the whole refiftance, let $a=y$, then $2\mu\pi n v \times \frac{1}{4} a^2 = \frac{\mu\pi n v a^2}{2}$; but πa^2 = a great circle of a fphere $=c^2$, which being fubftituted in the foregoing expreffion, gives the refiftance $= \frac{\mu n c^2 v}{2}$; but when fuch a circle or a cylinder of this thicknefs moves forward in a fluid in the direction of its length, its refiftance has been found $= u n c^2 v$; whence it is evident, that the refiftance of a globe is juft half the refiftance of a cylinder of the fame thicknefs, if it moves in the fame fluid as the globe in the direction of its axis, and with the fame velocity.

R E M A R K III.

Such a fluid as has been hitherto confidered is not to be found, confequently the refiftance of fluids which actually exifts, to bodies moving therein muft be different from what was found in the forgoing remarks. We fhall confine ourfelves chiefly to the air, whofe refiftance is the object of the prefent enquiry. We are firft to obferve, that the air is a compreffed fluid, and preffes equally on all parts of a body placed therein, fo that it is impoffible that this preffure can produce motion in a body which is at reft, but the cafe is different with refpect to a body in motion, where the impulfe of the ftroke againft the particles of the fluid muft be taken into the account, as well as the preffure. Indeed, if the circumambient fluid continued every where round the body of the fame weight, the diminution of the body's motion would proceed from the force of the ftroke only; and this is nearly the cafe in flow motions. But if the body move with great celerity it puts the air into a very fenfible motion,

motion, whereby the preffure is altered from what it was, and remains no longer the fame on every part of the body, and therefore the ftate of the body is altered, not only by the refiftance of the air, but alfo by its preffure. Particular regard is to be had to the hinder part of the moving body, for when it was at reft it was juft as much preffed forward as backward. But when the body has acquired a very quick motion, fo that the air cannot follow it clofely, there can be fcarcely any preffure on the hinder part, but the preffure on the fore-part is not caft off nor avoided in the fame manner, and therefore the refiftance on that part muft be confiderably increafed.

Hence it may be concluded, that if the velocity be diminifhed, the preffure on the fore-part will ftill be greater than on the hind-part; and, therefore, the refiftance will continue in this cafe greater than what the body fuffers from the ftroke only, becaufe the air incumbent on the fore-part of the body is heavier than that behind. There is a remarkable circumftance in air different from water, or other fluids; it is this, that it may be compreffed into a much leffer fpace, or expanded into a much greater than what it occupies in its natural ftate, and therefore it may vary greatly with refpect to its denfity. If, therefore, a body moves with great celerity through the air, and drives this before it, it is evident, that the air will be denfer before, and rarer behind it, and for this reafon the preffure of the air will be greateft on the fore-part, and will confpire with the refiftance occafioned by the greater number of the particles which oppofe the body's motion towards diminifhing its celerity; and fince the fluid becomes denfer before the body, as its celerity increafes, it is fufficiently evident, that the law of refiftance in very fwift motions is greater than that

 which

which is commonly received, agreeable to Mr. Robins's aſſertion.

Hence it appears, that a body moving in the air is obſtructed by two forces; the one ariſes from the ſtroke againſt the particles of the fluid, which is properly called reſiſtance, and the other from the unequal preſſure. Although both theſe forces muſt be taken into the account in determining the body's motion, yet it will be proper to conſider them ſeparately, as they proceed from different cauſes. We will firſt conſider the force of the ſtroke, by which the body looſes as much motion as it communicates to the particles of air, for juſt ſo much force as is requiſite to give motion to the particles of air, is exerted againſt the body towards the contrary part, that is, towards retarding its motion. Here it is eaſy to conceive, that this reſiſtance muſt be leſs than what it appeared to be by either of the two foregoing remarks. In the laſt caſe, where the elaſtic particles were ſuppoſed to bound away before the body, the expreſſion for the reſiſtance was found = to the weight of a cylinder of the fluid, whoſe length is $=4v$; but, in the firſt caſe, where the particles are ſuppoſed without elaſticity, and to be drove forward with the ſame velocity as the body, the length of ſuch a cylinder is $=2v$. When a body ſtrikes againſt the particles of air, they neither fly off directly before it, nor yet are puſhed on in the ſame direction with the ſame velocity as the body, but glide off ſideways, without acquiring any great motion, unleſs the body moves very ſwiftly.

