The three-dimensional Euler fluid equations: where do we stand?

J. D. Gibbon

Mathematics Department

Imperial College London, London SW7 2AZ

j.d.gibbon@imperial.ac.uk

"Euler Equations 250 years on"

19th June 2007

In honour of

Leonhard Euler (1707–1783)

(from Ecclesiasticus 44:1-15)

Let us now praise famous men

& our fathers that begat us.

The Lord hath wrought great glory by them
through his great power from the beginning ...

All these were honoured in their generations,
and were the glory of their times.

There be of them, that have left a name behind them,
that their praises might be reported ...

Summary of this lecture

1. Some elementary introductory remarks on :
   - 3D incompressible Euler fluid equations & its invariants
   - 2D and $2\frac{1}{2}$D Euler equations.
2. **The Euler singularity problem :**
   - **The Beale-Kato Majda (BKM) Theorem.**
   - **Numerical studies :** A brief history of investigations regarding the possible development of a finite time singularity in $\boldsymbol{\omega}$ in 3D Euler.
   - Work on the **direction of vorticity.**
3. **Ertel's Theorem** & its consequences : 3D Euler as a Lagrangian evolution equation. Euler & his brothers/sisters: ideal MHD, barotropic Euler, mixing.
4. **Quaternions** & their application to Euler (aero/astro/animation ideas) : Lagrangian frame dynamics of an Euler fluid particle & the pressure Hessian.
5. Review of work on the **restricted Euler equations.**

3D incompressible Euler equations

1. The 3D incompressible Euler equations in terms of the **velocity field** $\boldsymbol{u}(\boldsymbol{x},\, t)$:

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla\right)\boldsymbol{u} = -\nabla p\,; \qquad \text{div}\,\boldsymbol{u} = 0$$

div $\boldsymbol{u} = 0$ constrains the pressure p to obey $\{S = \frac{1}{2}(u_{i,j} + u_{j,i})$ strain matrix$\}$

$$-\Delta p = TrS^2 - \tfrac{1}{2}\boldsymbol{\omega}^2\,.$$

2. 3D incompressible Euler in terms of the **vorticity field** $\boldsymbol{\omega}(\boldsymbol{x},\, t) =$ curl $\boldsymbol{u}$:

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla\right)\boldsymbol{\omega} = \boldsymbol{\omega}\cdot\nabla\boldsymbol{u} = S\boldsymbol{\omega}\,.$$

3. **Three invariants :**

- **Energy :** $\int_V |\boldsymbol{u}|^2\, dV$ • **Circulation :** $\oint_C \boldsymbol{u}\cdot d\boldsymbol{r}$ – Kelvin's Theorem

- **Helicity :** $\int_V \boldsymbol{\omega}\cdot\boldsymbol{u}\, dV$: for a vector field in 3-space this is the standard measure of the extent to which the field lines wrap (coil) around one another.

- G. I. Barenblatt (1990) said in his review of

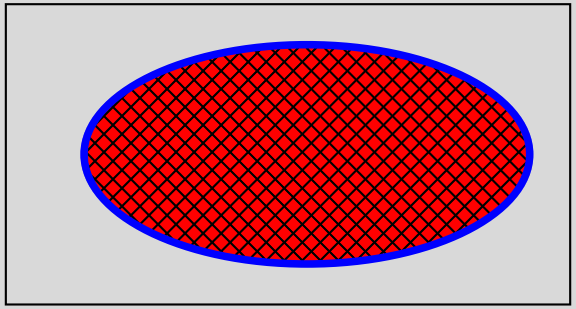
Topological Fluid Mechanics, Proceedings of the IUTAM Symposium, edited by **H. K. Moffatt and A. Tsinober,** CUP, 1990.

"The magnetic helicity-invariant integral characterizing the topology of a magnetic field, was discovered by the plasma physicist Woltjer (1958) ... The next developments in Topological Fluid Mechanics were made by Moffatt & Arnol'd. Moffatt recognized the topological nature of such invariants for magnetic fields ... whilst Arnold demonstrated their general nature for fields of general type. In 1969 Moffatt introduced the concepts of the helicity density & the helicity integral for vortex flows."

- **Arnol'd & Khesin,** Topological Methods in Hydrodynamics, 1998.
- **Moffatt (1969,78,85/6,92), Ricca & Moffat (1992).**

4. **Vortex sheets :** Sijue Wu, Wednesday 15:50-16:20, *Recent progress in the mathematical analysis of vortex sheets*.

2D incompressible Euler vortex patches



In the interior of $\Omega(t)$, $\boldsymbol{\omega} = \boldsymbol{\omega}_0 =$ const for all $t > 0$: in the exterior $\boldsymbol{\omega} = 0$. The **boundary**, initially Γ_0, evolves as Γ_t.

