Principles of the motion of fluids*

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The elements of the theory of the motion of fluids in general are treated here, the whole matter being reduced to this: given a mass of fluid, either free or confined in vessels, upon which an arbitrary motion is impressed, and which in turn is acted upon by arbitrary forces, to determine the motion carrying forward each particle, and at the same time to ascertain the pressure exerted by each part, acting on it as well as on the sides of the vessel. At first in this memoir, before undertaking the investigation of these effects of the forces, the Most Famous Author[1] carefully evaluates all the possible motions which can actually take place in the fluid. Indeed, even if the individual particles of the fluid are free from each other, motions in which the particles interpenetrate are nevertheless excluded, since we are dealing with fluids that do not permit any compression into a narrower volume. Thus it is clear that an arbitrary small portion of fluid cannot receive a motion other than the one which constantly conserves the same volume; even though meanwhile the shape is changed in any way. It would hold indeed, as long as no elementary portion would be compressed at any time into a smaller volume; furthermore[2] if the portion expanded into a larger volume, the continuity of the particles was violated, these were dispersed and no longer clung together, such a motion would no longer pertain to the science of the motion of fluids; but individual droplets would separately perform their motion. Therefore, this case being excluded, motion of the fluids must be restricted by this rule that each small portion must retain for ever the same volume; and this principle restricts the general expressions of motion for elements of the fluid. Plainly, considering an arbitrary small portion of the fluid, its individual points have to be carried by such a motion that, when at a moment of time they arrive at the next location, until then they occupy a volume equal to the previous one; thus if, as usual, the motion of a point is decomposed parallel to fixed orthogonal directions, it is necessary that a certain established relation hold between these three velocities, which the Author has determined in the first part.

In the second part the author proceeds to the determination of the motion of a fluid produced by arbitrary forces, in which matter the whole investigation reduces to this that the pressure with which the parts of the fluid at each point act upon one another shall be ascertained; which pressure is denoted most conveniently, as customary for water, by a certain height; this is to be understood thus, that the each element of the fluid sustains a pressure the same as if were pressed by a heavy column of the same fluid, whose height is equal to that amount. Thus, in such way in each point of the fluid the height referring to the state of the pressure will be given; since it is not equal to the one in the neighbourhood, it will perturb the motion of the elements. But this pressure depends as well on the forces acting on each element of the fluid, as on those, acting in the whole mass; thus, by the given forces, the pressure in each point and thereupon the acceleration of each element – or its retardation – can be assigned for the motion, all which determinations are expressed by the author through differential formulas. But, in fact, the full development of these formulas mostly involves the greatest difficulties. But nevertheless this whole theory has been reduced to pure analysis, and what remains to be completed in it depends solely upon subsequent progress in Analysis. Thus it is far from true that purely analytic researches are of no use in applied mathematics; rather, important additions in pure analysis are now required.

I. FIRST PART

1. Since liquid substances differ from solid ones by the fact that their particles are mutually independent of each other, they can also receive most diverse motions; the motion performed by an arbitrary particle of the fluid is not determined by the motion of the remaining particles to the point that it cannot move in any other way. The matter is very different in solid bodies, which, if they were inflexible, would not undergo any change in their shape; in whatsoever way they be moved, each of their particles would constantly keep the same location and distance with respect to other particles; it thus follows that, the motion of two or, if necessary, three of all the particles being known, the motion of any other particle can be defined; furthermore the motion of two or three particles of such a body cannot be chosen at will, but must be constrained in such a way that these particles preserve constantly their positions with respect to each other.[3]

2. But if, moreover, solid bodies are flexible, the motion of each particle is less constrained: because of bending, the distance as well as the relative position of each particle can

*This is an English adaptation by Walter Pauls of Euler's memoir 'Principia motus fluidorum' (Euler, 1756–1757). Updated versions of the translation may become available at `www.oca.eu/etc7/EE250/texts/euler1761eng.pdf`. For a detailed presentation of Euler's fluid dynamics papers, cf. Truesdell, 1954, which has also been helpful for this translation. Euler's work is discussed in the perspective of eighteenth century fluid dynamics research by Darrigol and Frisch, 2008. The help of O. Darrigol, U. Frisch, G. Grimberg and G. Mikhailov is also acknowledged.
Explanatory footnotes and references have been supplied where necessary; Euler's memoir had neither footnotes nor a list of references.

[1] Summaries, which at that time were not placed at the beginning of the corresponding paper, were published under the responsibility of the Academy; the presence of the words "Most Famous Author", rather common at the time, cannot be taken as evidence that Euler usually referred to himself in this way.

[2] In the original, we find "verum quoniam"; the litteral translation "since indeed" does not seem logically consistent.

[3] Here Euler refers to the motion of rigid solid bodies treated previously in Euler, 1750.

be subject to change. However, the manner itself of bending constitutes a certain law which various particles of such a body have to obey in their motion: certainly what has to be taken care of is that the parts that experience in their neighbourhood such a strong bending with respect to each other are neither torn apart from the inside nor penetrate into each other. Indeed, as we shall see, impenetrability is demanded for all bodies.

3. In fluid bodies, whose particles are united among themselves by no bond, the motion of each particle is much less restricted: the motion of the remaining particles is not determined from the motion of any number of particles. Even knowing the motion of one hundred particles, the future motion permitted to the remaining particles still can vary in infinitely many ways. From which it is seen that the motion of these fluid particles plainly does not depend on the motion of the remaining ones, unless it be enclosed by these so that it is constrained to follow them.

4. However, it cannot happen that the motion of all particles of the fluid suffers no restrictions at all. Furthermore, one cannot at will invent a motion that is conceived to occur for each particle. Since, indeed, the particles are impenetrable, it is immediately clear that a motion cannot be maintained in which some particles go through other particles and, accordingly, penetrate each other: also, because of this reason such motion certainly cannot be conceived to occur in the fluid. Therefore, infinitely many motions must be excluded; after their determination the remaining ones are grouped together. It is seen worthwhile to define them more accurately regarding the property which distinguishes them from the previous ones.

5. But before the motion by which the fluid is agitated at any place can be defined, it is necessary to see how every motion, which can definitely be maintained in this fluid, be recognized: these motions, here, I will call possible, which I will distinguish from impossible motions which certainly cannot take place. We must then find what characteristic is appropriate to possible motions, separating them from impossible ones. When this is done, we shall have to determine which one of all possible motions in a certain case ought actually to occur. Plainly we must then turn to the forces which act upon the water, so that the motion appropriate to them may be determined from the principles of mechanics.

6. Thus, I decided to inquire into the character of the possible motions, such that no violation of impenetrability can occur in the fluid. I shall assume the fluid to be such as never to permit itself to be forced into a lesser space, nor should its continuity be interrupted. Once the theory of fluids has been adjusted to fluids of this nature, it will not be difficult to extend it also to those fluids whose density is variable and which do not necessarily require continuity.[4]

7. If, thus, we consider an arbitrary portion in such a fluid, the motion, by which each of its particles is carried has to be set up so that at each time they occupy an equal volume. When this occurs in separate portions, any expansion into a larger volume, or compression into a smaller volume is prohibited. And, if we turn attention to this only property, we can have only such motion that the fluid is not permitted to expand or compress. Furthermore, what is said here about arbitrary portions of the fluid, has to be understood for each of its elements; so that the volume of its elements must constantly preserve its value.

8. Thus, assuming that this condition holds, let an arbitrary motion be considered to occur at each point of the fluid; moreover, given any element of the fluid, consider the brief translations of each of its boundaries. In this manner the volume, in which the element is contained after a very short time, becomes known. From there on, this volume is posed to be equal to the one occupied previously, and this equation will prescribe the calculation of the motion, in so far as it will be possible. Since all elements occupy the same volumes during all periods of time, no compression of the fluid, nor expansion can occur; and the motion is arranged in such a way that this becomes possible.

9. Since we consider not only the velocity[5] of the motion occurring at every point of the fluid but also its direction, both aspects are most conveniently handled, if the motion of each point is decomposed along fixed directions. Moreover, this decomposition is usually carried out with respect to two or three directions:[6] the former is appropriate for decomposition, if the motion of all points is completed in the same plane; but if their motion is not contained in the same plane, it is appropriate to decompose the motion following three fixed axes. Because the latter case is more difficult to treat, it is more convenient to begin the investigation of possible motions with the former case; once this has been done, the latter case will be easily completed.

10. First I will assign to the fluid two dimensions in such a way that all of its particles are now not only found with certainty in the same plane, but also their motion is performed in it. Let this plane be represented in the plane of the table (Fig. 1), let an arbitrary point l of the fluid be considered, its position being denoted by orthogonal coordinates $\mathrm{AL} = x$ and $\mathrm{L}l = y$. The motion is decomposed following these directions, giving a velocity $lm = u$ parallel to the axis AL and $ln = v$ parallel to the other axis AB: so that the true future velocity of this point is $\sqrt{(uu + vv)}$, and its direction with respect to the axis AL is inclined by an angle with the tangent $\frac{v}{u}$.

11. Since the state of motion, presented in a way which suits each point of the fluid, is supposed to evolve, the velocities u and v will depend on the position l of the point and will therefore be functions of the coordinates x and y. Thus, we put upon a differentiation

$$du = \mathrm{L}dx + ldy \quad \text{and} \quad dv = \mathrm{M}dx + mdy,$$

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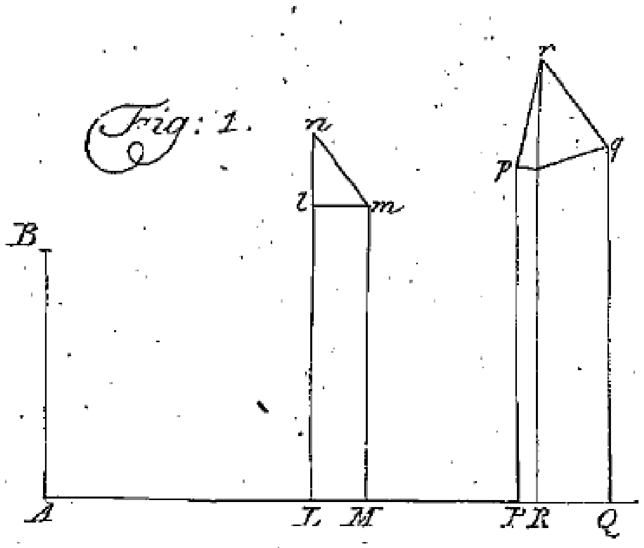
[4] See the English translation of "General laws of the motion of fluids" in these Proceedings.

[5] Meaning here the absolute value of the velocity.

[6] Depending on the dimension: Euler treats both the two- and the three-dimensional cases.



which differential formulas, since they are complete,[7] satisfy furthermore $\frac{d\mathrm{L}}{dy} = \frac{dl}{dx}$ and $\frac{d\mathrm{M}}{dy} = \frac{dm}{dx}$. Here it is noted that in such expression $\frac{d\mathrm{L}}{dy}$, the differential of L itself or dL, is understood to be obtained from the variability with respect to y; in similar manner in the expression dl/dx, for dl the differential of l itself has to be taken, which arises if we take x to vary.

12. Thus, it is in order to be cautious and not to take in such fractional expressions $\frac{d\mathrm{L}}{dy}$, $\frac{dl}{dx}$, $\frac{d\mathrm{M}}{dy}$, and $\frac{dm}{dx}$ the numerators dL, dl, dM, and dm as denoting the complete differentials of the functions L, l, M and m; but they designate such differentials constantly that are obtained from variation of only one coordinate, obviously the one, whose differential is represented in the denominator; thus, such expressions will always represent finite and well defined quantities. Furthermore, in the same way are understood L$= \frac{du}{dx}$, $l = \frac{du}{dy}$, M$= \frac{dv}{dx}$ and $m = \frac{dv}{dy}$; which notation of ratios has been used for the first time by the most enlightened *Fontaine*,[8] and I will also apply it here, since it gives a non negligible advantage of calculation.