Since, therefore, the particles of air are not put into ſo great a motion as was ſuppoſed in the two foregoing caſes, the reſiſtance muſt be leſs than the weight of a cylinder of either the length $4v$ or $2v$, and for this reaſon the reſiſt-

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ance of the air to a plane$=c^2$ moving with the velocity $\sqrt{v}$ perpendicular to the air, has been meaſured by the weight of a column of air, whoſe baſe is $=c^2$, and height $=v$. It has been found by experiment, that a body in water ſuffers a reſiſtance equal to the weight of a column of water whoſe height is $=v$, and ſince the particles of water and thoſe of air give way to a moving body in the ſame manner, it has been concluded, that the reſiſtance of both theſe fluids is to be treated of in the ſame manner.

In order to attain a clearer conception of this ſubject, we may obſerve, that when a body moves in a fluid with a given velocity, it is acted on by the ſame force as if it ſtood ſtill, and the fluid flowed upon it with the ſame velocity. Let us ſuppoſe a veſſel full of water with a hole in the bottom, and this hole to be preſſed cloſe with a finger, then ſhall the finger ſuſtain a preſſure equal to that of a column of water, whoſe baſe is equal to the hole, and height equal to that of the water in the veſſel. If now the finger be removed to a very ſmall diſtance, ſo that the water may ſpout freely upon it, one would be apt to conclude, that the finger ſuſtained the action of the ſame force as before. But the water ſpouts out with a velocity a body would acquire by falling through a height equal to that of the water in the veſſel; and, therefore, the force of the water-ſpout upon the finger will be = the weight of a column of water, whoſe baſe is the hole and height the ſame as a body muſt fall through to acquire the velocity of the water; and the foregoing opinion is ſtrengthened, namely, that the reſiſtance as well in air as water is equal to the weight of a cylinder of the fluid, whoſe height is v, namely, that by falling through which the velocity is acquired.

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We ſhall now diſcuſs this ſubject more particularly, by the help of the foregoing principles, namely, that the reſiſtance is as the force requiſite for communicating ſo much motion to the fluid as is produced by the motion of the body.

Let CD (Fig. 14.) be at reſt, and the fluid move againſt it in the direction A B with the velocity $\sqrt{b}$, or that which a body would acquire by falling through the ſpace b. It is evident, that if the particles of the fluid were to continue their motion without obſtruction, the body would be no way affected by ſuch a motion. But ſince the particles of the fluid upon approaching the body are forced to diverge, and undergo an alteration both in the velocity and direction, therefore the body muſt ſuſtain the action of a force ſufficient to produce both theſe changes. Let the fluid approach the obſtacle, and at A a run with the velocity $\sqrt{b}$, and there be forced to diverge ſideways in ſome other direction as A a M m. Now if a ray or ſmall ſtream of the fluid be ſuppoſed to run in the canal A a M m, in this caſe not only the direction is changed, but the canal becomes wider or narrower, according as the velocity of the ſtream diminiſhes or increaſes. Let the ſection of the canal A a $= a$ be infinitely ſmall, and let the whole fluid be ſuppoſed to conſiſt of an infinite number of ſuch ſmall ſtreams, and that each move in a ſeparate canal. Let the ſection M $m = z$, and the velocity of the fluid matter at M $m = \sqrt{v}$, now ſince the velocity of a fluid moving through any canal or pipe, is inverſely as the ſection of the ſame, we have $a : z :: \sqrt{v} : \sqrt{b}$, or $a\sqrt{b} = z\sqrt{v}$. Draw an axis A P perpendicular to A B, and put A P $= x$, P M $= y$. Let QN be parallel and infinitely

infinitely near to P M, ſo ſhall PQ $=$ MO $= \dot{x}$, O N $= \dot{y}$, MN $= \sqrt{\dot{v}^2+\dot{y}^2}$, and the quantity of the fluid in M N $n\,m$ will be expreſſed by $z\sqrt{\dot{x}^2+\dot{y}^2}$. Let moreover $\dot{y} = p\dot{x}$, then ſhall M N $= \sqrt{\dot{x}^2+\dot{y}^2} = \dot{x}\sqrt{1+p^2}$, and if R be the center of a circle, whoſe curvature is the ſame as that of the canal at M m, we ſhall have the radius of curvature MR $= -\frac{\dot{x}\times\overline{1+p^2}|^{\frac{3}{2}}}{\dot{p}}$ *.

