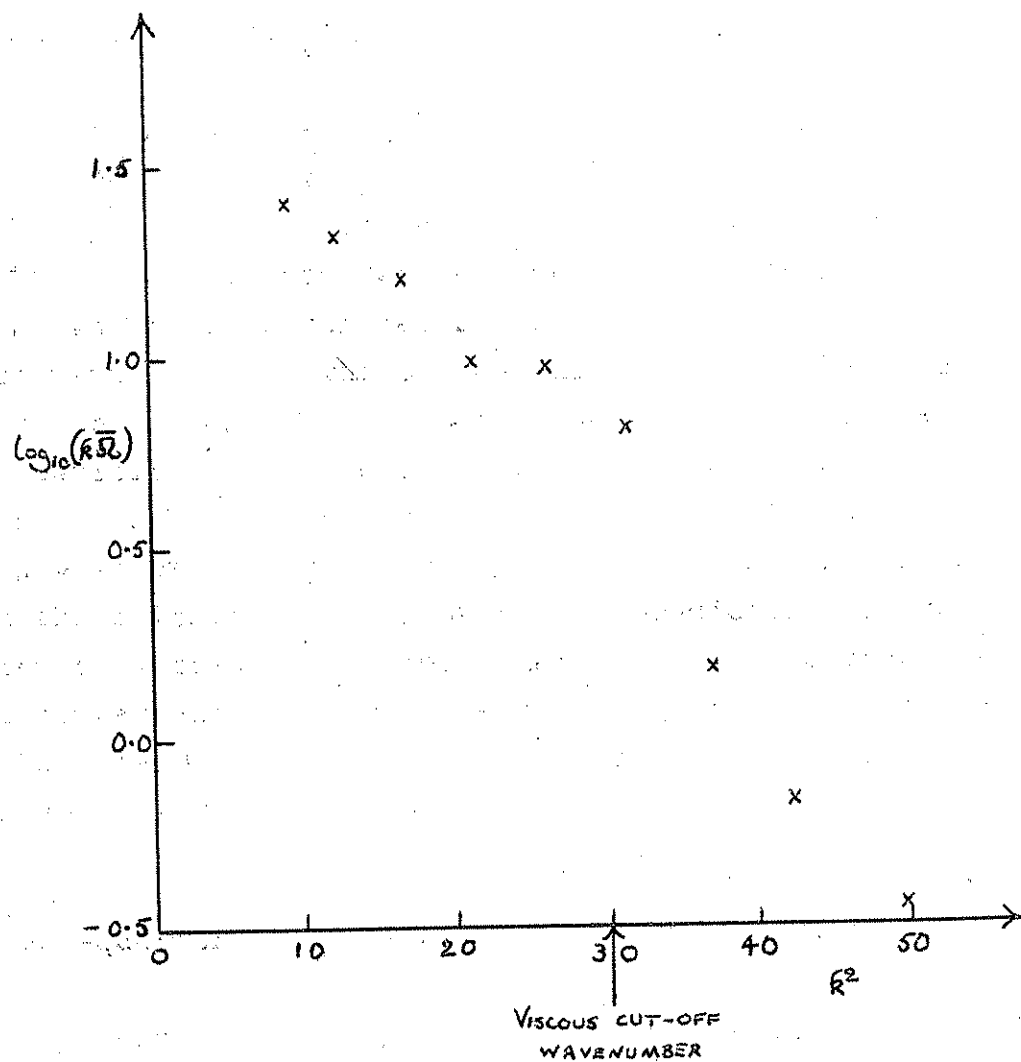


FIG. 6: THE VORTICITY SPECTRUM FOR $n=8$ AND $\nu=0$ FOR
 $R < 5.5$, $\nu=0.5$ FOR $R \geq 5.5$.

As confirmation of this argument a plot of the waveband spectra with half the bandwidth produces a quite chaotic variation with time in all the spectral functions whereas a doubling of the bandwidth produces the opposite effect. Thus figure 7, which is a plot of the energy spectra for the same calculation as that depicted in figure 3 but with double bandwidths, shows clearly a smoothing out of the fluctuations, especially in the smaller wavenumbers.