13. Since $du = \mathrm{L}dx + ldy$ and $dv = \mathrm{M}dx + mdy$, here it is appropriate to assign a pair of velocities to the point which is at an infinitely small distance from the point l; if the distance of such a point from the point l parallel to the axis AL is dx, and parallel to the axis AB is dy, then the velocity of this point parallel to the axis AL will be $u + \mathrm{L}dx + ldy$; furthermore, the velocity parallel to the other axis AB is $v + \mathrm{M}dx + mdy$. Thus, during the infinitely short time dt this point will be carried parallel to the direction of the axis AL the distance $dt(u + \mathrm{L}dx + ldy)$ and parallel to the direction of the other axis AB the distance $dt(v + \mathrm{M}dx + mdy)$.

14. Having noted these things, let us consider a triangular element lmn of water, and let us seek the location into which it is carried by the motion during the time dt. Let lm be the side parallel to the axis AL and let ln be the side parallel to the axis AB: let us also put $lm = dx$ and $ln = dy$; or let the coordinates of the point m be $x + dx$ and y; the coordinates of the point n be x and $y + dy$. It is plain, since we do not define the relation between the differentials dx and dy, which can be taken negative as well as positive, that in thought the whole mass of fluid may be divided into elements of this sort, so that what we determine for one in general will extend equally to all.

15. To find out how far the element lmn is carried during the time dt due to the local motion, we search for the points p, q and r, to which its vertices, or the points l, m and n are transferred during the time dt. Since

	of point l	of point m	of point n
Velocity w.r.t. AL=	u	$u + \mathrm{L}dx$	$u + ldy$
Velocity w.r.t. AB=	v	$v + \mathrm{M}dx$	$v + mdy$

in the time dt the point l reaches the point p, chosen such that:

$$\mathrm{AP} - \mathrm{AL} = udt \quad \text{and} \quad \mathrm{P}p - \mathrm{L}l = vdt.$$

Furthermore, the point m reaches the point q, such that

$$\mathrm{AQ} - \mathrm{AM} = (u + \mathrm{L}dx)dt \quad \text{and} \quad \mathrm{Q}q - \mathrm{M}m = (v + \mathrm{M}dx)dt.$$

Also, the point n is carried to r, chosen such that

$$\mathrm{AR} - \mathrm{AL} = (u + ldy)dt \quad \text{and} \quad \mathrm{R}r - \mathrm{L}n = (v + mdy)dt.$$

16. Since the points l, m and n are carried to the points p, q and r, the triangle lmn made of the joined straight lines pq, pr and qr, is thought to be arriving at the location defined by the triangle pqr. Because the triangle lmn is infinitely small, its sides cannot receive any curvature from the motion, and therefore, after having performed the translation of the element of water lmn in the time dt, it will conserve the straight and triangular form. Since this element lmn must not be either extended to a larger volume, nor compressed into a smaller one, the motion should be arranged so that the volume of the triangle pqr is rendered to be equal to the area of the triangle lmn.

17. The area of the triangle lmn, being rectangular at l, is $\frac{1}{2}dxdy$, value to which the area of the triangle pqr should be put equal. To find this area, the pair of coordinates of the points p, q and r must be examined, which are:

$$\begin{aligned} &\mathrm{AP} = x + udt; \quad \mathrm{AQ} = x + dx + (u + \mathrm{L}dx)dt; \\ &\mathrm{AR} = x + (u + ldy)dt; \quad \mathrm{P}p = y + vdt \\ &\mathrm{Q}q = y + (v + \mathrm{M}dx)dt, \quad \mathrm{R}r = y + dy + (v + mdy)dt \end{aligned}$$

Then, indeed, the area of the triangle pqr is found from the area of the succeeding trapezoids, so that

$$pqr = \mathrm{P}pr\mathrm{R} + \mathrm{R}rq\mathrm{Q} - \mathrm{P}pq\mathrm{Q}.$$

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[7] Exact differentials.

[8] A paper "Sur le calcul intégral" containing the notation $\frac{df}{dx}$ for the partial derivative of f with respect to x was presented by Alexis Fontaine des Bertins to the Paris Academy of Sciences in 1738, but it was published only a quarter of a century later (Fontaine, 1764). Nevertheless, Fontaine's paper was widely known among mathematicians from the beginning of the 1740s, and, particularly, was discussed in the correspondence between Euler, Daniel Bernoulli and Clairaut; cf. Euler, 1980: 65–246.

Since these trapezoids have a pair of sides parallel and perpendicular to the base AQ, their areas are easily found.

18. Plainly, these areas are given by the expressions

$$\mathrm{P}pr\mathrm{R} = \frac{1}{2}\mathrm{PR}(\mathrm{P}p + \mathrm{R}r)$$
$$\mathrm{R}rq\mathrm{Q} = \frac{1}{2}\mathrm{RQ}(\mathrm{R}r + \mathrm{Q}q)$$
$$\mathrm{P}pq\mathrm{Q} = \frac{1}{2}\mathrm{PQ}(\mathrm{P}p + \mathrm{Q}q)$$

By putting these together we find:

$$\Delta pqr = \frac{1}{2}\mathrm{PQ}.\mathrm{R}r - \frac{1}{2}\mathrm{RQ}.\mathrm{P}p - \frac{1}{2}\mathrm{PR}.\mathrm{Q}q$$

Let us set for brevity

$\mathrm{AQ} = \mathrm{AP} + \mathrm{Q}$; $\mathrm{AR} = \mathrm{AP} + \mathrm{R}$; $\mathrm{Q}q = \mathrm{P}p + q$; and $\mathrm{R}r = \mathrm{P}p + r$,

so that $\mathrm{PQ} = \mathrm{Q}$, $\mathrm{PR} = \mathrm{R}$, and $\mathrm{RQ} = \mathrm{Q} - \mathrm{R}$, and we have $\Delta pqr = \frac{1}{2}\mathrm{Q}(\mathrm{P}p + r) - \frac{1}{2}(\mathrm{Q} - \mathrm{R})\mathrm{P}p - \frac{1}{2}\mathrm{R}(\mathrm{P}p + q)$ or $\Delta pqr = \frac{1}{2}\mathrm{Q}.r - \frac{1}{2}\mathrm{R}.q$.

19. Truly, from the values of the coordinates represented before it follows that

$$\mathrm{Q} = dx + \mathrm{L}dxdt; \quad q = \mathrm{M}dxdt$$
$$\mathrm{R} = ldydt; \quad r = dy + mdydt,$$

upon the substitution of these values, the area of the triangle is obtained

$$pqr = \frac{1}{2}dxdy(1 + \mathrm{L}dt)(1 + mdt) - \frac{1}{2}\mathrm{M}l\,dxdydt^2, \quad \text{or}$$
$$pqr = \frac{1}{2}dxdy(1 + \mathrm{L}dt + mdt + \mathrm{L}mdt^2 - \mathrm{M}ldt^2).$$

This should be equal to the area of the triangle lmn, that is $= \frac{1}{2}dxdy$; hence we obtain the following equation

$$\mathrm{L}dt + mdt + \mathrm{L}mdt^2 - \mathrm{M}ldt^2 = 0 \quad \text{or}$$
$$\mathrm{L} + m + \mathrm{L}mdt - \mathrm{M}ldt = 0.$$

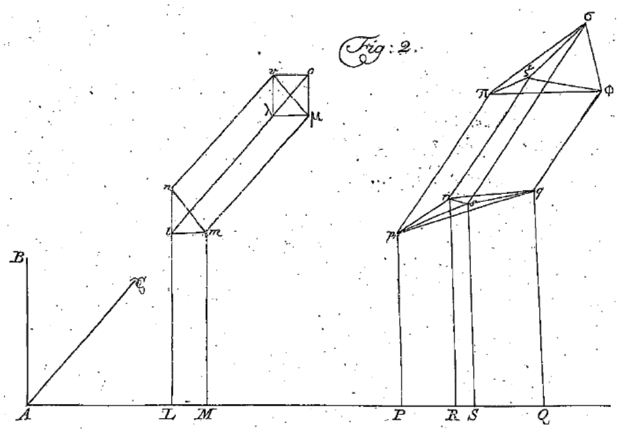
20. Since the terms $\mathrm{L}mdt$ and $\mathrm{M}ldt$ are negligible for finite L and m, we will have the equation $\mathrm{L} + m = 0$. Hence, for the motion to be possible, the velocities u and v of any point l have to be arranged such that after calculating their differentials

$$du = \mathrm{L}dx + ldy, \quad \text{and} \quad dv = \mathrm{M}dx + mdy,$$

one has $\mathrm{L} + m = 0$. Or, since $\mathrm{L} = \frac{du}{dx}$ and $m = \frac{dv}{dy}$, the velocities u and v, which are considered to occur at the point l parallel to the axes AL and AB, must be functions of the coordinates x and y such that $\frac{du}{dx} + \frac{dv}{dy} = 0$, and thus, the criterion of possible motions consists in this that $\frac{du}{dx} + \frac{dv}{dy} = 0$;[9] and unless this condition holds, the motion of the fluid cannot take place.

21. We shall proceed identically when the motion of the fluid is not confined to the same plane. Let us assume, to investigate this question in the broadest sense, that all particles of the fluid are agitated among themselves by an arbitrary motion, with the only law to be respected that neither condensation nor expansion of the parts occurs anywhere: in the same way, we seek which condition should apply to the velocities that are considered to occur at every point, so that motion be possible: or, which amounts to the same, all motions that are opposed to these conditions should be eliminated from the possible ones, this being the criterion of possible motions.

22. Let us consider an arbitrary point of the fluid λ. To represent its location we use three fixed axes AL, AB and AC orthogonal to each other (Fig. 2). Let the triple coordinates parallel to these axes be AL$= x$, L$l = y$ and $l\lambda = z$; which are obtained if firstly a perpendicular λl is dropped from the point λ to the plane determined by the two axes AL and AB; and then a perpendicular lL is drawn from the point l to the axis AL. In this manner the location of the point λ is expressed through three such coordinates in the most general way and can be adapted to all points of the fluid.



23. Whatever the later motion of the point λ, it can be resolved following the three directions $\lambda\mu$, $\lambda\nu$, λo, parallel to the axes AL, AB and AC. For the motion of the point λ we set

the velocity parallel to the direction $\lambda\mu = u$,
the velocity parallel to the direction $\lambda\nu = v$,
the velocity parallel to the direction $\lambda o = w$.

Since these velocities can vary in an arbitrary manner for different locations of the point λ, they will have to be considered as functions of the three coordinates x, y and z. After differentiating them, let us put to proceed

$$du = \mathrm{L}dx + ldy + \lambda dz$$
$$dv = \mathrm{M}dx + mdy + \mu dz$$
$$dw = \mathrm{N}dx + ndy + \nu dz.$$

Henceforth the quantities L, l, λ, M, m, μ, N, n, ν will be functions of the coordinates x, y and z.

[9] This is the two-dimensional incompressibility condition, which in a slightly different form has already been established by d'Alembert, 1752; cf. also Darrigol and Frisch, 2008:§ III.

24. Because these formulas are complete differentials, we obtain as above

$$\frac{d\mathrm{L}}{dy}=\frac{dl}{dx};\quad \frac{d\mathrm{L}}{dz}=\frac{d\lambda}{dx};\quad \frac{dl}{dz}=\frac{d\lambda}{dy}$$
$$\frac{d\mathrm{M}}{dy}=\frac{dm}{dx};\quad \frac{d\mathrm{M}}{dz}=\frac{d\mu}{dx};\quad \frac{dm}{dz}=\frac{d\mu}{dy}$$
$$\frac{d\mathrm{N}}{dy}=\frac{dn}{dx};\quad \frac{d\mathrm{N}}{dz}=\frac{d\nu}{dx};\quad \frac{dn}{dz}=\frac{d\nu}{dy},$$

where it is assumed that the only varying coordinate is that whose differential appears in the denominator.[10]

25. Thus, this point λ will be moved in the time dt by this threefold motion, which is considered to take place at the point X; hence it moves

$$\text{parallel to the axis AL the distance} = udt$$
$$\text{parallel to the axis AB the distance} = vdt$$
$$\text{parallel to the axis AC the distance} = wdt$$

The true velocity of the point λ, denoted by V, which clearly arises from the composition of this triple motion, is given in view of orthogonality of the three directions by $V=\sqrt{(uu+vv+ww)}$ and the elementary distance, which is travelled in time dt through its motion, will be Vdt.