The existence of weak solutions was proved by **Yudovich (1963)**. Euler vortex patches are such that when $\boldsymbol{u}$ is smooth, $\boldsymbol{\omega}$ is a particular solution of the 2D incompressible Euler equations ($\boldsymbol{\omega} = \text{curl}\,\boldsymbol{u}$)

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla\right)\boldsymbol{\omega} = 0\,, \qquad \boldsymbol{\omega}\cdot\nabla\boldsymbol{u} = 0\,.$$

Using harmonic analysis **Chemin (1993)** proved that if the boundary Γ_t is initially smooth (Γ_0 is C^r for $r > 1$) then it remains smooth for all $t > 0$. The theoretical bounds for the parametrization of Γ_t are $\exp\exp t$.

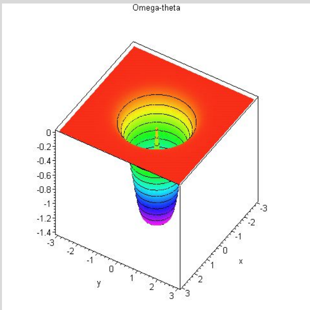
- $\exists$ a different version on this proof by **Bertozzi & Constantin (1993)**.
- See also Chapter 9 of the book by **Majda & Bertozzi (2000)**.

$2\frac{1}{2}$D Euler infinite energy singular solutions

$$\boldsymbol{U}_{3D}(x,y,z,t) = \{\boldsymbol{u}(x,y,t),\, z\gamma(x,y,t)\}$$ e.g. Burgers' vortex

on a domain that is infinite in z with a circular X-section Ω of radius L.

ω_θ



$$(\partial_t + \boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla p\,,$$

$$\text{div}\,\boldsymbol{u} = -\gamma\,,$$

$$(\partial_t + \boldsymbol{u}\cdot\nabla)\gamma + \gamma^2 = \frac{2}{\pi L^2}\int_\Omega \gamma^2\,dA\,.$$

1. A time-independent form of these equations was given by Oseen (1922).
2. Ohkitani/JDG (2000) derived the above & showed numerically $\gamma \to -\infty$.
3. Later, using Lagrangian arguments, Constantin (2000) proved analytically that $\gamma \to \pm\infty$ in a finite time in different parts of the X-section Ω.

- $\gamma \to +\infty$ the vortex is tube-like; as $\gamma \to -\infty$ ring-like: **infinite energy**.
- JDG/Moore/Stuart (2003) found a class of analytical singular solutions.

3D Euler: Why the interest in singularities?

- **Physically** their formation may signify the onset of turbulence & may be a mechanism for energy transfer to small scales (talk by Greg Eyink: *Dissipative anomalies in singular Euler flows,* Thursday 08:30-09:20).
- **Numerically** they require very special methods – a great challenge to CFD.
- **Mathematically** their onset would rule out a global existence result (talk of Peter Constantin: *The Euler blow-up problem,* Thursday at 09.30–10.00).

Beale-Kato-Majda-Theorem (1984): *There exists a global solution of the 3D Euler equations* $\boldsymbol{u} \in C([0,\,\infty]; H^s) \cap C^1([0,\,\infty]; H^{s-1})$ *for* $s \geq 3$ *if*

$$\int_0^T \|\boldsymbol{\omega}(\cdot\,,\tau)\|_{L^\infty(\Omega)}\,d\tau < \infty\,, \qquad \text{for every } T > 0.$$

Corollary to BKM Thm: If a singularity is observed in a numerical experiment

$$\|\boldsymbol{\omega}(\cdot\,,t)\|_{L^\infty(\Omega)} \sim (T-t)^{-\beta}$$

then $\beta \geq 1$ for the singularity to be genuine & not an artefact of the numerics.

Numerical search for singularities

(a revised & up-dated version of a list originally compiled by Rainer Grauer)

1. Morf, Orszag & Frisch (1980): Padé-approximation, complex time singularity of 3D Euler {see also Bardos *et al* (1976)}: Singularity: yes. (Pauls, Matsumoto, Frisch & Bec (2006) on complex singularities of 2D Euler).
2. Chorin (1982): Vortex-method. Singularity: yes.
3. Brachet, Meiron, Nickel, Orszag & Frisch (1983): Taylor-Green calculation. Saw vortex sheets and the suppression of singularity. Singularity: no.
4. Siggia (1984): Vortex-filament method; became anti-parallel. Singularity: yes.
5. Ashurst & Meiron/Kerr & Pumir (1987): Singularity: yes/no.
6. Pumir & Siggia (1990): Adaptive grid. Singularity: no.
7. Brachet, Meneguzzi, Vincent, Politano & P-L Sulem (1992): pseudospectral code, Taylor-Green vortex. Singularity: no.