Now given that fluctuations do occur in the waveband spectra, we may utilize them in investigating the behaviour of the transfer of vorticity to high wavenumbers. For, provided the transfer of vorticity through the spectrum may be thought of as a wholly coherent process, the maxima and minima in the fluctuations clearly mark in time the passage of an excess or deficit of mean-square vorticity through a waveband. Moreover if we assume that the uniform strain analysis of §2.3 is valid for some range of wavenumbers in the calculation then we have a prediction for this vorticity transfer, namely that vorticity is transferred conservatively through wavenumber space such that the wavenumber k associated with one particular quantity of vorticity behaves as

$$k \propto \exp \{ \sqrt{(\alpha^2 - \beta^2)} t \} \quad (3.4.1)$$

(see equation (2.2.25)).

Figures 8 and 9 are plots of the positions in wavenumber space and time of the maxima and minima occurring in the fluctuations for the calculations described in §3.3 (iii) and (iv) respectively. (It will be recalled that the positions of the extrema

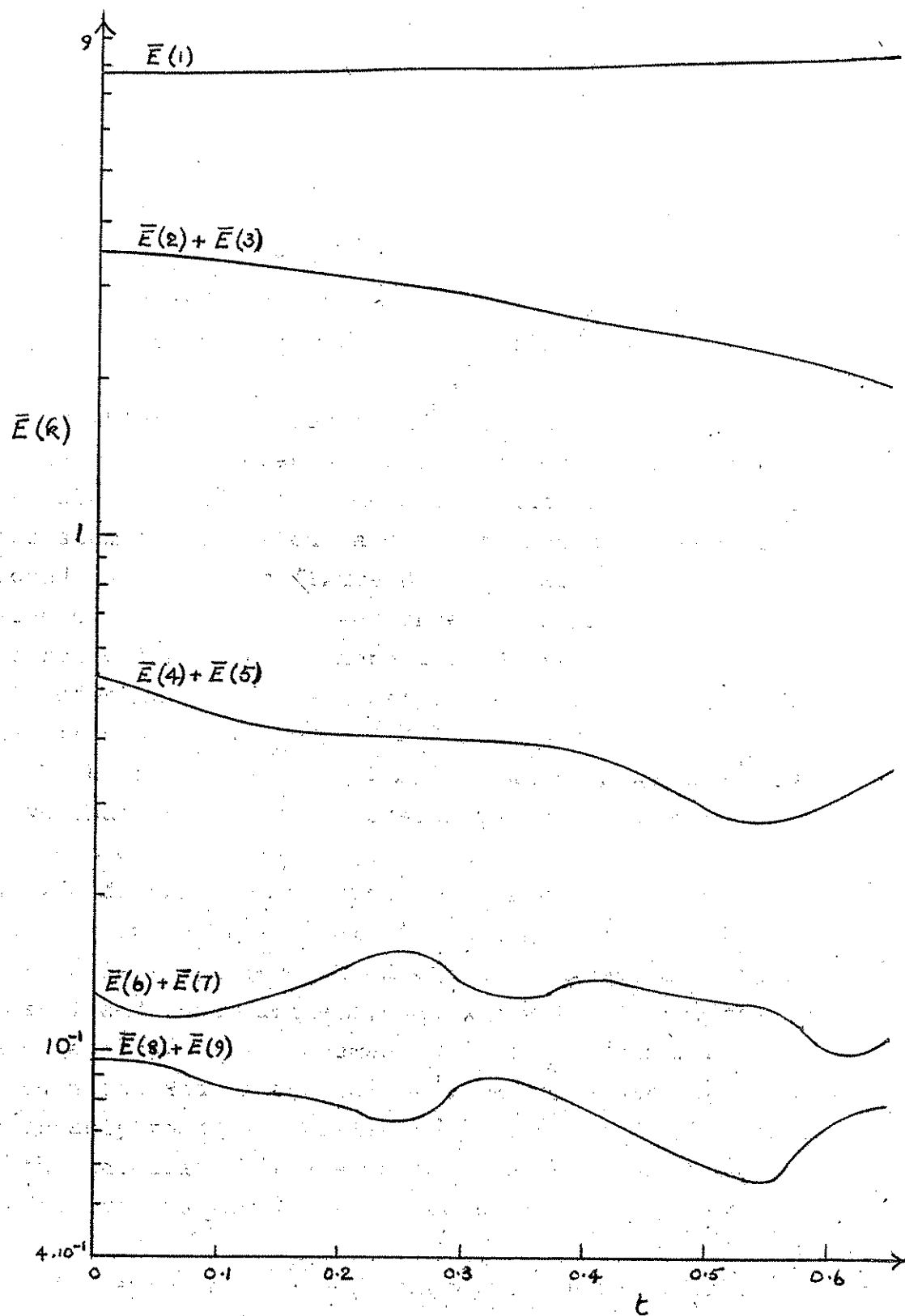


FIG. 7: THE BEHAVIOUR OF THE DOUBLE BANDWIDTH ENERGY SPECTRA WITH TIME FOR $\kappa = 10$ AND $\nu = 0.01$.

were identical in the two calculations in (iii) and were also identical in the two calculations in (iv).) The lines joining the various points on the graphs have been constructed to indicate the probable paths followed by particular quantities of mean-square vorticity. They have been drawn on observation of the general pattern of the points on the graphs and also of the size and shape of the fluctuations from the original spectral graphs, such as figure 3.

Both of these graphs indicate that (3.4.1) is a fairly reasonable basis for this transfer of vorticity. The value of $\sqrt{(\alpha^2 - \beta^2)}$ does fluctuate a little but it does seem to vary about a definite average value of around 3 for either figure 8 or 9. Moreover this value of 3 is of the right order of magnitude on theoretical grounds. There are two estimates of this rate of transfer: the first is based on the postulate that it must be of order $(\alpha^2)^{1/2}$ or $(\beta^2)^{1/2}$ and these are both of magnitude $(\Omega/2)^{1/2}$ whereas we have now shown that the rate of transfer is about $0.4\Omega^{1/2}$; the second is based on the postulate that this quantity must be determined by Kolmogoroff arguments in the equilibrium range so that it would have the value $F\varepsilon_\omega^{1/3}$ where F is a constant which would be expected to be of order 1 and which actually has the value 1 from these calculations. The agreement between the numerical calculation and theoretical analysis is therefore remarkably good, especially for the highest wavenumbers in the calculation where we would expect the effect of the cut-off wavenumber to be noticeable.

It is to be noted too that the coherency in the transfer of vorticity through the spectrum appears