26. Let us consider an arbitrary solid element of the fluid to see whereto it is carried during the time dt; since it amounts to the same, let us assign a quite arbitrary shape to that element, but of a kind such that the entire mass of the fluid can be divided into such elements; to investigate by calculation, let the shape be a right triangular pyramid, bounded by four vertices λ, μ, ν and o, so that for each one there are three coordinates

	of point λ	of point μ	of point ν	of point o
w.r.t. AL	x	$x+dx$	x	x
w.r.t. AB	y	y	$y+dy$	y
w.r.t. AC	z	z	z	$z+dz$

Since the base of this pyramid is $\lambda\mu\nu = lmn = \frac{1}{2}dxdy$ and the hight $\lambda o = dz$, its volume will be $=\frac{1}{6}dxdydz$.

27. Let us investigate, whereto these vertices λ, μ, ν and o are carried during the time dt: for which purpose their three velocities parallel to the directions of the three axes must be considered. The differential values of the velocities u, v and w are given by

Velocity	of point λ	of point μ	of point ν	of point o
w.r.t. AL	u	$u+\mathrm{L}dx$	$u+ldy$	$u+\lambda dz$
w.r.t. AB	v	$v+\mathrm{M}dx$	$v+mdy$	$v+\mu dz$
w.r.t. AC	w	$w+\mathrm{N}dx$	$w+ndy$	$w+odz$

28. If we let the points λ, μ, ν and o be transferred to the points π, Φ, ρ and σ in the time dt, and establish the three coordinates of these points parallel to the axes, the small displacement parallel to these axes will be

$$\begin{aligned}
\mathrm{AP}-\mathrm{A\,L} &= u\,dt\\
\mathrm{AQ}-\mathrm{A\,M} &= (u+\mathrm{L}\,dx)\,dt\\
\mathrm{AR}-\mathrm{A\,L} &= (u+l\ \,dy)\,dt\\
\mathrm{AS}-\mathrm{A\,L} &= (u+\lambda\,dz)\,dt\\
\hline
\mathrm{P}p-\mathrm{L}l &= v\,dt\\
\mathrm{Q}q-\mathrm{M}m &= (v+\mathrm{M}dx)\,dt\\
\mathrm{R}r-\mathrm{L}n &= (v+mdy)\,dt\\
\mathrm{S}s-\mathrm{L}l &= (v+\mu\,dz)\,dt\\
\hline
p\,\pi-l\,\lambda &= w\,dt\\
q\,\Phi-m\mu &= (w+\mathrm{N}\,dx)\,dt\\
r\,\rho-n\nu &= (w+n\,dy)\,dt\\
s\,\sigma-l\,o &= (w+\nu\,dz)\,dt
\end{aligned}$$

Thus the three coordinates for these four points π, Φ, ρ and σ will be

$$\begin{aligned}
\mathrm{AP}&=x+udt; & \mathrm{P}p&=y+vdt;\\
p\pi&=z+wdt\\
\mathrm{RQ}&=x+dx+(u+\mathrm{L}dx)dt; & \mathrm{Q}q&=y+(v+\mathrm{M}dx)dt;\\
q\Phi&=z+(w+\mathrm{N}dx)dt\\
\mathrm{AR}&=x+(u+ldy)dt; & \mathrm{R}r&=y+dy+(v+mdy)dt;\\
r\rho&=z+(w+ndy)dt\\
\mathrm{AS}&=x+(u+\lambda dz)dt; & \mathrm{S}s&=y+(v+\mu dz)dt;\\
s\sigma&=z+dz+(w+\nu dz)dt
\end{aligned}$$

29. Since after time dt has elapsed the vertices λ, μ, ν and o of the pyramid are transferred to the points π, Φ, ρ and σ, $\pi\Phi\rho\sigma$ defines a similar triangular pyramid. Due to the nature of the fluid the volume of the pyramid $\pi\Phi\rho\sigma$ should be equal to the volume of the pyramid $\lambda\mu\nu o$ put forward, that is $\frac{1}{6}dxdydz$. Thus, the whole matter is reduced to determining the volume of the pyramid $\pi\Phi\rho\sigma$. Clearly, it remains a pyramid, if the solid $pqr\pi\Phi\rho\sigma$ is removed from the solid $pqr\pi\Phi\rho\sigma$; the latter solid is a prism orthogonally incident to the triangular basis pqr, and cut by the upper oblique section $\pi\rho\Phi$.

30. The other solid $pqr\pi\Phi\rho\sigma$ can be divided by similarly into three prisms truncated in this manner, namely

$$\text{I. } pqrs\pi\Phi\sigma;\quad \text{II. } prs\pi\rho\sigma;\quad \text{III. } qrs\Phi\rho\sigma$$

This has to be accomplished in such a way that

$$\frac{1}{6}dxdydz = pqs\pi\Phi\sigma + prs\pi\rho\sigma + qrs\Phi\rho\sigma - pqr\pi\Phi\rho.$$

Since such a prism is orthogonally incident to its lower base, and furthermore has three unequal heights, its volume is found by multiplying the base by one third of the sum of these heights.

31. Thus, the volumes of these truncated prisms will be

$$\begin{aligned}
p\,q\,s\,\pi\,\Phi\sigma &= \frac{1}{3}p\,q\,s\,(p\,\pi+q\,\Phi+s\,\sigma)\\
p\,r\,s\,\pi\,\rho\,\sigma &= \frac{1}{3}p\,r\,s\,(p\,\pi+r\,\rho+s\,\sigma)\\
q\,r\,s\,\Phi\rho\,\sigma &= \frac{1}{3}q\,r\,s\,(q\,\Phi+r\,\rho+s\,\sigma)\\
p\,q\,r\,\pi\,\Phi\rho &= \frac{1}{3}p\,q\,r\,(p\,\pi+q\,\Phi+r\,\rho).
\end{aligned}$$

[10] The partial differential notation was so new that Euler had to remind the reader of its definition.

Since $pqr = pqs + prs + qrs$, the sum of the first three prisms will definitely be small, or

$$\frac{1}{6}dxdydz = -\frac{1}{3}p\pi.qrs - \frac{1}{3}q\Phi.prs - \frac{1}{3}r\rho.pqs + \frac{1}{3}s\sigma.pqr,$$

or

$$dxdydz = 2pqr.s\sigma - 2pqs.r\rho - 2prs.q\Phi - 2qrs.p\pi.$$

32. Thus, it remains to define the bases of these prisms: but before we do this, let us put

$$\begin{array}{lll} \mathrm{AQ} = \mathrm{AP} + \mathrm{Q}; & \mathrm{Q}q = \mathrm{P}p + q\,; & q\Phi = p\pi + \Phi; \\ \mathrm{AR} = \mathrm{AP} + \mathrm{R}; & \mathrm{R}r = \mathrm{P}p + r\,; & r\rho = p\pi + \rho\,; \\ \mathrm{AS} = \mathrm{AP} + \mathrm{S}\,; & \mathrm{S}s = \mathrm{P}p + s\,; & s\sigma = p\pi + \sigma\,, \end{array}$$

in order to shorten the following calculations. After the substitution of these values, the terms containing $p\pi$ will annihilate each other, and we shall have

$$dxdydz = 2pqr.\sigma - 2pqs.\rho - 2prs.\Phi$$

so that the value of the bases to be investigated is smaller.

33. Furthermore the triangle pqr is obtained by removing the trapezoid PpqQ from the figure PprqQ, the latter being the sum of the trapezoids PprR and RrqQ; from which it follows that

$$\Delta pqr = \frac{1}{2}\mathrm{PR}(\mathrm{P}p{+}\mathrm{R}r) + \frac{1}{2}\mathrm{RQ}(\mathrm{R}r{+}\mathrm{Q}q) - \frac{1}{2}\mathrm{PQ}(\mathrm{P}p{+}\mathrm{Q}q);$$

or, because of $\mathrm{PR} = \mathrm{R}$; $\mathrm{RQ} = \mathrm{Q} - \mathrm{R}$; and $\mathrm{PQ} = \mathrm{Q}$ we shall have

$$\Delta pqr = \frac{1}{2}\mathrm{R}(\mathrm{P}p - \mathrm{Q}q) + \frac{1}{2}\mathrm{Q}(\mathrm{R}r - \mathrm{P}p) = \frac{1}{2}\mathrm{Q}r - \frac{1}{2}\mathrm{R}q.$$

In the same manner we have

$$\Delta pqs = \frac{1}{2}\mathrm{PS}(\mathrm{P}p + \mathrm{S}s) + \frac{1}{2}\mathrm{SQ}(\mathrm{S}s + \mathrm{Q}q) - \frac{1}{2}\mathrm{PQ}(\mathrm{P}p + \mathrm{Q}q),$$

or

$$\Delta pqs = \frac{1}{2}\mathrm{S}(\mathrm{P}p + \mathrm{S}s) + \frac{1}{2}(\mathrm{Q} - \mathrm{S})(\mathrm{S}s + \mathrm{Q}q) - \frac{1}{2}\mathrm{Q}(\mathrm{P}p + \mathrm{Q}q),$$

from where it follows that:

$$\Delta pqs = \frac{1}{2}\mathrm{S}(\mathrm{P}p - \mathrm{Q}q) + \frac{1}{2}\mathrm{Q}(\mathrm{S}s - \mathrm{P}p) = \frac{1}{2}\mathrm{Q}s - \frac{1}{2}\mathrm{S}q.$$

And finally

$$\Delta prs = \frac{1}{2}\mathrm{PR}(\mathrm{P}p + \mathrm{R}r) + \frac{1}{2}\mathrm{RS}(\mathrm{R}r + \mathrm{S}s) - \frac{1}{2}\mathrm{PS}(\mathrm{P}p + \mathrm{S}s),$$

or

$$\Delta prs = \frac{1}{2}\mathrm{R}(\mathrm{P}p + \mathrm{R}r) + \frac{1}{2}(\mathrm{S} - \mathrm{R})(\mathrm{R}r + \mathrm{S}s) - \frac{1}{2}\mathrm{S}(\mathrm{P}p + \mathrm{S}s)$$

from where it follows that

$$\Delta prs = \frac{1}{2}\mathrm{R}(\mathrm{P}p - \mathrm{S}s) + \frac{1}{2}\mathrm{S}(\mathrm{R}r - \mathrm{P}p) = \frac{1}{2}\mathrm{S}r - \frac{1}{2}\mathrm{R}s.$$

34. After the substitution of these values we will obtain

$$dxdydz = (\mathrm{Q}r - \mathrm{R}q)\sigma + (\mathrm{S}q - \mathrm{Q}s)\rho + (\mathrm{R}s - \mathrm{S}r)\Phi;$$

thus the volume of the pyramid $\pi\Phi\rho\sigma$ will be

$$\frac{1}{6}(\mathrm{Q}r - \mathrm{R}q)\sigma + \frac{1}{6}(\mathrm{S}q - \mathrm{Q}s)\rho + \frac{1}{6}(\mathrm{R}s - \mathrm{S}r)\Phi.$$

From the values of the coordinates presented above in §. 28 follows

$$\begin{array}{lll} \mathrm{Q} = dx + \mathrm{L}dxdt & q = \mathrm{M}dxdt & \Phi = \mathrm{N}dxdt \\ \mathrm{R} = ldydt & r = dy + mdydt & \rho = ndydt \\ \mathrm{S} = \lambda dzdt & s = \mu dzdt & \sigma = dz + \nu dzdt. \end{array}$$