8. Kerr (1993, 2005): Chebyshev polynomials with anti-parallel initial conditions; resolution $512^2 \times 256$. Observed $\|\boldsymbol{\omega}\|_{L^\infty(\Omega)} \sim (T-t)^{-1}$. Singularity: yes. (poster by Bustamante & Kerr, Tuesday 17:00-18:00.)
9. Grauer & Sideris (1991): 3D axisymmetric swirling flow. Singularity: yes.
10. Boratav & Pelz (1994, 1995): Kida's high symmetry. Singularity: yes.
11. Pelz & Gulak (1997): Kida's high symmetry. Singularity: yes.
12. Grauer, Marliani & Germaschewski (1998): Singularity: yes.
13. Pelz (2001, 2003): Singularity: yes.
14. Kida has edited a memorial issue for Pelz in Fluid Dyn. Res., **36**, (2005) :
    - Cichowlas & Brachet: Singularity: no.
    - Pelz & Ohkitani: Singularity: no.
    - Gulak & Pelz: Singularity: yes.
15. Hou & Li (2006): Singularity: no. (talk by Hou on Thursday 10.00–10.30).

Discussion on Thursday 17:00-18:00

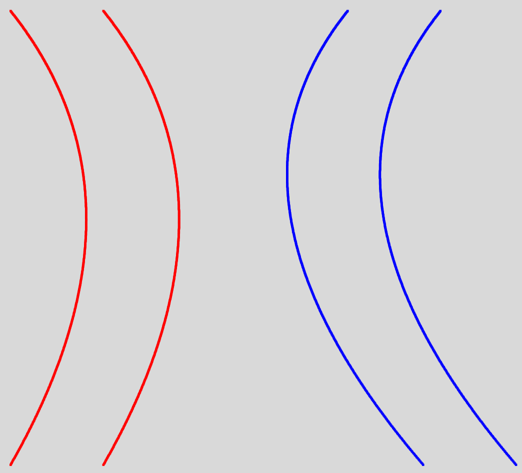
moderated by Claude Bardos & Edriss Titi

Singularities: why do we care?

My own suggestion for debate:

By concentrating on the 'yes/no' aspect of Euler solutions, have we over-emphasized the blow-up problem to the detriment of studying the subtle directional amplification mechanisms that produce violent growth of the vorticity in the first place?

Direction of vorticity: the work of CFM & DHY



a) Constantin, Fefferman & Majda (1996) discussed the idea of vortex lines being "smoothly directed" in a region of greatest curvature. They argued that if the velocity is finite in a ball $(B_{4\rho})$ & $\lim_{t\to T}\sup_{\boldsymbol{W}_0}\int_0^t \|\nabla\hat{\boldsymbol{\omega}}(\cdot\,,\tau)\|^2_{L^\infty(B_{4\rho})}\,d\tau < \infty$ then there can be no singularity at time T.

b) Deng, Hou & Yu (2006) take the arc length $L(t)$ of a vortex line L_t with $\hat{\boldsymbol{n}}$ the unit normal and κ the curvature. If $M(t) \equiv \max\left(\|\nabla\cdot\hat{\boldsymbol{\omega}}\|_{L^\infty(L_t)}, \|\kappa\|_{L^\infty(L_t)}\right)$ they argue that there will be no blow-up at time T if

1. $U_{\hat{\omega}}(t) + U_{\hat{n}}(t) \lesssim (T-t)^{-A}$ $\qquad A + B = 1,$
2. $M(t)L(t) \leq \text{const} > 0$
3. $L(t) \gtrsim (T-t)^{B}$.

See the talks: Constantin (Thurs 09.30–10.00) & Hou (Thurs 10.00–10.30).

The 3-dim Euler equations and Ertel's Theorem

$$\frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega}\cdot\nabla\boldsymbol{u} = S\boldsymbol{\omega} \qquad \text{incompressible Euler}$$

Ertel's Theorem (1942): If ω satisfies the 3D incompressible Euler equations then any arbitrary differentiable μ satisfies

$$\frac{D}{Dt}(\boldsymbol{\omega}\cdot\nabla\mu) = \boldsymbol{\omega}\cdot\nabla\left(\frac{D\mu}{Dt}\right)$$

Clearly, the operations $\left[\frac{D}{Dt},\, \boldsymbol{\omega}\cdot\nabla\right] = 0$ commute. Thus $\boldsymbol{\omega}\cdot\nabla(t) = \boldsymbol{\omega}\cdot\nabla(0)$ is a Lagrangian invariant & is "frozen in" (Cauchy 1859).

- Ertel (1942); Truesdell & Toupin (1960); Beltrami (1871);
- Ohkitani (1993); Kuznetsov & Zakharov (1997);
- Viudez (2001); Bauer (2000).