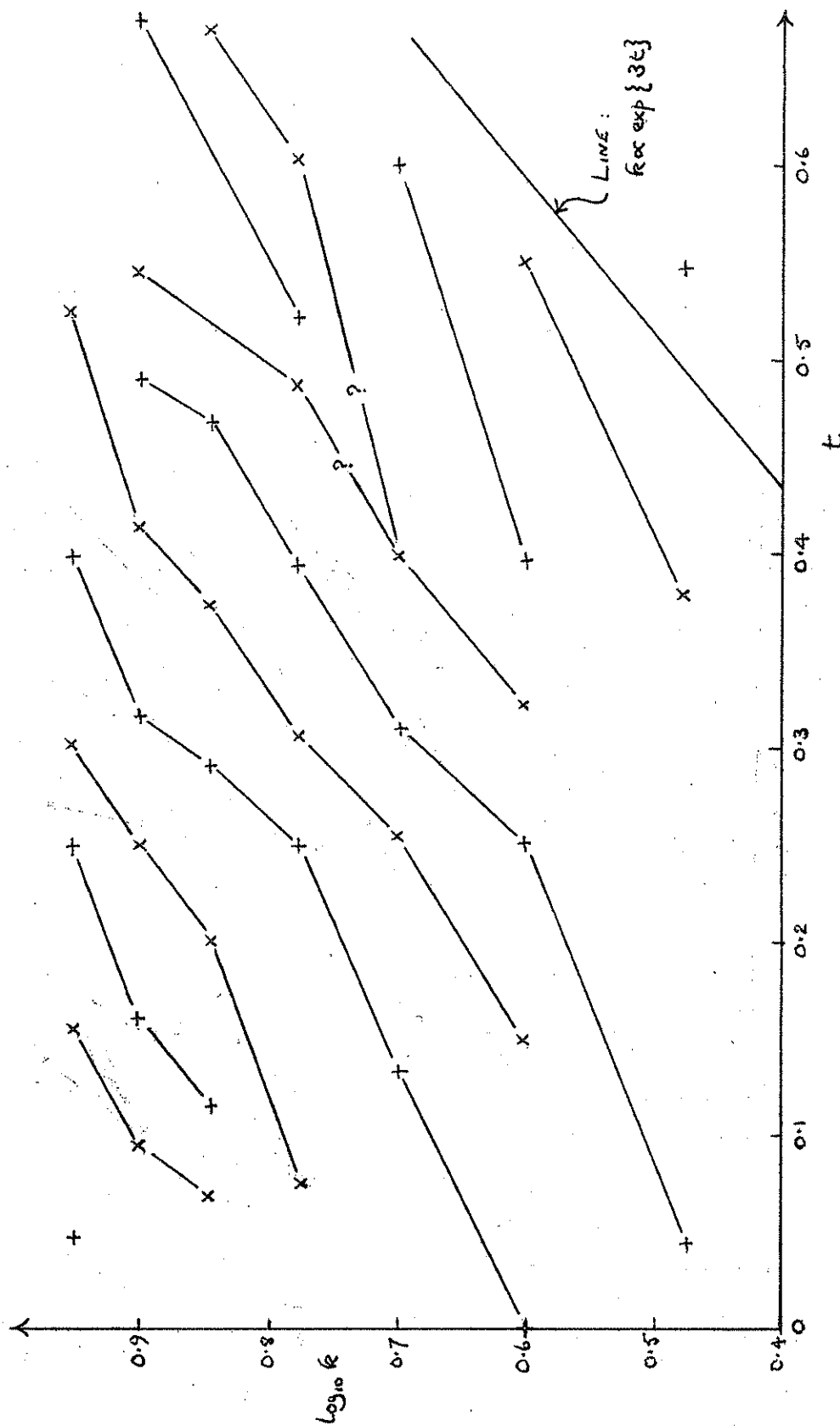


FIG. 8: THE OCCURRENCES OF THE MAXIMA (+) AND MINIMA (x) IN THE FLUCTUATIONS OF THE $\bar{E}(r)$ WITH TIME FOR THE $n=10$ CALCULATIONS.