35. Since here we have

$$\begin{aligned} \mathrm{Q}r - \mathrm{R}q &= dxdy(1 + \mathrm{L}dt + mdt + \mathrm{L}mdt^2 - \mathrm{M}ldt^2) \\ \mathrm{S}q - \mathrm{Q}s &= dxdz(-\mu dt - \mathrm{L}\mu dt^2 + \mathrm{M}\lambda dt^2) \\ \mathrm{R}s - \mathrm{S}r &= dydz(-\lambda dt - m\lambda dt^2 + l\mu dt^2) \end{aligned}$$

the volume of the pyramid $\pi\Phi\rho\sigma$ is found to be expressed as

$$\frac{1}{6}dxdydz \left\{ \begin{array}{llll} 1 & +\mathrm{L}\,dt & +\mathrm{L}m\,dt^2 & +\mathrm{L}m\nu\,dt^3 \\ & +mdt & -\mathrm{M}l\,dt^2 & -\mathrm{M}l\nu\,dt^3 \\ & +\nu\,dt & +\mathrm{L}\nu\,dt^2 & -\mathrm{L}n\mu\,dt^3 \\ & & +m\nu\,dt^2 & +\mathrm{M}n\lambda\,dt^3 \\ & & -n\mu\,dt^2 & -\mathrm{N}m\lambda\,dt^3 \\ & & -\mathrm{N}\lambda\,dt^2 & +\mathrm{N}l\mu\,dt^3 \end{array} \right\},$$

which (volume), since it must be equal to that of the pyramid $\lambda\mu\nu o = \frac{1}{6}dxdydz$, will satisfy, after performing a division by dt the following equation[11]

$$\begin{aligned} 0 =& \mathrm{L} + m + \nu + dt(\mathrm{L}m + \mathrm{L}\nu + m\nu - \mathrm{M}l - \mathrm{N}\lambda - n\mu) \\ &+ dt^2(\mathrm{L}m\nu + \mathrm{M}n\lambda + \mathrm{N}l\mu - \mathrm{L}n\mu - \mathrm{M}l\nu - \mathrm{N}l\mu). \end{aligned}$$

36. Discarding infinitely small terms, we get this equation:[12] $\mathrm{L} + m + \nu = 0$, through which is determined the relation between u, v and w, so that the motion of the fluid be possible. Since $\mathrm{L} = \frac{du}{dx}$, $m = \frac{dv}{dy}$ and $\nu = \frac{dw}{dz}$, at an arbitrary point of the fluid λ, whose position is defined by the three coordinates x, y and z, and the velocities u, v and w are assigned in the same manner to be directed along these same coordinates, the criterion of possible motions is such that

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0.$$

This condition expresses that through the motion no part of the fluid is carried into a greater or or lesser space, but perpetually

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[11] This is the calculation to which Euler refers in his later French memoir Euler, 1755.

[12] This is the three-dimensional incompressibility condition.

the continuity of the fluid as well as the identical density is conserved.

37. This property is to be interpreted further so that at the same instant it is extended to all points of the fluid: of course, the three velocities of all the points must be such functions of the three coordinates x, y and z that we have $\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0$: in this way the nature of those functions defines the motion of every point of the fluid at a given instant. At another time the motion of the same points may be howsoever different, provided that at an arbitrary point of time the property holds for the whole fluid. Up to now I have considered the time simply as a constant quantity.

38. If however, we also wish to consider time as variable so that the motion of the point λ whose position is given by the three coordinates AL$= x$, L$l = y$ and $l\lambda = z$ has to be defined after the elapsed time t, it is certain that the three velocities u, v and w depend not only on the coordinates x, y and z but additionally on the time t, that is they will be functions of these four quantities x, y, z and t; furthermore, their differentials are going to have the following form

$$du = \mathrm{L}dx + ldy + \lambda dz + \mathfrak{L}dt;$$
$$dv = \mathrm{M}dx + mdy + \mu dz + \mathfrak{M}dt;$$
$$dw = \mathrm{N}dx + ndy + \nu dz + \mathfrak{N}dt;$$

Meanwhile we shall always have $\mathrm{L} + m + \nu = 0$, having in view that at every arbitrary instant the time t is considered to be constant, or $dt = 0$. Howsoever the functions u, v and w vary with time t, it is necessary that at every moment of time the following holds:

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0.$$

Since the condition expresses that any arbitrary portion of the fluid is carried in a time dt into a volume equal to itself, the same will have to happen, due to the same condition, in the next time interval, and therefore in all the following time intervals.

II. SECOND PART

39. Having presented what pertains to all possible motions, let us now investigate the nature of the motion which can really occur in the fluid. Here, besides the continuity of the fluid and the constancy of its density, we will also have to consider the forces which act on every element of the fluid. When the motion of any element is either non-uniform or varying in its direction, the change of motion must be in accordance with the forces acting on this element. The change of motion becomes known from known forces, and the preceding formulas contain this change; we will now deduce new conditions[13] which single out the actual motion among all those possible up to this point.

40. Let us arrange this investigation in two parts as well; at first let us consider all motions being performed in the same plane. Let AL $= x$, L$l = y$ be, as before, the defining coordinates of the position of an arbitrary point l; now, after the elapsed time t, the two velocities of the point l parallel to the axes AL and AB are u and v: since the variability of time has to be taken into account, u and v will be functions of x, y and t themselves. In respect of which we put

$$du = \mathrm{L}dx + ldy + \mathfrak{L}dt \quad \text{and} dv = \mathrm{M}dx + mdy + \mathfrak{M}dt$$

and we have established above that because of the former condition encountered above, we have $\mathrm{L} + m = 0$.

41. After an elapsed small time interval dt the point l is carried to p, and it has travelled a distance udt parallel to the axis AL, a distance vdt parallel to the other axis AB. Hence, to obtain the increments in the velocities u and v of the point l which are induced during time dt, for dx and dy we must write the distance udt and vdt, from which will arise these true increments of the velocities

$$du = \mathrm{L}udt + lvdt + \mathfrak{L}dt \quad \text{and} \quad dv = \mathrm{M}udt + mvdt + \mathfrak{M}dt.$$

Therefore the accelerating forces, which produce these accelerations are

$$\text{Accel. force w.r.t. AL} = 2(\mathrm{L}u + lv + \mathfrak{L})$$
$$\text{Accel. force w.r.t. AB} = 2(\mathrm{M}u + mv + \mathfrak{M})$$

to which therefore the forces acting upon the particle of water ought to be equal.[14]

42. Among the forces which in fact act upon the particles of water, the first to be considered is gravity; its effect, however, if the plane of motion is horizontal, amounts to nothing. Yet if the plane is inclined, the axis AL following the inclination, the other being horizontal, gravity generates a constant accelerating force parallel to the axis AL, let it be α. Next we must not neglect friction, which often hinders the motion of water, and not a little. Although its laws have not yet been explored sufficiently, nevertheless, following the law of friction for solid bodies, probably we shall not wander too far astray if we set the friction everywhere proportional to the pressure with which the particles of water press upon one another.[15]

43. First, must be brought into the calculation the pressure with which the particles of water everywhere mutually act upon each other, by means of which every particle is pressed together on all sides by its neighbours; and in so far as this pressure is not everywhere equal, to that extent motion is communicated to that particle.[16] The water simply will be put

[13] Here Euler probably has in mind the condition of potentiality, which he will obtain in §§. 47 and 54 for the two-dimensional case and in §. 60 for the three-dimensional case.

[14] The unusual factors of 2 in the previous equations have to do with a choice of units which soon became obsolete; cf. Truesdell, 1954; Mikahailov, 1999.

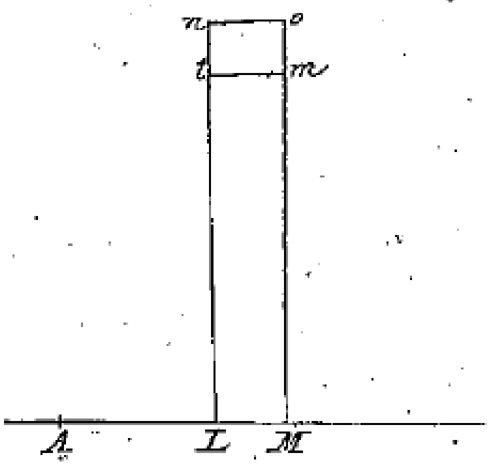
[15] It is actually not clear why Euler takes the friction force proportional to the pressure.

[16] Here Euler makes full use of the concept of internal pressure, cf. Darrigol and Frisch, 2008.

everywhere into a state of compression similar to that which still water experiences when stagnating at a certain depth. This depth is most conveniently employed for representing the pressure at an arbitrary point l of the fluid. Therefore let that height, or depth, expressing the state of compression at l, be p, a certain function of the coordinates x and y, and should the pressure at l vary also with the time, the time will also enter into the function p.

44. Thus let us set $dp = \mathrm{R}dx + rdy + \mathfrak{R}dt$, and let us consider a rectangular element of water, $lmno$, whose sides are $lm = no = dx$ and $ln = mo = dy$, whose area is $dxdy$ (Fig. 3). The pressure at l is p, the pressure at m is $p+\mathrm{R}dx$, at n it is $p+rdy$ and at o it is $p+\mathrm{R}dx+rdy$. Thus the side lm is pressed by a force $= dx(p + \frac{1}{2}\mathrm{R}dx)$, while the opposite side no will be pressed by a force $= dx(p+\frac{1}{2}\mathrm{R}dx+rdy)$; therefore by these two forces the element $lmno$ will be impelled in the direction ln by a force $= -rdxdy$. Moreover, in a similar manner from the forces $dy(p + \frac{1}{2}rdy)$ and $dy(p + \mathrm{R}dx + \frac{1}{2}rdy)$, which act on the sides ln and mo will result a force $= -\mathrm{R}dxdy$ impelling the element in the direction lm.

Fig: 3.



45. Thus will originate an accelerating force parallel to $lm = -\mathrm{R}$ and an accelerating force parallel to $ln = -r$, of which the one directed along the force of gravity α gives $\alpha - \mathrm{R}$. Having ignored friction so far, we obtain the following equations:[17]

$$\alpha - \mathrm{R} = 2Lu + 2lv + 2\mathfrak{L} \quad \text{or} \quad \mathrm{R} = \alpha - 2\mathrm{L}u - 2lv - 2\mathfrak{L}$$
$$-r = 2\mathrm{M}u + 2mv + 2\mathfrak{M} \quad \text{and} \quad r = -2\mathrm{M}u - 2mv - 2\mathfrak{M}$$

from which we gather that

$$dp = \alpha dx - 2(\mathrm{L}u + lv + \mathfrak{L})dx - 2(\mathrm{M}u + mv + \mathfrak{M})dy + \mathfrak{R}dt,$$

a differential which must be complete or integrable.

46. Because the term $\alpha\, dx$ is integrable by itself and nothing is determined for $\mathfrak{R}$, from the nature of complete differentials it is necessary that the following holds in the notation already employed:

$$\frac{d.\mathrm{L}u + lv + \mathfrak{L}}{dy} = \frac{d.\mathrm{M}u + mv + \mathfrak{M}}{dx}.$$

Since $\frac{du}{dx} = \mathrm{L}$, $\frac{du}{dy} = l$; $\frac{dv}{dx} = \mathrm{M}$, and $\frac{dv}{dy} = m$ it follows that

$$\mathrm{L}l + \frac{ud\mathrm{L}}{dy} + lm + \frac{vdl}{dy} + \frac{d\mathfrak{L}}{dy} = \mathrm{ML} + \frac{ud\mathrm{M}}{dx} + m\mathrm{M} + \frac{vdm}{dx} = \frac{d\mathfrak{M}}{dx}$$

which is reduced to this form:

$$(\mathrm{L} + m)(l - \mathrm{M}) + u\left(\frac{d\mathrm{L}}{dy} - \frac{d\mathrm{M}}{dx}\right) + v\left(\frac{dl}{dy} - \frac{dm}{dx}\right) + \frac{d\mathfrak{L}}{dy} - \frac{\mathfrak{M}}{dx} = 0$$

47. In fact, since we knew $\mathrm{L}dx + ldy + \mathfrak{L}dt$ and $\mathrm{M}dx + mdy + \mathfrak{M}dt$ to be complete differentials,

$$\frac{d\mathrm{L}}{dy} = \frac{dl}{dx}; \quad \frac{dm}{dx} = \frac{d\mathrm{M}}{dy}; \quad \frac{d\mathfrak{L}}{dy} = \frac{dl}{dt} \quad \text{and} \quad \frac{d\mathfrak{M}}{dx} = \frac{d\mathrm{M}}{dt}$$

after the substitution of which values we have the following equation

$$(\mathrm{L} + m)(l - \mathrm{M}) + u\left(\frac{dl - d\mathrm{M}}{dx}\right) + v\left(\frac{dl - d\mathrm{M}}{dy}\right) + \frac{dl - d\mathrm{M}}{dt} = 0.$$

Plainly, this is satisfied if $l = \mathrm{M}$: so that $\frac{du}{dy} = \frac{dv}{dx}$. Since this condition requires that $\frac{du}{dy} = \frac{dv}{dx}$,[18] it appears finally that the differential formula $udx + vdy$ must be complete; in this lies the criterion of actual motion.