Two important consequences of Ertel's Theorem

(i) Take $\mu = \rho$, the fluid density, which satisfies $D\rho/Dt = 0$ (Boussinesq). Then

$$\frac{D}{Dt}(\boldsymbol{\omega}\cdot\nabla\rho) = \boldsymbol{\omega}\cdot\nabla\left(\frac{D\rho}{Dt}\right) = 0$$

$\boldsymbol{\omega}\cdot\nabla\rho$ is the (conserved) potential vorticity – very important in GFD :
– see Hoskins, McIntyre, & Robertson (1985).

(ii) $\mu = \boldsymbol{u}$: the *vortex stretching vector* $\boldsymbol{\omega}\cdot\nabla\boldsymbol{u} = S\boldsymbol{\omega}$ obeys (Ohkitani 1993)

$$\frac{D(\boldsymbol{\omega}\cdot\nabla\boldsymbol{u})}{Dt} = \boldsymbol{\omega}\cdot\nabla\left(\frac{D\boldsymbol{u}}{Dt}\right) = -P\,\boldsymbol{\omega}$$

where the Hessian matrix of the pressure is defined as

$$P = \{p_{,ij}\} = \left\{\frac{\partial^2 p}{\partial x_i\,\partial x_j}\right\}.$$

Express 3D Euler as a Lagrangian evolution equation

Consider the class of Lagrangian evolution equations for a 3-vector $\boldsymbol{w}(\boldsymbol{x}, t)$

$$\frac{D\boldsymbol{w}}{Dt} = \boldsymbol{a}(\boldsymbol{x}, t) \qquad\qquad \frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla$$

transported by a divergence-free velocity field $\boldsymbol{u}$, where $\boldsymbol{a}$ satisfies

$$\frac{D\boldsymbol{a}}{Dt} = \boldsymbol{b}(\boldsymbol{x}, t)\,.$$

Using Ertel's Theorem, for Euler we identify the quartet of vectors:

$$\{\boldsymbol{u},\, \boldsymbol{w},\, \boldsymbol{a},\, \boldsymbol{b}\} \equiv (\boldsymbol{u},\, \boldsymbol{\omega},\, S\boldsymbol{\omega},\, -P\boldsymbol{\omega})$$

The $\operatorname{div}\boldsymbol{u} = 0$ constraint becomes the nonlocal relation

$$Tr\, P = \tfrac{1}{2}\omega^2 - Tr\, S^2\,.$$

Quartets $\{\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{a}, \boldsymbol{b}\}$ for Euler & his brothers/sisters

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla\,, \qquad \text{ideal MHD (Elsasser)}\begin{cases}\frac{D^{\pm}}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{v}^{\pm}\cdot\nabla \\ \boldsymbol{v}^{\pm} = \boldsymbol{u} \pm \boldsymbol{B}\end{cases}$$

System	$\boldsymbol{u}$	$\boldsymbol{w}$	$\boldsymbol{a}$	(Ertel) $\boldsymbol{b}$	BKM
incom Euler	$\boldsymbol{u}$	$\boldsymbol{\omega}$	$S\boldsymbol{\omega}$	$-P\boldsymbol{\omega}$	$\int_0^T \|\boldsymbol{\omega}\|_\infty\,dt$
barotr Euler	$\boldsymbol{u}$	$\boldsymbol{\omega}/\rho$	$(\boldsymbol{\omega}/\rho)\cdot\nabla\boldsymbol{u}$	$-(\omega_j/\rho)\partial_j(\rho\partial_i p)$	$\cdots$
MHD	$\boldsymbol{v}^{\pm}$	$\boldsymbol{v}^{\mp}$	$\boldsymbol{B}\cdot\nabla\boldsymbol{v}^{\mp}$	$-P\boldsymbol{B}$	$\int_0^T(\|\boldsymbol{\omega}\|_\infty + \|\boldsymbol{J}\|_\infty)\,dt$
Mixing	$\boldsymbol{u}$	$\boldsymbol{\delta\ell}$	$\boldsymbol{\delta\ell}\cdot\nabla\boldsymbol{u}$	$-P\boldsymbol{\delta\ell}$	$\int_0^T(\|\boldsymbol{\omega}\|_\infty + \|\boldsymbol{\delta\ell}\|_\infty)\,dt$

With $\boldsymbol{J} = \text{curl}\,\boldsymbol{B}$, the BKM-result for ideal MHD,

$$\int_0^T (\|\boldsymbol{\omega}\|_\infty + \|\boldsymbol{J}\|_\infty)\,dt < \infty\,,$$

is due to Caflisch, Klapper & Steel (1998).

Application of quaternions to 'Euler quartets'

Lord Kelvin (William Thompson) said :

> **"Quaternions came from Hamilton after his best work had been done, & though beautifully ingenious, they have been an unmixed evil to those who have touched them in any way"**

O'Connor & Robertson (1998), http://www-groups.dcs.st-and.ac.uk/ history/Mathematicians/Hamilton.html

Kelvin was wrong because quaternions are now used in :

- the avionics & robotics industries to track objects undergoing sequences of tumbling rotations ; see *"Quaternions & rotation Sequences: a Primer with Applications to Orbits, Aerospace & Virtual Reality"*, by J. B. Kuipers, Princeton Univ. Press, (1999).
- **the computer animation/vision industry ;** see *"Visualizing quaternions"*, by A. J. Hanson, M. K. Elsevier (2006).