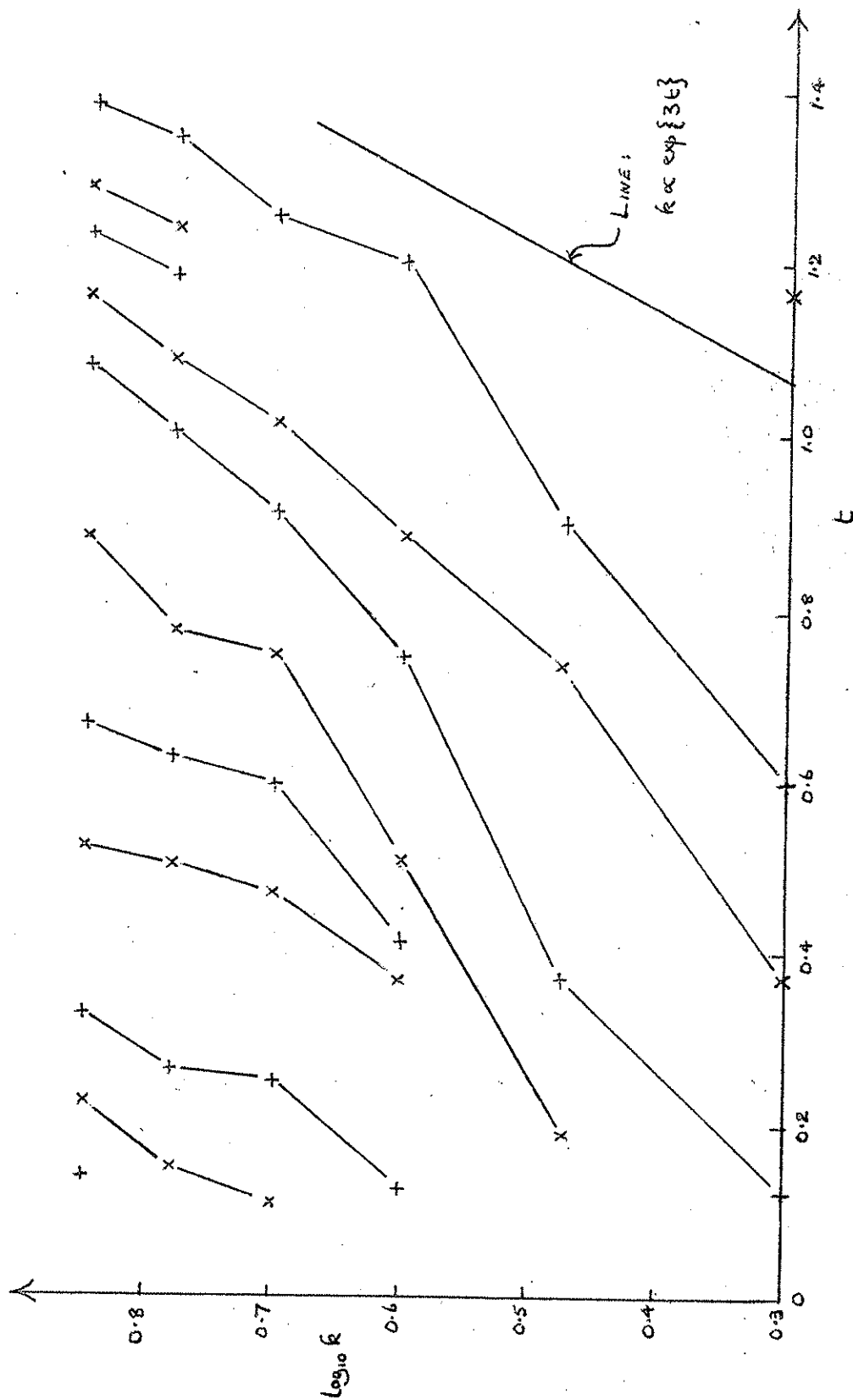


FIG. 9: THE OCCURRENCES OF THE MAXIMA (+) AND MINIMA (x) IN THE FLUCTUATIONS OF THE $\bar{E}(k)$ WITH TIME FOR THE $n=8$ CALCULATIONS.

even at fairly small wavenumbers near the vorticity-containing range. The uniform strain theory may therefore have some degree of validity in providing the basis for a dimensional argument which would hold for all wavenumbers in the inertial and viscous sub-ranges.

Finally we note that if a quantity of 'excess vorticity' covers a band of wavenumbers of width δk at time t then the uniform strain analysis predicts that

$$\delta k \propto \exp \{ \sqrt{(\alpha^2 - \beta^2)} t \}$$

(see (2.2.21)). Thus δk will always be eventually larger than the bandwidths used above for the spectrum functions. Consequently we must expect the higher wavenumber waveband spectra to have more marked fluctuations than those with smaller wavenumbers. Figure 7 illustrates this behaviour to a notable degree.

3.5 THE BEHAVIOUR OF THE ENERGY CONTAINING RANGE

In all the calculations the energy and vorticity containing ranges covered the first three wavebands. Within these wavebands there was a total of only 22 independent Fourier modes. The discreteness of the representation of the turbulence within this range was therefore most marked and the model could not be expected to behave as a freely developing, unbounded, turbulent flow.