48. This criterion is independent from the preceding one, which was provided by the continuity of the fluid and its uniform constant density. Therefore even if the fluid in motion changes its density, as happens in the motion of elastic fluids such as air, this property will hold nonetheless, namely $udx + vdy$ has to be a complete differential. In other words, the velocities u and v must always be functions of the coordinates x and y, together with time t, in such a way that when the time is taken constant the formula $udx + vdy$ admits an integration.

49. We shall now determine the pressure p itself, which is absolutely necessary for perfectly determining the motion of the fluid. Since we have found that $\mathrm{M} = l$ we have

$$dp = \alpha dx - 2u(\mathrm{L}dx + ldy) - 2v(ldx + mdy) - 2\mathfrak{L}dx - 2\mathfrak{M}dy + \mathfrak{R}dt.$$

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[17] Here the so-called Euler equations of incompressible fluid dynamics appear for the first time, but the notation and the units are not very modern, in contrast to the memoir he will write three years later (Euler, 1755).

[18] Here there are two problems. The minor problem is a typographical error in the published version ($\frac{du}{dx}$ instead of $\frac{dv}{dx}$), which is not present in a 1752 copy of the manuscript (not in Euler's hand), henceforth referred to as Euler, 1752. A more serious problem is that Euler here repeats the mistake of d'Alembert, 1752 who confused a sufficient condition – the vanishing of the vorticity – with a necessary one.

Moreover $\mathrm{L}dx + ldy = du - \mathfrak{L}dt$; $ldx + mdy = dv - \mathfrak{M}dt$; hence we have

$$dp = \alpha dx -$$
$$2udu - 2vdv + 2\mathfrak{L}udt + 2\mathfrak{M}vdt - 2\mathfrak{L}dx - 2\mathfrak{M}dy + \mathfrak{N}dt$$

Therefore, if we wish to ascertain for the present time the pressure at each point of the fluid, with no account of its variation in time, we shall have to consider this equation

$$dp = \alpha dx - 2udu - 2vdv - 2\mathfrak{L}dx - 2\mathfrak{M}dy,$$

and in our notation $\mathfrak{L} = \frac{du}{dt}$ and $\mathfrak{M} = \frac{dv}{dt}$.[19] Hence

$$dp = \alpha dx - 2udu - 2vdv - 2\frac{du}{dt}dx - 2\frac{dv}{dt}dy,$$

in the integration of which the time is to be taken constant.

50. This equation is integrable by hypothesis, and is indeed understood as such, if we consider the criterion of the motion which, as we have seen, consists in that $udx + vdy$ be a complete differential when the time t is taken constant. Let therefore S be its integral, which consequently will be a function of x, y and t themselves. For $dt = 0$ we obtain $d\mathrm{S} = udx + vdy$, while assuming the time t variable as well, let us write

$$d\mathrm{S} = udx + vdy + \mathrm{U}dt,$$

on which account we obtain $\frac{du}{dt} = \frac{d\mathrm{U}}{dx}$ and $\frac{dv}{dt} = \frac{d\mathrm{U}}{dy}$. Then, in fact $\mathrm{U} = \frac{d\mathrm{S}}{dt}$.

51. After inserting these values we will obtain

$$\frac{du}{dt}.dx + \frac{dv}{dt}.dy = \frac{d\mathrm{U}}{dx}.dx + \frac{d\mathrm{U}}{dy}.dy$$

and this differential formula is manifestly integrated at constant time t to give U. For this to become clearer, let us set $d\mathrm{U} = \mathrm{K}dx + kdy$; thus $\frac{d\mathrm{U}}{dx} = \mathrm{K}$ and $\frac{d\mathrm{U}}{dy} = k$, so that $\frac{d\mathrm{U}}{dx}.dx + \frac{d\mathrm{U}}{dy} = \mathrm{K}dx + kdy = d\mathrm{U}$. Since its integral is $\mathrm{U} = \frac{d\mathrm{S}}{dt}$, we shall have

$$dp = \alpha dx - 2udu - 2vdv - 2d\mathrm{U}$$

from where it appears by integration:

$$p = \mathrm{Const.} + \alpha x - uu - vv - \frac{2d\mathrm{S}}{dt}$$

with a given function S of the coordinates x, y and t themselves, whose differential, for $dt = 0$ is $udx + vdy$.

52. In order to understand better the nature of these formulas, let us consider the true velocity of the point l, which is $\mathrm{V} = \sqrt{(uu + vv)}$. And the pressure will be: $p = \mathrm{Const.} + \alpha x - \mathrm{VV} - \frac{2d\mathrm{S}}{dt}$: in which the last term $d\mathrm{S}$ denotes the differential of $\mathrm{S} = \int(udx + vdy)$ itself, where the time t is allowed to vary.

53. If we now wish to also take friction into account, let us set it proportional to the pressure p. While the point l travels the element ds, the retarding force arising from the friction is $= \frac{p}{f}$; so that, setting $\frac{d\mathrm{S}}{dt} = \mathrm{U}$, our differential equation will be for constant t

$$dp = \alpha dx - \frac{p}{f}ds - \mathrm{V}d\mathrm{V} - 2d\mathrm{U},$$

from where we obtain by integration, taking e for the number whose hyperbolic[20] logarithm is $= 1$,

$$p = e^{\frac{-s}{f}} \int e^{\frac{s}{f}}(\alpha dx - 2\mathrm{V}d\mathrm{V} - 2d\mathrm{V}) \quad \text{or}$$
$$p = \alpha x - \mathrm{VV} - 2\mathrm{U} - \frac{1}{f}e^{\frac{-s}{f}} \int e^{\frac{s}{f}}(\alpha x - \mathrm{VV} - 2\mathrm{U})ds.$$

54. The criterion of the motion which drives the fluid in reality consists in this that, fixing the time t, the differential $udx + vdy$ has to be complete: also continuity and constant uniform density demand that $\frac{du}{dx} + \frac{dv}{dy} = 0$, hence it follows too that this differential $udy - vdx$ will have to be complete.[21] From where both velocities u and v jointly must be functions of the coordinates x and y with the time t in such a way that both differential formulas $udx + vdy$ and $udy - vdx$[22] be complete differentials.

55. Let us set up the same investigation in general, giving the point λ three velocities directed parallel to the axes AL, AB, AC. Let u, v, w denote these functions, which depend on coordinates x, y, z, besides t. After a differentiation we obtain

$$du = \mathrm{L}dx + ldy + \lambda dz + \mathfrak{L}dt$$
$$dv = \mathrm{M}dx + mdy + \mu dz + \mathfrak{M}dt$$
$$dw = \mathrm{N}dx + ndy + \nu dz + \mathfrak{N}dt.$$

Although here the time t is also taken as variable, nonetheless for the motion to be possible, by the preceding condition[23] we have $\mathrm{L} + m + \nu = 0$, or, which reexpresses the same

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0,$$

a condition on which the present examination does not depend.

56. After the passage of time interval dt the point λ is carried to π, and it travels a distance udt parallel to the axis AL, a distance vdt parallel to the axis AB and a distance wdt parallel to the axis AC. Thus the three velocities of the point which has moved from λ to π will be:

parallel to AL $= u + \mathrm{L}u\,dt + l\,v\,dt + \lambda w\,dt + \mathfrak{L}\,dt$;
parallel to AB $= v + \mathrm{M}u\,dt + mv\,dt + \mu w\,dt + \mathfrak{M}dt$;
parallel to AC $= w + \mathrm{N}u\,dt + n\,v\,dt + \nu\,w\,dt + \mathfrak{N}\,dt$,

[19] The printed version has $L = \frac{dv}{dt}$ instead of $\mathfrak{L} = \frac{du}{dt}$. Euler, 1752 is correct.

[20] Natural

[21] The published version has $udx + vdy$, a mistake not present in Euler, 1752.

[22] Previous mistake repeated in the published version.

[23] From Part I.

and the accelerations parallel to the same directions will be

$$\begin{aligned}
\text{par. AL} &= 2(\mathrm{L}u + l\ v + \lambda w + \mathfrak{L});\\
\text{par. AB} &= 2(\mathrm{M}u + mw + \mu w + \mathfrak{M});\\
\text{par. AC} &= 2(\mathrm{N}u + nv + \nu w + \mathfrak{N}).
\end{aligned}$$

57. If we take the axis AC to be vertical, in such a way that the remaining two AL and AB are horizontal, the accelerating force due to gravity arises parallel to the axis AC with the value -1. Then indeed, denoting the pressure at λ by p, its differential, at constant time is

$$dp = \mathrm{R}\,dx + rdy + \rho dz,$$

from which we obtain the three accelerating forces

$$\text{par. AL} = \mathrm{R};\quad \text{par. AB} = -r;\quad \text{par. AC} = -\rho$$

which are in fact easily collected in the same manner as was done in §§. 44 and 45, so that it is not necessary to repeat the same computation. Hence we obtain the following equations[24]

$$\begin{aligned}
\mathrm{R} &= -2(\mathrm{L}u + lv + \lambda w + \mathfrak{L})\\
r &= -2(\mathrm{M}u + mv + \mu w + \mathfrak{M})\\
\rho &= -1 - 2(\mathrm{N}u + nv + \nu w + \mathfrak{N})
\end{aligned}$$

58. Since the differential formula $dp = \mathrm{R}dx + rdy + \rho dz$ has to be a complete differential, we have

$$\frac{d\mathrm{R}}{dy} = \frac{dr}{dx};\quad \frac{d\mathrm{R}}{dz} = \frac{d\rho}{dx};\quad \frac{dr}{dz} = \frac{d\rho}{dy}.$$

After a differentiation and a division by -2 the following three equations are obtained[25]

$$\begin{aligned}
&\text{I}\begin{cases} \frac{ud\mathrm{L}}{dy} + \frac{vdl}{dy} + \frac{wd\lambda}{dy} + \frac{d\mathfrak{L}}{dy} + \mathrm{L}l + lm + \lambda n = \\ \frac{ud\mathrm{M}}{dx} + \frac{vdm}{dx} + \frac{wd\mu}{dx} + \frac{d\mathfrak{M}}{dx} + \mathrm{ML} + m\mathrm{M} + \mu\mathrm{N} \end{cases}\\
&\text{II}\begin{cases} \frac{ud\mathrm{L}}{dz} + \frac{vdl}{dz} + \frac{wd\lambda}{dz} + \frac{d\mathfrak{L}}{dz} + \mathrm{L}\lambda + l\mu + \lambda\nu = \\ \frac{ud\mathrm{N}}{dx} + \frac{vdn}{dx} + \frac{wd\nu}{dx} + \frac{d\mathfrak{N}}{dx} + \mathrm{NL} + n\mathrm{M} + \nu\mathrm{N} \end{cases}\\
&\text{III}\begin{cases} \frac{ud\mathrm{M}}{dz} + \frac{vdm}{dz} + \frac{wd\mu}{dz} + \frac{d\mathfrak{M}}{dz} + \mathrm{M}\lambda + m\mu + \mu\nu = \\ \frac{ud\mathrm{N}}{dy} + \frac{vdn}{dy} + \frac{wd\nu}{dy} + \frac{d\mathfrak{N}}{dy} + \mathrm{N}l + nm + \nu n \end{cases}
\end{aligned}$$