What are quaternions? (Hamilton 1843)

Quaternions are constructed from a scalar p & a 3-vector $\boldsymbol{q}$ by forming the tetrad

$$\mathfrak{p} = [p,\, \boldsymbol{q}] = pI - \boldsymbol{q}\cdot\boldsymbol{\sigma}\,, \qquad \boldsymbol{q}\cdot\boldsymbol{\sigma} = \sum_{i=1}^{3} q_i\,\sigma_i$$

based on the Pauli spin matrices that obey the relations $\sigma_i\sigma_j = -\delta_{ij} - \epsilon_{ijk}\sigma_k$

$$\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Thus quaternions obey the multiplication rule (associative but non-commutative)

$$\mathfrak{p}_1 \circledast \mathfrak{p}_2 = [p_1p_2 - \boldsymbol{q}_1\cdot\boldsymbol{q}_2\,,\, p_1\boldsymbol{q}_2 + p_2\boldsymbol{q}_1 + \boldsymbol{q}_1\times\boldsymbol{q}_2]\,.$$

Quaternions have inverses: let $\hat{\mathfrak{p}} = [p,\, \boldsymbol{q}]$ be a unit quaternion with $p^2 + q^2 = 1$ then its inverse is $\hat{\mathfrak{p}}^* = [p,\, -\boldsymbol{q}]$.

Quaternions, Cayley-Klein parameters & Rotations

Let $\hat{\mathfrak{p}} = [p,\, \boldsymbol{q}]$ be a unit quaternion; for a pure quaternion $\mathfrak{r} = [0,\, \boldsymbol{r}]$ the transformation $\mathfrak{r} \to \mathfrak{R} = [0,\, \boldsymbol{R}]$

$$\mathfrak{R} = \hat{\mathfrak{p}} \circledast \mathfrak{r} \circledast \hat{\mathfrak{p}}^* = [0,\, (p^2 - q^2)\boldsymbol{r} + 2p(\boldsymbol{q} \times \boldsymbol{r}) + 2\boldsymbol{q}(\boldsymbol{r} \cdot \boldsymbol{q})] \equiv O(\theta\,, \hat{\boldsymbol{n}})\boldsymbol{r}\,,$$

gives the **Euler-Rodrigues** formula for the rotation $O(\theta\,, \hat{\boldsymbol{n}})$ by an angle θ of $\boldsymbol{r}$ about its normal $\hat{\boldsymbol{n}}$. **Cayley-Klein parameters** ($SU(2)$) are the elements of

$$\hat{\mathfrak{p}} = \pm[\cos\tfrac{1}{2}\theta,\; \hat{\boldsymbol{n}}\sin\tfrac{1}{2}\theta]$$

If $\hat{\mathfrak{p}} = \hat{\mathfrak{p}}(t)$ then

$$\dot{\mathfrak{R}}(t) = (\dot{\hat{\mathfrak{p}}} \circledast \hat{\mathfrak{p}}^*) \circledast \mathfrak{R} - ((\dot{\hat{\mathfrak{p}}} \circledast \hat{\mathfrak{p}}^*) \circledast \mathfrak{R})^*\,,$$

$$\boxed{\dot{\boldsymbol{R}} = \boldsymbol{\Omega}_0(t) \times \boldsymbol{R} \qquad \text{as in rigid body result}}$$

The angular velocity is $\boldsymbol{\Omega}_0(t) = 2\mathrm{Im}\,(\dot{\hat{\mathfrak{p}}} \circledast \hat{\mathfrak{p}}^*)$: (see Whittaker (1944) & Marsden & Ratiu 2003).

It don't mean a thing if it ain't got that swing!

(Duke Ellington)

Consider the general Lagrangian evolution equation for a 3-vector $\boldsymbol{w}$ such that

$$\frac{D\boldsymbol{w}}{Dt} = \boldsymbol{a}(\boldsymbol{x},\, t) \qquad\qquad \frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla$$

transported by a velocity field $\boldsymbol{u}$. Define the scalar α_a the 3-vector $\boldsymbol{\chi}_a$ as

$$\alpha_a = |\boldsymbol{w}|^{-1}(\hat{\boldsymbol{w}}\cdot\boldsymbol{a})\,, \qquad\qquad \boldsymbol{\chi}_a = |\boldsymbol{w}|^{-1}(\hat{\boldsymbol{w}}\times\boldsymbol{a})\,,$$

for $|\boldsymbol{w}| \neq 0$. The parallel/perp decomposition $\boldsymbol{a} = \alpha_a\boldsymbol{w} + \boldsymbol{\chi}_a\times\boldsymbol{w}$ means that

$$\boxed{\frac{D|\boldsymbol{w}|}{Dt} = \alpha_a|\boldsymbol{w}|\,, \qquad\qquad \frac{D\hat{\boldsymbol{w}}}{Dt} = \boldsymbol{\chi}_a\times\hat{\boldsymbol{w}}\,.}$$