Now to make the particles of the fluid M N $m\,n$ move in the direction of this curve, the force in the direction M R muſt be to the weight of theſe particles as $2\,v$ to $-\frac{\dot{x}\times\overline{1+p^2}|^{\frac{3}{2}}}{\dot{p}}$, that is, (if the weight of the particles be expreſſed by their bulk $z\dot{x}\sqrt{1+p^2}$) $-\frac{\dot{x}\times\overline{1+p^2}|^{\frac{3}{2}}}{\dot{p}} : 2\,v :: z\dot{x}\sqrt{1+p^2} : -\frac{2vz\dot{p}}{1+p^2}$ $=$ the force in the direction M R. Moreover as the width of the canal M m increaſes the velocity diminiſhes; to this end a force is required, which acting in the direction m S, obſtructs the progreſſive motion of the fluid at m, let this force $=$ T, then $z\dot{x}\sqrt{1+p^2}\times\dot{v} = -\,\mathrm{T}\dot{x}\sqrt{1+p^2}$; or T $= -\,z\dot{v}$. Theſe two forces M R and m S are exerted at every point M, to change the courſe of the fluid through the canal A a M m. Now to determine the joint effect of theſe forces we ſhall reſolve each into the conſtant directions BA and AP. The firſt force M R which was $-\frac{2vz\dot{p}}{1+p^2}$, gives the force

* Since $\dot{y} = p\dot{x}$, and $\dot{x}$ conſtant, we have $\ddot{y} = \dot{p}\dot{x}$, and if MN $= \dot{w} = \dot{x}\sqrt{1+p^2}$, then will MR $= -\frac{\dot{w}^3}{\dot{x}\ddot{y}} = \frac{-\dot{x}^3\times\overline{1+p^2}|^{\frac{3}{2}}}{\dot{x}^2\dot{p}} = \frac{-\dot{x}\times\overline{1+p^2}|^{\frac{3}{2}}}{\dot{p}}$.

$-\frac{2vz\dot{p}}{1+p^2\times\sqrt{1+p^2}}$ in the direction BA, and $-\frac{2vzp\dot{p}}{1+p^2\times\sqrt{1+p^2}}$ in the direction A P, the other force m S, which was $-z\dot{v}$, gives $-\frac{zp\dot{v}}{\sqrt{1+p^2}}$ in the direction BA, and $\frac{z\dot{v}}{\sqrt{1+p^2}}$ in the direction PA; therefore the whole force exerted on the particles MNmn in the direction BA is $\frac{-2vz\dot{p}}{1+p^2\times\sqrt{1+p^2}}-\frac{zp\dot{v}}{\sqrt{1+p^2}}$, and in the direction AQ is $\frac{-2vzp\dot{p}}{1+p^2\times\sqrt{1+p^2}}+\frac{z\dot{v}}{\sqrt{1+p^2}}$, but we have already ſhewn that $z=\frac{a\sqrt{b}}{\sqrt{v}}$, conſequently the force which urges the particles in MNmn in the direction BA is $=\frac{-2a\dot{p}\sqrt{bv}}{1+p^2\times\sqrt{1+p^2}}-\frac{ap\dot{v}\sqrt{b}}{\sqrt{v\times 1+p^2}}$, the fluent of this expreſſion is $=\frac{-2ap\sqrt{bv}}{\sqrt{1+p^2}}+C$ = the whole force exerted in the direction BA, to cauſe an alteration in the motion of the fluid matter in the canal AaMm, when the value of C is properly aſſigned. In order to determine the value of C, we are to obſerve, that when AP=0 (in which caſe p is infinitely great) and $v=b$, the whole force muſt vaniſh, put therefore $p=\infty$, and then ſhall $C-2ab=0$, and conſequently $C=2ab$; therefore to produce a change in the motion of the fluid contained in the canal AaMm, a forces is required in the direction $BA=2ab-\frac{2ap\sqrt{bv}}{\sqrt{1+p^2}}=2ab\times\overline{1-\frac{p\sqrt{v}}{\sqrt{b\times 1+p^2}}}$. But $\frac{p}{\sqrt{1+p^2}}$ is the coſine of the angle MNO or the angle mSB, the radius being 1: hence the force is $2ab\times\overline{1-\frac{\sqrt{v}}{\sqrt{b}}\times\text{coſ.}\,m\text{SB}}$