59. Moreover, because of the nature of the complete differentials, we have

$$\begin{aligned}
&\frac{d\mathrm{L}}{dy} = \frac{dl}{dx};\quad \frac{dm}{dx} = \frac{d\mathrm{M}}{dy};\quad \frac{d\lambda}{dy} = \frac{dl}{dz};\\
&\frac{d\mu}{dx} = \frac{d\mathrm{M}}{dz};\quad \frac{d\mathfrak{L}}{dy} = \frac{dl}{dt};\quad \frac{d\mathfrak{M}}{dx} = \frac{d\mathrm{M}}{dt}\\
&\frac{d\mathrm{L}}{dz} = \frac{d\lambda}{dx};\quad \frac{dl}{dz} = \frac{d\lambda}{dy};\quad \frac{dn}{dx} = \frac{d\mathrm{N}}{dy};\\
&\frac{d\nu}{dx} = \frac{d\mathrm{N}}{dz};\quad \frac{d\mathfrak{L}}{dz} = \frac{d\lambda}{dt};\quad \frac{d\mathfrak{N}}{dx} = \frac{d\mathrm{N}}{dt};\\
&\frac{d\mathrm{M}}{dz} = \frac{d\mu}{dx};\quad \frac{d\mathrm{N}}{dy} = \frac{dn}{dx};\quad \frac{dm}{dz} = \frac{d\mu}{dy};\\
&\frac{d\nu}{dy} = \frac{dn}{dz};\quad \frac{d\mathfrak{M}}{dz} = \frac{d\mu}{dt};\quad \frac{d\mathfrak{N}}{dy} = \frac{dn}{dt},
\end{aligned}$$

after substituting of which values those three equations will be transformed into these[26]

$$\begin{aligned}
&\left(\frac{dl - d\mathrm{M}}{dt}\right) + u\left(\frac{dl - d\mathrm{M}}{dx}\right) + v\left(\frac{dl - d\mathrm{M}}{dy}\right) +\\
&w\left(\frac{dl - d\mathrm{M}}{dz}\right) + (l - \mathrm{M})(\mathrm{L} + m) + \lambda n - \mu\mathrm{N} = 0,\\
&\left(\frac{d\lambda - d\mathrm{N}}{dt}\right) + u\left(\frac{d\lambda - d\mathrm{N}}{dx}\right) + v\left(\frac{d\lambda - d\mathrm{N}}{dy}\right) +\\
&w\left(\frac{d\lambda - d\mathrm{N}}{dz}\right) + (\lambda - \mathrm{N})(\mathrm{L} + \nu) + l\mu - n\mathrm{M} = 0,\\
&\left(\frac{d\mu - dn}{dt}\right) + u\left(\frac{d\mu - dn}{dx}\right) + v\left(\frac{d\mu - dn}{dy}\right) +\\
&w\left(\frac{d\mu - dn}{dz}\right) + (\mu - n)(m + \nu) + \mathrm{M}\lambda - \mathrm{N}l = 0.
\end{aligned}$$

60. Now it is manifest that these three equations are satisfied by the following three values

$$l = \mathrm{M};\quad \lambda = \mathrm{N};\quad \mu = n$$

in which is contained the criterion furnished by the consideration of the forces. Here therefore follows that in the notation chosen we have[27]

$$\frac{du}{dy} = \frac{dv}{dx};\quad \frac{du}{dz} = \frac{dw}{dx};\quad \frac{dv}{dz} = \frac{dw}{dy}$$

these conditions moreover are the same as those which are required in order that the formula $udx + vdy + wdz$ be a complete differential. From which this criterion consists in that the three velocities u, v and w have to be functions of x, y and z together with t in such a manner that for fixed constant time the formula $udx + vdy + wdz$ admits an integration.

61. Taking the time t constant or $dt = 0$, we have

$$\begin{aligned}
du &= \mathrm{L}dx + \mathrm{M}dy + \mathrm{N}dz\\
dv &= \mathrm{M}dx + mdy + ndz\\
dw &= \mathrm{N}dx + ndy + \nu dz
\end{aligned}$$

---

[24] These are the three dimensional Euler equations.

[25] The printed version contains mistakes not present in Euler, 1752: in the formula labelled II, instead of L there is $\mathfrak{L}$; in the formula labelled III there is a v instead of u.

[26] These are the equations for the vorticity.

[27] Here Euler repeats the mistake of assuming that the only solution is zero-vorticity flow; in Euler, 1755 this will be corrected.

moreover, for R, r and ρ the values are

$$\begin{aligned} \mathrm{R} &= -2(\mathrm{L}u + \mathrm{M}v + \mathrm{N}w + \mathfrak{L}) \\ r &= -2(\mathrm{M}u + mv + nw + \mathfrak{M}) \\ \rho &= -1 - 2(\mathrm{N}u + nv + \nu w + \mathfrak{N}) \end{aligned}$$

Regarding the pressure p, we obtain the following equation

$$\begin{aligned} &dp = -dz \\ &- 2u(\mathrm{L}dx + \mathrm{M}dy + \mathrm{N}dz) = -dz - 2udu - 2vdv - 2wdw \\ &- 2v(\mathrm{M}dx + mdy + ndz) \qquad\qquad - 2\mathfrak{L}dx - 2\mathfrak{M} - 2\mathfrak{N}dz \\ &- 2w(\mathrm{N}dx + ndy + \nu dz) \\ &- 2\mathfrak{L}dx - 2\mathfrak{M}dy - 2\mathfrak{N}dz \end{aligned}$$

62. Since in truth $\mathfrak{L} = \frac{du}{dt}$; $\mathfrak{M} = \frac{dv}{dt}$; $\mathfrak{N} = \frac{dw}{dt}$, we obtain by integration

$$p = \mathrm{C} - z - uu - vv - ww - 2\int \left(\frac{du}{dt}dx + \frac{dv}{dt}dy + \frac{dw}{dt}dz \right)$$

By the previously ascertained condition $udx + vdy + wdz$ is integrable. Let us denote its integral by S, which can also involve the time t; taking also the time t variable, we have

$$d\mathrm{S} = udx + vdy + wdz + \mathrm{U}dt,$$

and we have $\frac{du}{dt} = \frac{d\mathrm{U}}{dx}$; $\frac{dv}{dt} = \frac{d\mathrm{U}}{dy}$; $\frac{dw}{dt} = \frac{d\mathrm{U}}{dz}$. From where, with time generally taken constant, it can be assumed in the above integral that

$$\frac{d\mathrm{U}}{dx}dx + \frac{d\mathrm{U}}{dy}dy + \frac{d\mathrm{U}}{dz}dz = d\mathrm{U},$$

and we obtain[28]

$$\begin{aligned} p &= \mathrm{C} - z - uu - vv - ww - 2\mathrm{U}, \quad \text{or} \\ p &= \mathrm{C} - z - uu - vv - ww - 2\frac{d\mathrm{S}}{dt} \end{aligned}$$

63. Thus, $uu+vv+ww$ is manifestly expressing the square of the true velocity of the point λ, so that, if the true velocity of this point is denoted V, the following equation is obtained for the pressure[29]

$$p = \mathrm{C} - z - \mathrm{VV} - \frac{2d\mathrm{S}}{dt}.$$

To use this, firstly one must seek the integral S of the formula $udx+vdy+wdz$ which should be complete. This is differentiated again, taking only the time t as variable. After division by dt, one obtains the value of the formula $\frac{d\mathrm{S}}{dt}$, which enters into the expression for the state of the pressure p.

64. But before we may add here the previous criterion, regarding possible motion, the three velocities u, v and w must be such functions of the three coordinates x, y and z, and of time t that, firstly, $udx + vdy + wdz$ be a complete differential and, secondly, that the condition $\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0$ holds. The whole motion of fluids endowed with invariable density is subjected to these two conditions.

Furthermore, if we take also the time t to be variable, and the differential formula $udx+vdy+wdz+\mathrm{U}dt$ is a complete differential, the state of the pressure at any point λ, expressed as an altitude p, will be given by

$$p = \mathrm{C} - z - uu - vv - ww - 2\mathrm{U},$$

if only the fluid enjoys the natural gravity and the plane BAL is horizontal.

65. Suppose we had attributed another direction to the gravity or even adopted arbitrary variable forces acting on the particles of the fluid. Differences would arise in the values of the pressure, but the law which the three velocities of the fluid have to obey would not suffer any changes. Thus, whatever the acting forces, the three velocities u, v and w have to satisfy the conditions that the differential formula $udx + vdy + wdz$ be complete and that $\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0$ should hold. Therefore, the three velocities u, v and w can be fixed in infinitely many ways while satisfying the two conditions; and then it is possible to prescribe the pressure at every point of the fluid.[30]

66. However, much more difficult would be the following question: given the acting forces and the pressure at all places, to determine the motion of the fluid at all points. Indeed, we would then have some equations[31] of the form $p = \mathrm{C} - z - uu - vv - ww - 2\mathrm{U}$, from which the relation of the functions u, v and w would have to be defined in such a way that not only the equations themselves would be satisfied , but also the previously contributed rules[32] would have to be obeyed; this work would certainly require the greatest force of calculation. It is fitting therefore to inquire in general into the nature of functions proper to satisfy both criteria.

67. Most conveniently therefore let us begin with the characterization of the integral quantity S, whose differential is $udx + vdy + wdz$, when time is held constant. Let thus S be a function of x, y and z, the time t being contained in constant quantities. When S is differentiated, the coefficients of the differentials dx, dy and dz are the velocities u, v and w which at the present time suit the point of fluid λ, whose coordinates are x, y and z. The question thus arises here to find the functions S of x, y and z such that $\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0$; now, since we have $u = \frac{dS}{dx}$, $v = \frac{dS}{dy}$ and $w = \frac{dS}{dz}$ it follows that $\frac{dd\mathrm{S}}{dx^2} + \frac{dd\mathrm{S}}{dy^2} + \frac{dd\mathrm{S}}{dz^2} = 0$.[33]

68. Since it is not plain how this can be handled in general, I shall consider certain rather general cases. Let

$$\mathrm{S} = (\mathrm{A}x + \mathrm{B}y + \mathrm{C}z)^n.$$

[28] The published version has a ds in the denominator, instead of the correct dt, found in Euler, 1752.

[29] This is basically the Bernoulli pressure law for potential flow.

[30] Many statements in this paragraph are rendered invalid by the generally incorrect assumption of potential flow.

[31] The plural is here used probably because this relation has to be satisfied at all points.

[32] Incompressibility and potentiality.

[33] This is what will later be called Laplace's equation.

We have

$$\frac{d\mathrm{S}}{dx} = n\mathrm{A}(\mathrm{A}x + \mathrm{B}y + \mathrm{C}z)^{n-1} \text{ and}$$
$$\frac{dd\mathrm{S}}{dx^2} = n(n-1)\mathrm{AA}(\mathrm{A}x + \mathrm{B}y + \mathrm{C}z)^{n-2}$$

and the expressions for $\frac{dd\mathrm{S}}{dy^2}$ and $\frac{dd\mathrm{S}}{dz^2}$ will be similar. Thus we have to satisfy

$$n(n-1)(\mathrm{A}x + \mathrm{B}y + \mathrm{C}z)^{n-2}(\mathrm{AA} + \mathrm{BB} + \mathrm{CC}) = 0$$

which is plainly satisfied when either $n = 0$ or $n = 1$. Thus we have the solutions $\mathrm{S} = \mathrm{Const.}$ and $\mathrm{S} = \mathrm{A}x + \mathrm{B}y + \mathrm{C}z$, where the constants A, B and C are arbitrary.

