2D Symmetry Plane Equations from 3D Euler

Miguel D. Bustamante   Helgi Haraldsson
Robert Kerr
Mathematics Institute, University of Warwick
Coventry CV4 7AL, United Kingdom

19 December 2006

Abstract submitted to EE250

In the context of 3D Euler equations possessing a symmetry plane (see [1, 2]) we introduce a new set of 2D equations on the symmetry plane:

$$\frac{D}{Dt}\omega = \alpha\,\omega\,, \qquad \begin{aligned} \frac{D}{Dt}\boldsymbol{\kappa}_{2D} &= \nabla_{2D}\alpha + (\boldsymbol{\kappa}_{2D}\cdot\nabla_{2D})\boldsymbol{u}_{2D} - 2\,\alpha\,\boldsymbol{\kappa}_{2D}, \\ \triangle_{2D}\alpha + \alpha_{,zz} &= -\hat{\boldsymbol{\omega}}\cdot(\nabla_{2D}\times\omega\boldsymbol{\kappa}_{2D}). \end{aligned} \tag{1}$$

The out-of-plane coordinate is denoted by z, so the symmetry plane indicated by the subscript $2D$ corresponds to $z = 0$. All vector fields are parallel to the symmetry plane, except for $\hat{\boldsymbol{\omega}}$, which is pointing out of plane. On the symmetry plane $\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \boldsymbol{u}_{2D}\cdot\nabla_{2D}$ is the $2D$ convective derivative and α is the stretching rate in the out-of-plane direction $\hat{\boldsymbol{z}}$. The remaining quantities in the symmetry plane come from the full three-dimensional vorticity $\boldsymbol{\omega}$ and are its magnitude $\omega = |\boldsymbol{\omega}|$, its direction vector $\hat{\boldsymbol{\omega}} \equiv \boldsymbol{\omega}/\omega$, and the curvature vector of vortex lines $\boldsymbol{\kappa} = (\hat{\boldsymbol{\omega}}\cdot\nabla)\hat{\boldsymbol{\omega}}$. The $2D$ velocity $\boldsymbol{u}_{2D}$ is obtained from α and ω by solving $\alpha = -\nabla_{2D}\cdot\boldsymbol{u}_{2D}$ and $\omega = \hat{\boldsymbol{\omega}}\cdot(\nabla_{2D}\times\boldsymbol{u}_{2D})$.

The system of equations (1) can be closed (and this is the only approximation to be made) by assuming that the second out-of-plane derivative of the stretching vanishes: $\alpha_{,zz} = 0$. This implies $\triangle_{2D}\alpha = \nabla_{2D}\cdot(\hat{\boldsymbol{\omega}}\times\omega\boldsymbol{\kappa}_{2D})$ which can be solved for the curvature:

$$\omega\,\boldsymbol{\kappa}_{2D} = -\hat{\boldsymbol{\omega}}\times\nabla_{2D}\alpha\,. \tag{2}$$

In order to validate the above assumption, figure 1 (left) shows, on the symmetry plane, gray scale contours of ω and line contours of α taken from a full 3D calculation [3]. There is clearly a direction of high gradients of α, parallel in the symmetry plane, making some angle with respect to the vertical, as there is a direction of weaker gradients. Though not shown, the out-of-plane term $\alpha_{,zz}$ has the same order of magnitude as the square of the weaker gradient just mentioned. Thus the condition $\alpha_{,zz} << \triangle_{2D}\alpha$ is satisfied, with a factor of around 1/400.

In order to demonstrate the potential of these new 2D equations, we have done a calculation in a vortex dipole approximation. Preliminary results are consistent with a full 3D vortex filament calculation [4], showing the expected scaling laws and time dependence for the collapse of a vortex dipole. These results, plotted in figure 1 (right), show the following scaling laws: for the maximum value of stretching, $\alpha \propto (T-t)^{-1}$. For the maximum value of approaching velocity squared of the interacting vortices, $u^2 \propto (T-t)^{-1}$. Finally, for the maximum value of vortex curvature squared, $\boldsymbol{\kappa}_{2D}^2 \propto (T-t)^{-1}$. Our calculation goes further. It gives an estimate of the ratio $\omega_{pk}/\alpha_{pk} \approx 16.0$ where $\omega_{pk} = \max_{(x,y)\in\mathbb{R}^2}\omega(x,y,0,t)$ is the maximum value of the modulus of the vorticity and α_{pk} is the value of the stretching at the point where the vorticity

is maximum. Our estimate is consistent, within this approximation, with the full 3D Euler calculations by [5, 6].

Moreover, applying eq.(2) to the full 3D data in figure 1 (left) gives the direction of $\boldsymbol{\kappa}_{2D}$ to be along contours of α, denoted by dashed and solid lines in the figure. The results from our vortex dipole approximation (figure not shown) are consistent with this direction.

Now that our approximate 2D system, eqs.(1),(2) has been validated, we plan to study it numerically using the initial condition of anti-parallel vortices from an existing fully three-dimensional Euler calculation [5]. Further validations would then consist of comparisons with that calculation. It is worth mentioning that our approximation is consistent with the conservation of total energy when the energy flux to and from a neighborhood of the symmetry plane is taken into account. Finally, we plan to study a variational formulation of our new equations.

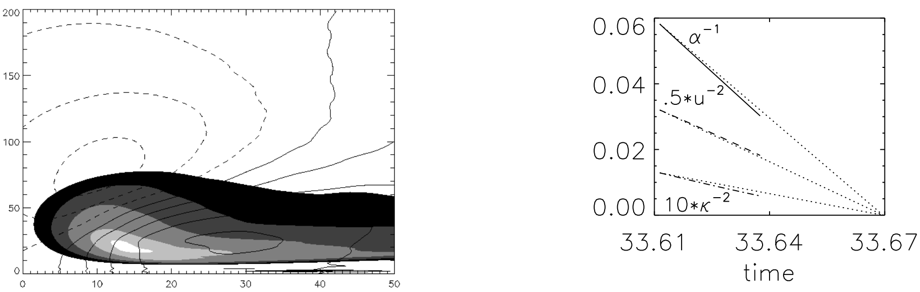

Figure 1: (Left): Plot of vorticity contours (gray scale) and stretching contours (lines) on the symmetry plane. (Right): Time dependence of quantities from vortex dipole approximation: maximum stretching (solid line), approaching velocity of vortices (dashed), and maximum curvature (dash-dotted). The dotted lines are just indicators.

References

[1] Wm. T. Ashurst and D. I. Meiron. Numerical study of vortex reconnection. *Phys. Rev. Lett.*, 58(16):1632–1635, Apr 1987.

[2] Alain Pumir and Robert M. Kerr. Numerical simulation of interacting vortex tubes. *Phys. Rev. Lett.*, 58(16):1636–1639, Apr 1987.

[3] Robert M. Kerr. Velocity and scaling of collapsing Euler vortices. *Physics of Fluids*, 17(7):075103, 2005.

[4] Alain Pumir and Eric D. Siggia. Vortex dynamics and the existence of solutions to the Navier–Stokes equations. *Physics of Fluids*, 30(6):1606–1626, 1987.

[5] Robert M. Kerr. Evidence for a singularity of the three-dimensional, incompressible Euler equations. *Physics of Fluids A: Fluid Dynamics*, 5(7):1725–1746, 1993.

[6] Rainer Grauer, Christiane Marliani, and Kai Germaschewski. Adaptive mesh refinement for singular solutions of the incompressible Euler equations. *Phys. Rev. Lett.*, 80(19):4177–4180, May 1998.