However, just as we expected the model to show

a transfer of vorticity to high wavenumbers, we might expect the model to try to adjust itself so that there is a continuous shift of the energy, or vorticity, containing range towards the smallest possible wavenumbers - as was predicted in the similarity theory in §1.7. This does in fact seem to have happened in all the above calculations since $\bar{E}(1)$ was always a monotonically increasing function of time, except perhaps for the initial period of adjustment. In addition $\bar{E}(2)$ and $\bar{E}(3)$, which cover the range just beyond the maximum energy containing wavenumbers, had a tendency to decrease in time (see figure 7). Thus this computational model may, with a larger range of wavenumbers, be capable of testing the conditions for validity of the similarity theory of §1.7 and also provide some information about the general behaviour of the energy containing range for an unbounded turbulent flow.

3.6 THE DECAY OF TURBULENCE

If the similarity theory of §1.7 is valid for these calculations then the decay of the vorticity is given by

$$\Omega = \Omega_0 + \frac{A}{t^2} \quad (3.6.1)$$

(see equation (1.7.7)) where A and Ω_0 are constants and Ω_0 is to be interpreted as a measure of the extent to which a similarity theory covers the whole vorticity spectrum.

Now although all the calculations had truncated viscous dissipation ranges, the actual rate of dissipation

was probably determined by the state of the energy and vorticity containing ranges. Therefore as the latter ranges did not overlap the dissipation range one could suppose that the decay of the turbulence would behave as though an equilibrium range existed and would perhaps even obey (3.6.1).

Figures 10 and 11 show the vorticity decay curves for the calculations described in §3.3 (iii) and (iv) respectively. The accuracy with which the decay law (3.6.1) with $\Omega_0 = 0$ is followed is very striking indeed, particularly as one would have expected an appreciable value for Ω_0 in view of the non-similar structure in the small wavenumbers.

The decay laws for the four curves are

$$\begin{aligned}\Omega^{-1/2} &= 0.030(t + 4.44) \quad \text{for } n=10, \nu=0.01, \\ \Omega^{-1/2} &= 0.032(t + 4.26) \quad \text{for } n=10, \nu=0.1, \\ \Omega^{-1/2} &= 0.030(t + 4.82) \quad \text{for } n=8, \nu=0.5 \\ &\quad \text{and } \nu=0.25.\end{aligned}$$

All these calculations had similar initial flows and we have now shown that they all obey the same relationship (see §1.7)

$$\varepsilon_\omega^{1/3} = 0.4 \Omega^{1/2}$$

despite the fact that ν varied from 0.01 to 0.5. This is therefore very strong evidence indeed for the existence of a cascade mechanism for vorticity in two dimensional turbulence.