- α_a **is the 'growth rate'** (Constantin 1994).
- χ_a **is the 'swing' rate :** $\chi_a = 0$ **when** w **and** a **align.**
- **For Euler when** $w = \omega$ **aligns with** $S\omega$ **(straight tube/sheet)** $\Rightarrow \chi = 0$**.**

How quaternions apply to quartets

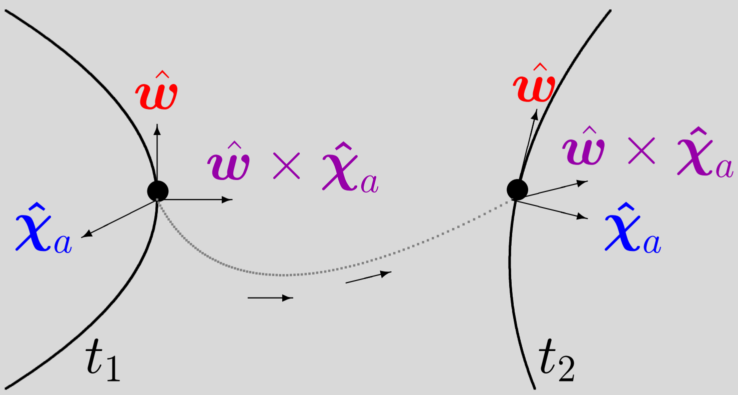
Based on the quartet $\{\boldsymbol{u},\, \boldsymbol{w},\, \boldsymbol{a},\, \boldsymbol{b}\}$ define the **quaternions**

$$\mathfrak{q}_a = [\alpha_a,\, \boldsymbol{\chi}_a] \qquad\qquad \mathfrak{w} = [0,\, \boldsymbol{w}]$$

The parallel/perp decomposition allows us to re-write $D\boldsymbol{w}/Dt = \boldsymbol{a}$ as

$$\frac{D\mathfrak{w}}{Dt} = \mathfrak{q}_a \circledast \mathfrak{w}\,,$$

with the ortho-normal "quaternion frame" as: $\{\hat{\boldsymbol{w}}\,,\ \ \hat{\boldsymbol{\chi}}_a\,,\ \ (\hat{\boldsymbol{w}} \times \hat{\boldsymbol{\chi}}_a)\}$



The frame orientation is a visual diagnostic in addition to the path.

Result : (JDG/Holm 06) If the members of the quartet $\{\boldsymbol{u},\ \boldsymbol{w},\ \boldsymbol{a},\ \boldsymbol{b}\}$ satisfy

$$\frac{D\boldsymbol{w}}{Dt} = \boldsymbol{a}(\boldsymbol{x},\, t) \qquad\qquad \frac{D\boldsymbol{a}}{Dt} = \boldsymbol{b}(\boldsymbol{x},\, t)$$

(i) $\mathfrak{q}_a$ and $\mathfrak{q}_b$ satisfy the Riccati equation ($|\boldsymbol{w}| \neq 0$),

$$\frac{D\mathfrak{q}_a}{Dt} + \mathfrak{q}_a \circledast \mathfrak{q}_a = \mathfrak{q}_b\,;$$

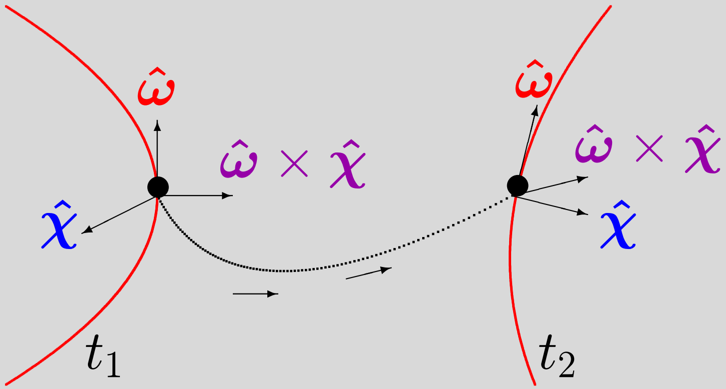
(ii) the ortho-normal frame $\{\hat{\boldsymbol{w}}\,,\ \hat{\boldsymbol{\chi}}_a\,,\ (\hat{\boldsymbol{w}} \times \hat{\boldsymbol{\chi}}_a)\}$ satisfies