69. But if n is neither 0, nor 1, we necessarily have: $\mathrm{AA} + \mathrm{BB} + \mathrm{CC} = 0$: and then S is given by

$$\mathrm{S} = (\mathrm{A}x + \mathrm{B}y + \mathrm{C}z)^n$$

for any value of the exponent n; even the time t itself will possibly enter in n. Furthermore we can add up arbitrarily many such S and obtain yet another solution.[34] The function

$$\mathrm{S} = \alpha + \beta x + \gamma y + \delta z + \epsilon(\mathrm{A}x + \mathrm{B}y + \mathrm{C}z)^n +$$
$$\zeta(\mathrm{A}'x + \mathrm{B}'y + \mathrm{C}'z)^{n'} + \eta(\mathrm{A}''x + \mathrm{B}''y + \mathrm{C}''z)^{n''} +$$
$$\theta(\mathrm{A}'''x + \mathrm{B}'''y + \mathrm{C}'''z)^{n'''} \text{ etc.}$$

will satisfy the condition only if we have:

$$\mathrm{AA} + \mathrm{BB} + \mathrm{CC} = 0; \quad \mathrm{A}'\mathrm{A}' + \mathrm{B}'\mathrm{B}' + \mathrm{C}'\mathrm{C}' = 0;$$
$$\mathrm{A}''\mathrm{A}'' + \mathrm{B}''\mathrm{B}'' + \mathrm{C}''\mathrm{C}'' = 0 \text{ etc.}$$

70. Here suitable values are given for S in which the coordinates x, y, z have either one, or two, or three, or four dimensions[35]

I. $\mathrm{S} = \mathrm{A}$

II. $\mathrm{S} = \mathrm{A}x + \mathrm{B}y + \mathrm{C}z$

III. $\mathrm{S} = \mathrm{A}xx + \mathrm{B}yy + \mathrm{C}zz + 2\mathrm{D}xy + 2\mathrm{E}xz + 2\mathrm{F}yz$
with $\mathrm{A} + \mathrm{B} + \mathrm{C} = 0$

IV. $\mathrm{S} = \mathrm{A}x^3 + \mathrm{B}y^3 + \mathrm{C}z^3 + 3\mathrm{D}xxy + 3\mathrm{F}xxz + \mathrm{H}yyz + 6\mathrm{K}xyz + 3\mathrm{E}xyy + 3\mathrm{G}xzz + 3\mathrm{I}yzz$
with $\mathrm{A} + \mathrm{E} + \mathrm{G} = 0$; $\mathrm{B} + \mathrm{D} + \mathrm{I} = 0$; $\mathrm{C} + \mathrm{F} + \mathrm{H} = 0$

V.

$$\begin{aligned} &+ \mathrm{A}x^4 + 6\mathrm{D}xxyy + 4\mathrm{G}x^3y + 4\mathrm{H}xy^3 + 12\mathrm{N}xxyz \\ \mathrm{S} = &+ \mathrm{B}y^4 + 6\mathrm{E}xxzz + 4\mathrm{I}x^3z + 4\mathrm{K}xz^3 + 12\mathrm{O}xyyz \\ &+ \mathrm{C}z^4 + 6\mathrm{F}yyzz + 4\mathrm{L}y^3z + 4\mathrm{M}yz^3 + 12\mathrm{P}xyzz \end{aligned}$$

with
$$\begin{aligned} &\mathrm{A} + \mathrm{D} + \mathrm{E} = 0 \quad \mathrm{G} + \mathrm{H} + \mathrm{P} = 0 \\ &\mathrm{B} + \mathrm{D} + \mathrm{F} = 0 \quad \mathrm{I} + \mathrm{K} + \mathrm{O} = 0 \\ &\mathrm{C} + \mathrm{E} + \mathrm{F} = 0 \quad \mathrm{L} + \mathrm{M} + \mathrm{N} = 0 \end{aligned}$$

71. Hence it is clear how these formulas are to be obtained for any order. First, simply give to the various terms the numerical coefficients which belong to them from the law of permutation, or, equivalently, which arise when the trinomial $x + y + z$ is raised to that same power. Let indefinite letters A, B, C, etc., be adjoined to the numerical coefficients. Then, ignoring the coefficients, observe whenever there occur three terms of the type $\mathrm{LZ}x^2 + \mathrm{MZ}y^2 + \mathrm{NZ}z^2$ having a common factor Z formed from the variables. Whenever this occurs, set the sum of the literal coefficients $\mathrm{L} + \mathrm{M} + \mathrm{N}$ equal to zero. For example, for the fifth power we have

$$\begin{aligned} \mathrm{S} =& \mathrm{A}x^5 + 5\mathrm{D}x^4y + 5\mathfrak{D}x^4z + 10\mathrm{G}x^3yy + \mathfrak{G}x^3zz + \\ & 20\mathrm{K}x^3yz + 30\mathrm{N}xyyzz + \\ & \mathrm{B}x^5 + 5\mathrm{E}x^4y + 5\mathfrak{E}x^4z + 10\mathrm{H}x^3yy + \mathfrak{H}x^3zz + \\ & 20\mathrm{L}x^3yz + 30\mathrm{O}xyyzz \\ & + \mathrm{C}x^5 + 5\mathrm{F}x^4y + 5\mathfrak{F}x^4z + 10\mathrm{I}x^3yy + \mathfrak{I}x^3zz + \\ & 20\mathrm{M}x^3yz + 30\mathrm{P}xyyzz \end{aligned}$$

and the following determinations of the coefficient letters are obtained

$$\begin{aligned} &\mathrm{A} + \mathrm{G} + \mathfrak{G} = 0; \quad \mathrm{D} + \mathrm{H} + \mathrm{O} = 0; \quad \mathfrak{D} + \mathrm{I} + \mathrm{P} = 0; \\ &\mathrm{B} + \mathrm{H} + \mathfrak{H} = 0; \quad \mathrm{E} + \mathrm{G} + \mathrm{N} = 0; \quad \mathfrak{G} + \mathrm{F} + \mathrm{P} = 0; \\ &\mathrm{K} + \mathrm{L} + \mathrm{M} = 0; \\ &\mathrm{C} + \mathrm{I} + \mathfrak{F} = 0; \quad \mathrm{F} + \mathfrak{G} + \mathrm{N} = 0; \quad \mathfrak{F} + \mathfrak{H} + \mathrm{O} = 0. \end{aligned}$$

In the same way for the sixth order such determinations will give 15, for the seventh 21, for the eighth 28 and so on.

72. In the very first formula $\mathrm{S} = \mathrm{A}$ the coordinates x, y and z are clearly not intertwined. Thus the three velocities u, v and w are equal to zero, and hence this describes a quiet state of fluid. Also the pressure at an arbitrary point for different times will be able to vary in an arbitrary manner. Indeed A is an arbitrary function of time and, for a given time t, the pressure at the point λ is $p = \mathrm{C} - \frac{2d\mathrm{A}}{dt} - z$. Through this formula is revealed the state of the fluid, when it is subjected at an arbitrary instant to arbitrary forces, which nevertheless balance each other, so that no motion in the fluid can arise from them: where it happens, if the fluid is enclosed in a vase from which it can nowhere escape, it is also compressed by suitable forces inside.

73. Moreover, the second formula $\mathrm{S} = \mathrm{A}x + \mathrm{B}y + \mathrm{C}z$, after differentiation, gives these three velocities to the point λ:

$$u = \mathrm{A}; \quad v = \mathrm{B} \quad \text{and} \quad w = \mathrm{C}.$$

Thus simultaneously, all points of the fluid are carried by an identical motion in the same direction. From which the whole fluid moves in the same manner as a solid body, carried only by a forward motion. But at different times the velocities as well as the direction of this motion are able to be varied in an arbitrary way, depending on what the extrinsic driving forces require. Therefore, the pressure at the point λ at the time t on which A, B, C depend, is[36] $p = \mathrm{C} - z - \mathrm{AA} - \mathrm{BB} - \mathrm{CC} -$

[34] In modern terms, Euler is here using the linear character of the Laplace equation.

[35] In modern terms we would say “which are polynomials in x, y, z of degrees up to four”.

[36] The printed version, but not Euler, 1752, has a missing BB in the formula.

$2x\frac{d\mathrm{A}}{dt} - 2y\frac{d\mathrm{B}}{dt} - 2z\frac{d\mathrm{C}}{dt}$.

74. The third formula $\mathrm{S} = \mathrm{A}xx + \mathrm{B}yy + \mathrm{C}zz + 2\mathrm{D}xy + 2\mathrm{E}xz + 2\mathrm{F}yz$, where $\mathrm{A} + \mathrm{B} + \mathrm{C} = 0$, gives the following three velocities[37] of the point λ: $u = 2\mathrm{A}x + 2\mathrm{D}y + 2\mathrm{E}z$; $v = 2\mathrm{B}y + 2\mathrm{D}x + 2\mathrm{F}z$; $w = 2\mathrm{C}z + 2\mathrm{E}x + 2\mathrm{F}y$, or $w = 2\mathrm{E}x + 2\mathrm{F}y - 2(\mathrm{A} + \mathrm{B})z$. Here, at a given instant, different points of the fluid are carried by different motions; moreover, in the time development an arbitrary motion of a given point is permitted, because A, B, D, E, F can be arbitrary functions of the time t. Finally, a much greater variety can take place, if more elaborate values are given to the function S.

75. In the second case the motion of the fluid was corresponding to the forward motion of a solid body in which, plainly, at any instant the different parts are carried by a motion equal and parallel to itself. In other cases the motion of the fluid could be suspected to correspond to solid-body motion, either rotational or anomalous. It suffices to put forward such a hypothesis – beyond the second case – to find that it cannot take place. Indeed, in order to happen, not only would it be necessary that the pyramid $\pi\Phi\rho\sigma$ would be equal,[38] but also similar to the pyramid $\lambda\mu\nu o$, or that the following holds

$$\begin{aligned}
&\pi\Phi = \lambda\mu = dx = \sqrt{(\mathrm{QQ} + qq + \Phi\Phi)}\\
&\pi\rho = \lambda\nu = dy = \sqrt{(\mathrm{RR} + rr + \rho\rho)}\\
&\pi\sigma = \lambda o = dz = \sqrt{(\mathrm{SS} + ss + \sigma\sigma)}\\
&\Phi\rho = \mu\nu = \sqrt{(dx^2 + dy^2)} =\\
&\sqrt{((\mathrm{Q} - \mathrm{R})^2 + (q - r)^2 + (\Phi - \rho)^2)}\\
&\Phi\sigma = \mu o = \sqrt{(dx^2 + dz^2)} =\\
&\sqrt{((\mathrm{Q} - \mathrm{S})^2 + (q - s)^2 + (\Phi - \sigma)^2)}\\
&\rho\sigma = \nu o = \sqrt{(dy^2 + dz^2)} =\\
&\sqrt{((\mathrm{R} - \mathrm{S})^2 + (r - s)^2 + (\rho - \sigma)^2)},
\end{aligned}$$

where we applied the values taken from §. 32.

76. Then the three latter equations, combined with the former, are reduced to these:

$$\mathrm{QR} + qr + \Phi\rho = 0;\ \mathrm{QS} + qs + \Phi\sigma = 0 \text{ and } \mathrm{RS} + rs + \rho\sigma = 0.$$

Moreover, if the values assigned in §. 34 are substituted for the letters Q, R, S, q, r, s, Φ, ρ, σ and the higher-order terms for the rests are neglected, the three former will give

$$\begin{aligned}
1 &= 1 + 2\mathrm{L}dt; \quad l + \mathrm{M} = 0;\\
1 &= 1 + 2mdt; \quad \lambda + \mathrm{N} = 0;\\
1 &= 1 + 2\nu dt; \quad \mu + n = 0,
\end{aligned}$$

so that we have $\mathrm{L} = 0$ $m = 0$ and $\nu = 0$, $\mathrm{M} = -l$, $\mathrm{N} = -\lambda$ and $n = -\mu$.