3.7 CONCLUSIONS

Even though we have only been able to perform a

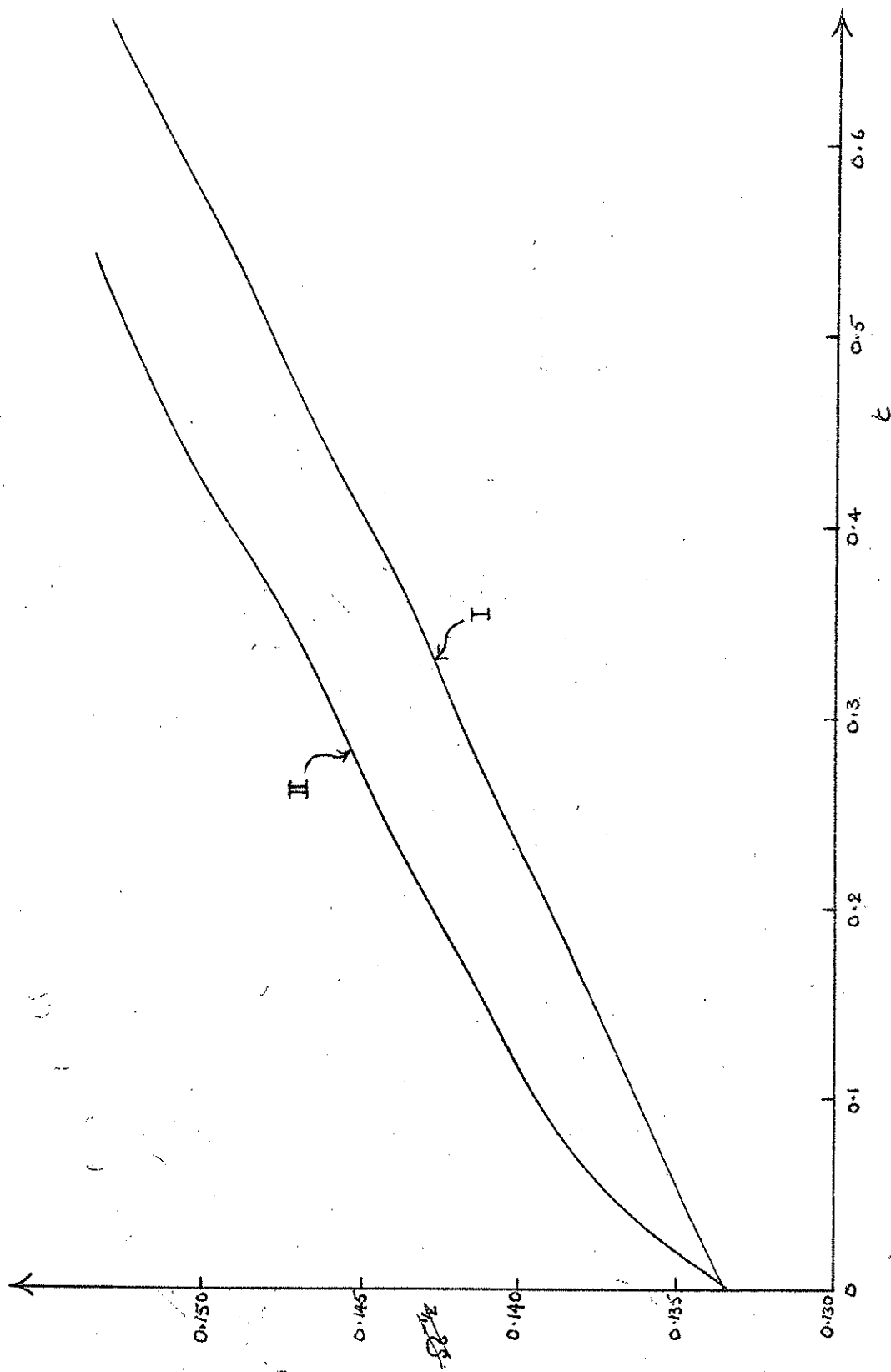


FIG. 10: THE VORTICITY DECAY CURVES FOR $n = 10$ AND (I) $\nu = 0.01$, (II) $\nu = 0.1$.