$$\frac{D\hat{\boldsymbol{w}}}{Dt} = \boldsymbol{\mathcal{D}}_{ab} \times \hat{\boldsymbol{w}}\,,$$
$$\frac{D(\hat{\boldsymbol{w}} \times \hat{\boldsymbol{\chi}}_a)}{Dt} = \boldsymbol{\mathcal{D}}_{ab} \times (\hat{\boldsymbol{w}} \times \hat{\boldsymbol{\chi}}_a)\,,$$
$$\frac{D\hat{\boldsymbol{\chi}}_a}{Dt} = \boldsymbol{\mathcal{D}}_{ab} \times \hat{\boldsymbol{\chi}}_a\,,$$

where the Darboux angular velocity vector $\boldsymbol{\mathcal{D}}_{ab}$ is defined as

$$\boldsymbol{\mathcal{D}}_{ab} = \boldsymbol{\chi}_a + \frac{c_b}{\chi_a}\hat{\boldsymbol{w}}\,, \qquad\qquad c_b = \hat{\boldsymbol{w}} \cdot (\hat{\boldsymbol{\chi}}_a \times \boldsymbol{\chi}_b)\,.$$

3-dim Euler: frame dynamics of an Euler fluid particle



Quartet : $(\boldsymbol{u},\, \boldsymbol{\omega},\, S\boldsymbol{\omega},\, -P\boldsymbol{\omega})$

$$\boxed{\frac{D\mathfrak{q}}{Dt} + \mathfrak{q} \circledast \mathfrak{q} + \mathfrak{q}_p = 0\,, \qquad TrP = \tfrac{1}{2}\omega^2 - TrS^2\,.}$$

Stationary values – **Burgers' vortices** (Moffatt, Kida, Ohkitani 1994).

$$\alpha = \gamma_0 = \text{const}\,, \qquad \boldsymbol{\chi} = 0\,, \qquad \alpha_p = -\gamma_0^2$$

Off-diagonal elements of P change rapidly near regions of high curvature across which χ_p and α_p change rapidly.

Restricted Euler equations: modelling the Hessian P

The gradient matrix $M_{ij} = \partial u_j/\partial x^i$ satisfies $(\mathrm{tr}\, P = -\mathrm{tr}\,(M^2))$

$$\frac{DM}{Dt} + M^2 + P = 0\,, \qquad \mathrm{tr}\, M = 0\,.$$

Several attempts have been made to model the Lagrangian averaged pressure Hessian by introducing **a constitutive closure.** This idea goes back to

Léorat (1975); Vieillefosse (1984); Cantwell (1992)

who assumed that the Eulerian pressure Hessian P is isotropic. This results in the *restricted Euler equations* with

$$P = -\tfrac{1}{3} I\, \mathrm{tr}\,(M^2)\,, \qquad \mathrm{tr}\, I = 3\,.$$

- **Constantin's distorted Euler equations (1986) :** Euler can be written as

$$\frac{\partial M}{\partial t} + M^2 + Q(t)Tr(M^2) = 0\,,$$

where $Q(t)$ is a matrix $Q_{ij} = R_iR_j$ with the Riesz transforms $R_i = (-\Delta)^{-1/2}\partial_i$.
'Distorted Euler equns' appear by replacing $Q(t)$ by $Q(0) \Rightarrow$ rigorous blow up.

• **Tetrad model** of Chertkov, Pumir & Shraiman (1999), recently developed by Chevillard & Meneveau (2006). Underlying its mean flow features is the assumption that the Lagrangian pressure Hessian is isotropic.

For P to transform as a Riemannian metric and satisfy $\mathrm{tr}\,P = -\mathrm{tr}\,(M^2)$

$$P = -\frac{G}{\mathrm{tr}\,G}\mathrm{tr}\,(M^2)\,, \qquad \mathsf{G}(t) = I \quad \Rightarrow \quad \text{restricted Euler}\,.$$

$\mathrm{tr}\,P = -\mathrm{tr}\,(M^2)$ is satisfied for *any* choice of $G = G^T$

$$P = -\left[\sum_{\beta=1}^{N} c_\beta \frac{\mathsf{G}_\beta}{\mathrm{tr}\,\mathsf{G}_\beta}\right]\mathrm{tr}\,(M^2)\,, \qquad \text{with} \qquad \sum_{\beta=1}^{N} c_\beta = 1\,,$$

so long as an evolutionary flow law is provided for each of the symmetric tensors $\mathsf{G}_\beta = \mathsf{G}_\beta^T$ with $\beta = 1, \ldots, N$. Any choice of flow laws for G would also determine the evolution of the driving term $\mathfrak{q}_b$ in the Riccati equation.