77. Thus, the three velocities of this point λ would have to be compared to the condition that the following hold[39]

$$\begin{aligned}
du &= ldy + \lambda dz;\\
dv &= -ldx + \mu dz;\\
dw &= -\lambda dx - \mu dy.
\end{aligned}$$

But the second condition demands a motion of the fluid such that $l = \mathrm{M}$, $\lambda = \mathrm{N}$ and $n = \mu$; hence all the coefficients l, λ and μ vanish; also the velocities u, v and w will take the same value everywhere in the fluid. Therefore it is plain that the motion of the fluid cannot correspond to solid-body motion other than pure translational.

78. To ascertain the effect of the forces which act from the outside upon the fluid, it is first necessary to determine those forces[40] which are required for effecting the motion which we have assumed to exist in the fluid. These are equivalent to the forces which in fact work upon the fluid; furthermore we have seen above in §. 56 that three accelerating forces are required, which are here repeated. If an element of fluid is conceived here, whose volume, or mass is $dxdydz$, the moving forces required for the motion are

$$\begin{aligned}
&\text{par. AL} = 2dxdydz(\mathrm{L}u + lv + \lambda w + \mathfrak{L}) =\\
&2dxdydz\left(u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz} + \frac{du}{dt}\right);\\
&\text{par. AB} = 2dxdydz(\mathrm{M}u + mv + \mu w + \mathfrak{M}) =\\
&2dxdydz\left(u\frac{dv}{dx} + v\frac{dv}{dy} + w\frac{dv}{dz} + \frac{dv}{dt}\right);\\
&\text{par. AC} = 2dxdydz(\mathrm{N}u + nv + \nu w + \mathfrak{N}) =\\
&2dxdydz\left(u\frac{dw}{dx} + v\frac{dw}{dy} + w\frac{dw}{dz} + \frac{dw}{dt}\right),
\end{aligned}$$

so that by triple integration the components of the total forces which must act on the whole mass of fluid may be obtained.

79. But since the second condition requires that $udx + vdy + wdz$ be a complete differential, whose integral is S, let us put as before, with time allowed to vary, $d\mathrm{S} = udx + vdy + wdz + \mathrm{U}dt$. Since $\frac{du}{dy} = \frac{dv}{dx}$; $\frac{du}{dz} = \frac{dw}{dx}$; $\frac{du}{dt} = \frac{d\mathrm{U}}{dx}$ those three moving forces emerge:[41]

$$\begin{aligned}
\text{par. AL} &= 2dxdydz\left(\frac{udu + vdv + wdw + d\mathrm{U}}{dx}\right)\\
\text{par. AB} &= 2dxdydz\left(\frac{udu + vdv + wdw + d\mathrm{U}}{dy}\right)\\
\text{par. AL} &= 2dxdydz\left(\frac{udu + vdv + wdw + d\mathrm{U}}{dz}\right)
\end{aligned}$$

80. Let us set now $uu + vv + ww + 2\mathrm{U} = \mathrm{T}$. The function T depends on the coordinates x. y, z; take it at a given instant

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[37] In both the printed version and in Euler, 1752, the first velocity component is mistakenly denoted by α.

[38] In volume.

[39] In the printed version, but not in Euler, 1752, there are several sign mistakes.

[40] Here, internal forces are meant.

[41] There is a misprint in the printed version, w instead of $+$.

of time t:[42]

$$d\mathrm{T} = \mathrm{K}dx + kdy + \kappa dz.$$

The three moving forces of the element $dxdydz$ are[43]

$$\text{par. AL} = \mathrm{K}dxdydz$$
$$\text{par. AB} = kdxdydz$$
$$\text{par. AC} = \kappa dxdydz$$

and by triple integration these formulas ought to be extended throughout the mass of the fluid; thus forces equivalent to all[44] and their directions may be obtained. Truly this discussion is for a later investigation, which I shall not deepen here.

81. Furthermore, the quantity $\mathrm{T} = uu + vv + ww + 2\mathrm{U}$, which is analyzed in this calculation, furnishes a simpler formula for expressing the pressure through the height p; we have indeed $p = \mathrm{C} - z - \mathrm{T}$ when the particles of the fluid are pressed upon solely by the gravity. But if an arbitrary particle λ is acted upon by three accelerating forces which are Q, q and Φ, acting parallel to the directions of the axes AF, AB and AC, respectively, after a calculation similar to the previous one has been carried out, the pressure will be given by

$$p = \mathrm{C} + \int (\mathrm{Q}dx + qdy + \Phi dz) - \mathrm{T}.$$

Thus it is plain that the differential $\mathrm{Q} + qdy + \Phi dz$ must be complete, as otherwise a state of equilibrium, or at least a possible one, could not exist. That this condition must be imposed on the acting forces Q, q and Φ was shown very clearly by the most famous Mr. Clairaut.[45]

82. Here are, therefore, the principles of the entire doctrine of the motion of fluids, which, even if they at first sight may seem insufficiently fruitful, nevertheless embrace almost everything treated both in hydrostatics and in hydraulics, so that these principles must be regarded as having very broad extent. For this to appear more clearly, it is worthwhile to show how the precepts learned in hydrostatics and hydraulics follow.

83. Let us therefore consider first a fluid in a state of rest, so that we have $u = 0$, $v = 0$ and $w = 0$; in view of $\mathrm{T} = 2\mathrm{U}$, the pressure in an arbitrary point λ of the fluid is

$$p = \mathrm{C} + \int (\mathrm{Q}dx + qdy + \Phi dz) - 2\mathrm{U}.$$

Here, U is a function of the time t itself which we take as constant. Indeed, we investigate the pressure at a given time; the quantity U can be included in the constant C, so that we obtain

$$p = \mathrm{C} + \int (\mathrm{Q}dx + qdy + \Phi dz)$$

where Q, q an Φ are the forces acting on the particle of water λ, parallel to the axes AL, AB and AC.

84. The pressure p can only depend on the position of the point λ that is on the coordinates x, y and z; it is thus necessary that $\int(\mathrm{Q}dx + qdy + \Phi dz)$ be a prescribed function of them, which therefore admits integration. Thus it is firstly clear that in the manner indicated the fluid cannot be sustained in equilibrium, unless the forces acting on each element of the fluid are such that the differential formula $\mathrm{Q}dx + qdy + \Phi dz$ is complete. Thus, if its integral is denoted P, the pressure at λ will be $p = \mathrm{C} + \mathrm{P}$. Therefore, if the only force present is gravity, impelling parallel to the direction CA, we shall have $p = \mathrm{C} - z$; hence, if the pressure is fixed at one point λ, the constant C can be obtained. From which the pressure at a given time will be defined completely at all points of the fluid.

85. However, with time passing, the pressure at a given place can change; and this plainly occurs, if variability is assumed for the forces impelling on the water, whose calculation cannot be made from those forces which are assumed to act on each element of the fluid,[46] but in such a way that they keep each other in equilibrium and produce no motion. But if, moreover, these forces are not subject to any change, the letter C will indeed denote a constant quantity, not depending on time t; and at a given location λ we will always find the same pressure $p = \mathrm{C} + \mathrm{P}$.

86. It is possible to determine the extremal shape of a fluid in a permanent state, when it is not subjected to any force.[47] Certainly, at the extreme surface of the fluid at which the fluid is left to itself and not contained within the walls of the vase in which it is enclosed, the pressure must be zero. Thus we shall obtain the following equation: $\mathrm{P} = \mathrm{const}$; the shape of the external surface of the fluid is then expressed through a relation between the three coordinates x, y and z. And if for the external circumference held $\mathrm{P} = \mathrm{E}$, since $\mathrm{C} = -\mathrm{E}$, in another arbitrary internal location λ the pressure would be $p = \mathrm{P} - \mathrm{E}$. In this manner, if the particles of the fluid are driven by gravity only, and because $p = \mathrm{C} - z$, the following will hold at for the external surface $z = \mathrm{C}$; from which the external free surface is perceived to be horizontal.

87. Next, everything which has so far been brought out concerning the motion of a fluid through tubes is easily derived from these principles. The tubes are usually regarded as very narrow, or else are assumed to be such that through any section normal to the tube the fluid flows across with equal motion: from there originates the rule, that the speed of the fluid at any place in the tube is reciprocally proportional to its amplitude. Let therefore λ be an arbitrary point of such a tube, of which the shape is expressed by two equations relating the three coordinates x, y and z, so that thereupon for any abscissa x the two remaining coordinates y and z can be defined.

88. Let henceforth the cross section of this tube at λ be rr;

[42] There is a misprint: u instead of κ.
[43] Here is again a misprint: k instead of κ.
[44] The pressure forces.
[45] Clairaut, 1743.
[46] That is the internal pressure forces.
[47] Here, Euler will comment on the shape of the free (extreme) surface of a fluid contained in an open vessel.

in another fixed location of the tube, where the cross-secion is ff, let the velocity at the present time be $\Taurus$; now after the time dt has elapsed, let the velocity become $\Taurus + d\Taurus$, so that $\Taurus$ is a function of time t, and similarly with $\frac{d\Taurus}{dt}$. Hence the true velocity of the fluid at λ will be at the present time $V = \frac{ff\Taurus}{rr}$. Since now y and z are obtained from the shape of the tube, we have $dy = \eta dx$ and $dz = \theta dx$; thus the three velocities of the point λ in the fluid, parallel to directions AL, AB and AC, are

$$u = \frac{ff\Taurus}{rr}\,\frac{1}{\sqrt{(1+\eta\eta+\theta\theta)}};\; v = \frac{ff\Taurus}{rr}\,\frac{\eta}{\sqrt{(1+\eta\eta+\theta\theta)}};$$
$$w = \frac{ff\Taurus}{rr}\,\frac{\theta}{\sqrt{(1+\eta\eta+\theta\theta)}},$$

and hence, $uu+vv+ww = \mathrm{VV} = \frac{f^4\Taurus\Taurus}{r^4}$: and rr is function of x itself, thus of the dependent variables y and z.

89. Since $udx+vdy+wdz$ must be a complete differential, the integral of which is denoted $= S$, we have:

$$d\mathrm{S} = \frac{ff\Taurus}{rr}\,\frac{dx(1+\eta\eta+\theta\theta)}{\sqrt{(1+\eta\eta+\theta\theta)}} = \frac{ff\Taurus}{rr}\,dx\sqrt{(1+\eta\eta+\theta\theta)}.$$

Moreover, $dx\sqrt{(1+\eta\eta+\theta\theta)}$ expresses the element of the tube itself; if we denote it by ds, we shall obtain $d\mathrm{S} = \frac{ff\Taurus\, ds}{rr}$: although $\Taurus$ is a function of the time,[48] here we fix the time and, furthermore, the quantities s and rr do not depend on time but only on the shape of the tube; thus we have $\mathrm{S} = \Taurus \int \frac{ffds}{rr}$.

90. Turning now to the pressure p which is found at the point of the tube λ, the quantity U has to be considered; it arises from the differentiation of the quantity S, if the time only is considered as variable, so that we have $\mathrm{U} = \frac{d\mathrm{S}}{dt}$. Thus, since the integral formula $\int \frac{ffds}{rr}$ does not involve time t, on the one hand we shall have $\frac{d\mathrm{S}}{dt} = \mathrm{U} = \frac{d\mathrm{U}}{dt}\int\frac{ffds}{rr}$, and on the other hand it will follow from §. 80 that:

$$\mathrm{T} = \frac{f^4\Taurus\Taurus}{r^4} + \frac{2d\Taurus}{dt}\int\frac{ffds}{rr}.$$

Therefore, after introducing arbitrary actions of forces Q, q and Φ, the pressure at λ will be

$$p = \mathrm{C} + \int(Q\,dx + q\,dy + \Phi\,dz) - \frac{f^4\Taurus\Taurus}{r^4} - \frac{2d\Taurus}{dt}\int\frac{ffds}{rr}$$

[48] As was stated in §. 88.

This is that same formula which is commonly written for the motion of a fluid through tubes; but now much more widely valid, since arbitrary forces acting on the fluid are assumed here, while this formula is commonly restricted to gravity alone. Meanwhile it is in order to remember that the three forces Q, q and Φ must be such that the differential formula $Q\,dx + q\,dy + \Phi\,dz$ be complete, that is, admit integration.

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