and

and this is the force by which the body CD is impelled in the direction AB; but $2ab$ expreſſes the weight of a cylinder of the fluid matter, whoſe baſe is $=Aa=a$, and height $=2b$; where $\sqrt{b}$ expreſſes the velocity with which the fluid ruſhes againſt the body, or, which comes to the ſame purpoſe, the body againſt the fluid. This force, and conſequently the reſiſtance, depends partly on the direction Sm, in which the particles of the fluid diverge ſideways, and partly on their velocity after the ſtroke.

Hence it appears, that if the fluid matter near the body diverges ſideways, ſo as to make a right angle with its natural direction AB, or that the angle mSB=90°, then ſhall the foregoing force $=2ab$; and if this happens to all the fluid matter which lies in the way of the body, we have the very ſame expreſſion for the reſiſtance as we found in the firſt caſe of the foregoing remark. The ſame thing happens when the fluid loſes all its motion by the ſtroke; but if the fluid matter after the ſtroke return in the ſame direction, and with the ſame velocity, then ſhall the angle mSB=180° and $\sqrt{v}=\sqrt{b}$, therefore the force will be $=4ab$, the ſame as was found in the other caſe of the foregoing remark.

If the velocity and direction of each thread Aa of the fluid which moves againſt the body BD, were known, the force of its action againſt the body might be determined. For this purpoſe, it is not neceſſary to know the width and the curvature every where of the canal AaMm, in which the ſmall ſtream Aa moves, but it is ſufficient if we know it in the laſt point of the canal; ſince the force of the ſtream Aa Mm in the direction AB is expreſſed by $2ab\times\overline{1-\frac{\sqrt{v}}{\sqrt{b}}\times\text{coſ.}\,m\text{SB}}$, or becauſe $\frac{\sqrt{v}}{\sqrt{b}}=\frac{a}{z}$; by $2ab\times$

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$\overline{1 - \frac{a}{z}} \times$ cof. mSB, where z denotes the width of the canal in the laft point M, and the angle mSB is determined by the pofition of the laft increment MNnm. It remains therefore only to fix upon the point which is to be efteemed the laft of the canal; if we go fo far that the fluid may pafs by the body, and attain its firft direction and velocity, then fhall $z = a$, and the angle mSB vanifh, and therefore its cof $= 1$, then fhall the force acting on the body in the direction AB$= 2ab \times \overline{1 - 1} = 0$, and the body fuffers no refiftance. Hence it appears, that for air or water, we are not to take the point of the canal for the laft, where the motion behind the body correfponds exactly with that at the beginning of the canal. In examining into the reafon of this, we are to confider more attentively the origin of this force which acts upon the body; we have feen that the force of the part of the canal AaMm exerted on the body in the direction AB is $= 2ab \times \overline{1 - \frac{a}{z}} \times$ cof. mSB, which confequently will increafe fo long as the angle mSB increafes, that is, as S moves from Aa, or fo long as the canal bends from the body, the force acting on it in the direction AB will increafe.

If the canal change its direction and bend the contrary way, and tend towards the body from M to D (Fig. 15) the angle MSB becomes continually lefs, and the force diminifhes, fo that when Dd becomes parallel to Aa, and of the fame width the force becomes $= 0$.

If the canal takes fuch a courfe it will be neceffary to confider each part AM and MD feparately: the firft part AM bending from, and the other MD towards the body. From the firft AM arifes a force which in the direction AB is