Related talks/posters/discussions at this meeting

1. For analytic work on *The Euler blow-up problem* see the talk of Peter Constantin on Thursday at 09.30–10.00.
2. For numerical work see:
   - the poster of Miguel Bustamente & Bob Kerr, Tuesday 17:00-18:00.
   - the talk *Blow-up or no blow-up? The interplay between theory and numerics* by Tom Hou on Thursday at 10.00–10.30.
   - the talk *A geometrical study of 3D incompressible Euler flows with Clebsch potentials* by K. Ohkitani on Thursday at 11:00-11:30.
3. **Discussion Session:** *Singularities: what do we care?* moderated by Claude Bardos & Edriss Titi, Thursday 17:00-18:00.
4. For Onsager's conjecture *Dissipative anomalies in singular Euler flows*, see the talk by Greg Eyink, Thursday 08:30-09:20.
5. For work on vortex sheets see the talk of Sijue Wu on Wednesday 15:50-16:20 *Recent progress in the mathematical analysis of vortex sheets*.

References

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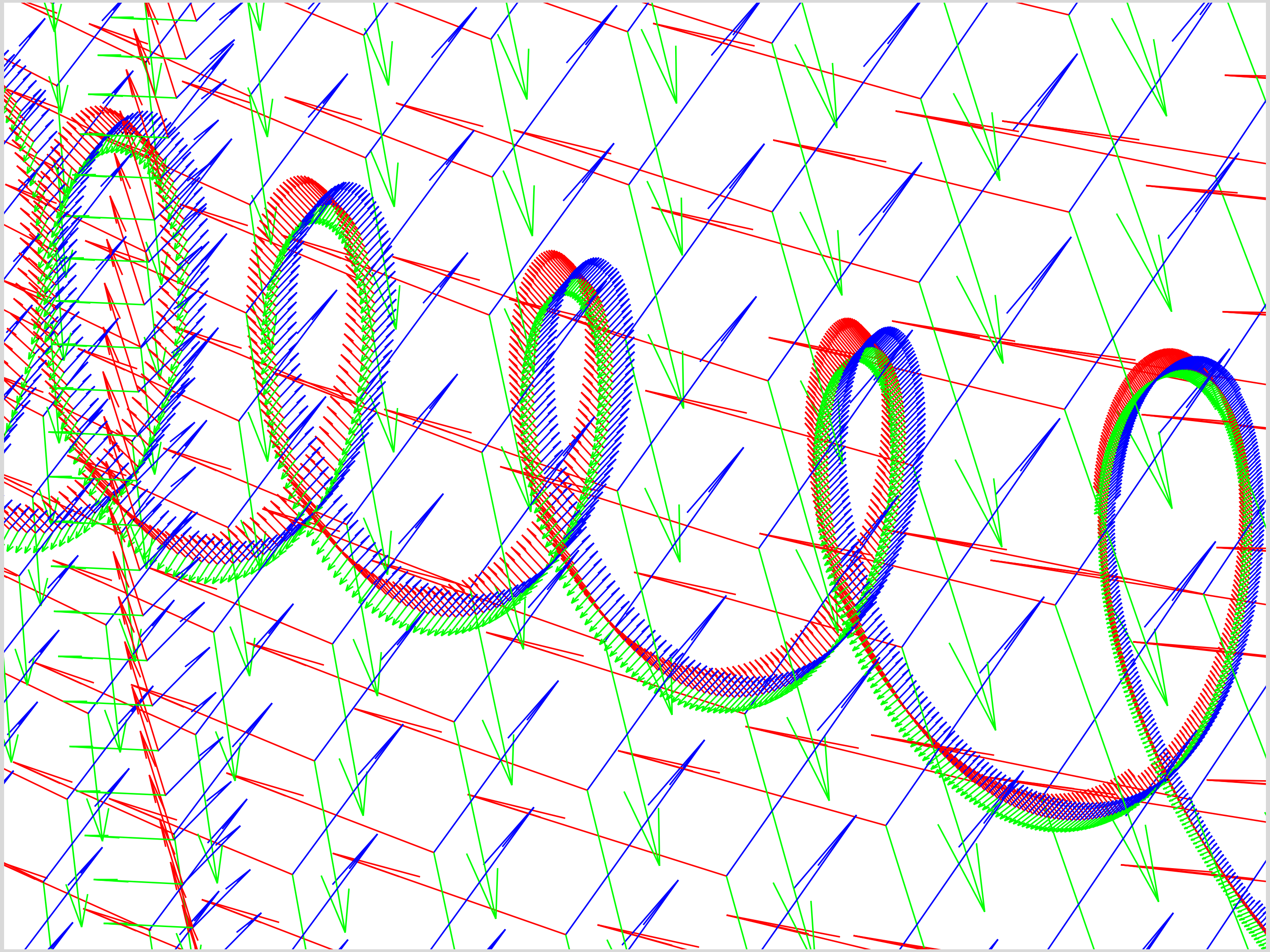

Figure 1: $\boldsymbol{u}$-**field is Arter-flow: Computation by Matthew Dixon.**