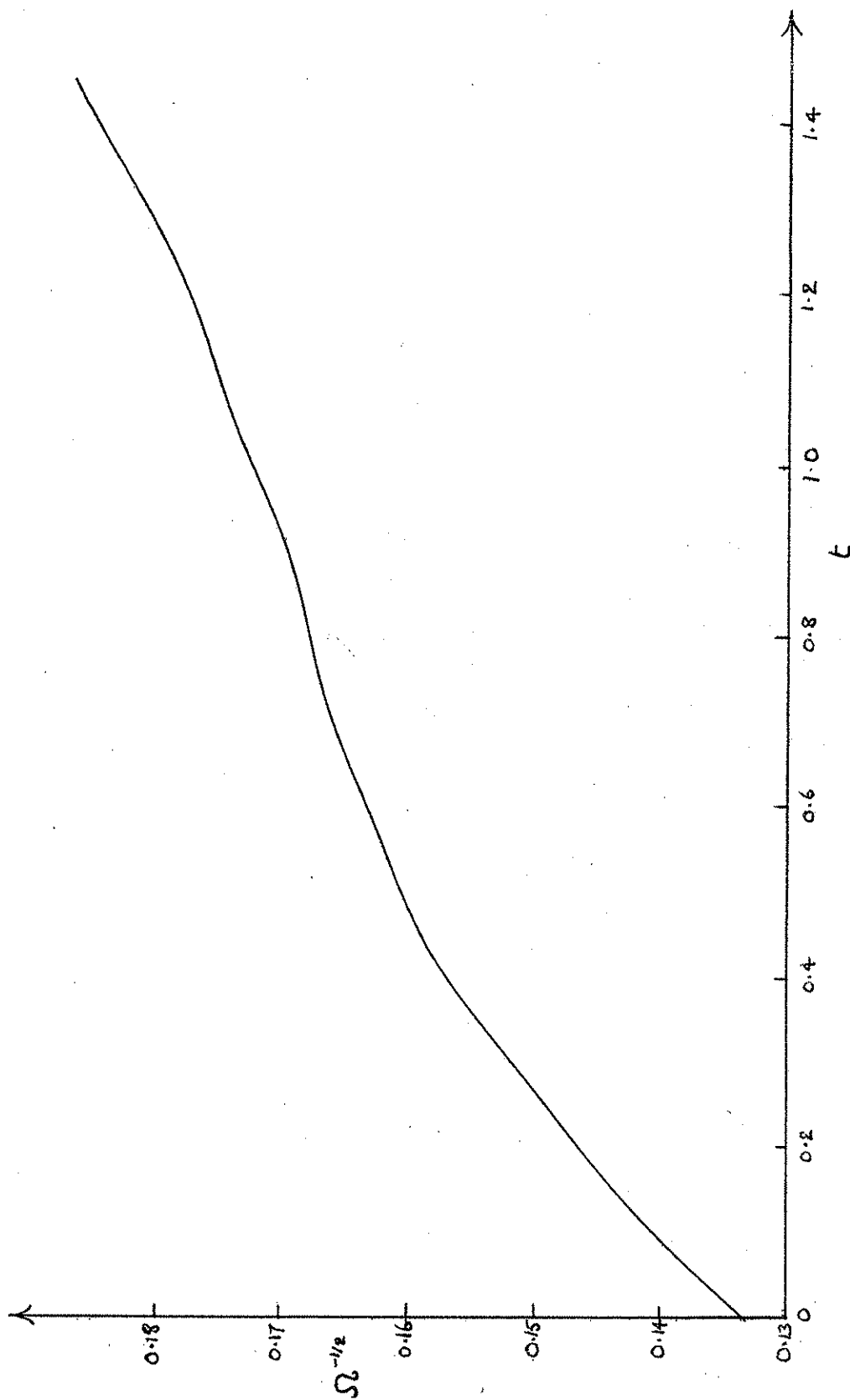


FIG. 11: THE VORTICITY DECAY CURVE FOR THE $n=8$ CALCULATIONS. (INITIAL PERIOD OF ADJUSTMENT, ACCORDING TO THE BEHAVIOUR OF THE $\bar{E}(R)$, WAS CONFINED TO $t < 0.4$.)

limited range of calculations with this computational model there is every indication that this method can be the basis of a useful model for high Reynolds number turbulence. The results we have achieved so far cannot be said to have confirmed absolutely the general outlines of the theory of two dimensional turbulence, since they may have been obtained by quite fortuitous conditions within the particular computations performed, but they do lend some support particularly to the cascade mechanism for vorticity, to the equilibrium theory, to the uniform strain theory, and to some aspects of the similarity theory for the decay of turbulence.

It is not impossible for this model to be developed and refined beyond that described here for, although the above calculations were performed almost at the limit of the accuracy and reliability of the EDSAC 2 machine, more modern computers which have greater speed and storage capacities should be capable of handling more Fourier modes in the calculation. Moreover it may be possible to start approximating the non-linear term on the basis of experience gained with this computational model and consequently to shorten the amount of calculation involved. Thus, for example, the argument in §3.2 (ii) and the results we have obtained both indicate that the interactions of the larger wavenumber modes may have a negligible effect in the evaluation of the non-linear term for small wavenumber modes so that we could ignore these terms altogether in a future computation. It may also be possible to vary the value of the time step δt in the computation for varying wavenumber k when the range of wavenumbers is extended: one would expect

the time scale of the variations to decrease as k increases so that σ_t could have large values for small k and small values for large k . Both these refinements could shorten the length of calculation considerably and it may then be possible to perform a large number of calculations with more Fourier modes and thus provide a very useful method for studying homogeneous turbulence.

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