is expreffed by $2ab \times \overline{1 - \frac{a}{z}} \times$ cof. MSB. The other part DM produces a force which is oppofite to the firft, and inclines the body to move back in the direction BA. Now, becaufe no motion can be produced in a body without fome force acting upon it, this laft force can affect the body only when the preffure of the fluid behind it becomes fufficiently ftrong to urge the body forward. Now the preffure of air and water being different on the fore and hind part, that on the fore part being commonly ftrongeft, it appears that the force arifing from the part MD can have little or no effect on the body. Therefore the force arifing from the whole canal AMD, which really acts upon the body, is very nearly the fame as that arifing from the part AM, and may therefore be expreffed by $2ab \times \overline{1 - \frac{a}{z}} \times$ cof. MSA. It appears from hence, that if the fluid was of fuch a ftructure, that the force arifing from the part MD, which urges the body in the direction BA could be fully exerted on the body, it would fuffer nothing from the impulfe of fuch a fluid, and confequently would not be refifted by it. This can happen only in the cafe when the matter is infinitely fluid, and alfo compreffed by an infinite force. Perhaps the fubtile celeftial matter in which the planets and comets move is of this kind, and may be the reafon why no diminution is obferved in their motions. But that the air, water, and other known fluids are not of this nature, appears by their very great refiftance, and fince, in thefe, the part MD which bends towards the body, can have very little effect on its motion, the other part AM, which bends the contrary way exerts its whole force without oppofition, and this is the reafon of its great refiftance.

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But in order to judge more accurately of this reſiſtance, we ſhall conſider a cylinder againſt which a fluid moves with a given velocity. Let O P α Q (Fig. 16.) be half the cylinder, A O Q its axis. Now we ſhall conſider this half only, ſince whatever we find true concerning it will be the ſame of the other half alſo. Let us conceive this cylinder to be placed in a canal, whoſe half width is A H, and let air or water flow through this canal with a given velocity againſt the body. Let us imagine this fluid matter to be divided into an infinite number of ſmall ſtreams AB, BC, CD, DE, &c. and let us conſider what courſe each of theſe will take about the body. It will appear, that they muſt bend nearly as is repreſented in the figure. Let $a, b, c, d, e, f,$ &c. be the points where theſe ſtreams begin to return towards the body. Compute the force of theſe ſtreams to theſe points $a, b, c,$ &c. in the direction A O, the ſum of all theſe forces will give the whole reſiſtance. It appears that the firſt ſtream A A $a\,b$ bends nearly in a right angle, and therefore its reſiſtance will be $= 2\,a\,b$, where a is the ſection and b the ſpace fallen to acquire the velocity of the ſtream. The next ſtream B C $b\,c$ does not bend ſo much, and therefore its force will be leſs, the force of the others C D, D E, E F, &c. become leſs as they are more remote from the center A, till it becomes quite imperceptible; whence it appears, that the whole force muſt be leſs than the weight of a cylinder of the fluid matter of the ſame thickneſs as the body, and height $2\,b$, where b is the ſpace fallen to acquire the velocity of the body.

What is ſaid of a cylinder, or ſuch a body whoſe anterior part is a plane impinging on the particles of the fluid at right angles, may be eaſily applied to the other figures. If the fore-part be not flat, but ſloped or ſharp, it is evident the

the reſiſtance will be leſs than in the foregoing caſe; for the ſtream A B will not bend at right angles, and the curvature of thoſe more remote from A will conſtantly diminiſh as before. We cannot therefore agree with Mr. Robins, when he ſays, that if the tranſverſe ſection remain the ſame, the change of figure will ſcarcely affect the velocity; and I think it probable, that the reſiſtance which bodies of different figures ſuffer in a compreſſed fluid may be determined by the foregoing rule; the only difference is, that the reſiſtance in the preſent caſe is much leſs than in the former, and there is reaſon to preſume, that the reſiſtance of a globe is half that of a cylinder, whoſe tranſverſe ſection is equal to a great circle of the globe; and we find, that if we take the weight of a cylinder of the fluid matter of the ſame thickneſs as the cylindrical body moving forward in the direction of its axis, and height equal to that a body muſt fall through to acquire the velocity of the body for the reſiſtance this ſuppoſition will agree beſt with experiments.

R E M A R K IV.

We have hitherto treated of the reſiſtance of the body ariſing from its ſtroke againſt the particles of the fluid; but, as we have already obſerved, there is a particular circumſtance which may increaſe the reſiſtance, namely, when the fluid matter preſſes the moving body more on the fore than hind part. So long as the body ſuſtains the ſame preſſure on every part, which happens when the motion is ſlow, the body ſuffers no other reſiſtance than what ariſes from the ſtroke of the body againſt the particles of the fluid, which