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Back to the primordial universe by a Monge–Ampère–Kantorovich mass transportation method

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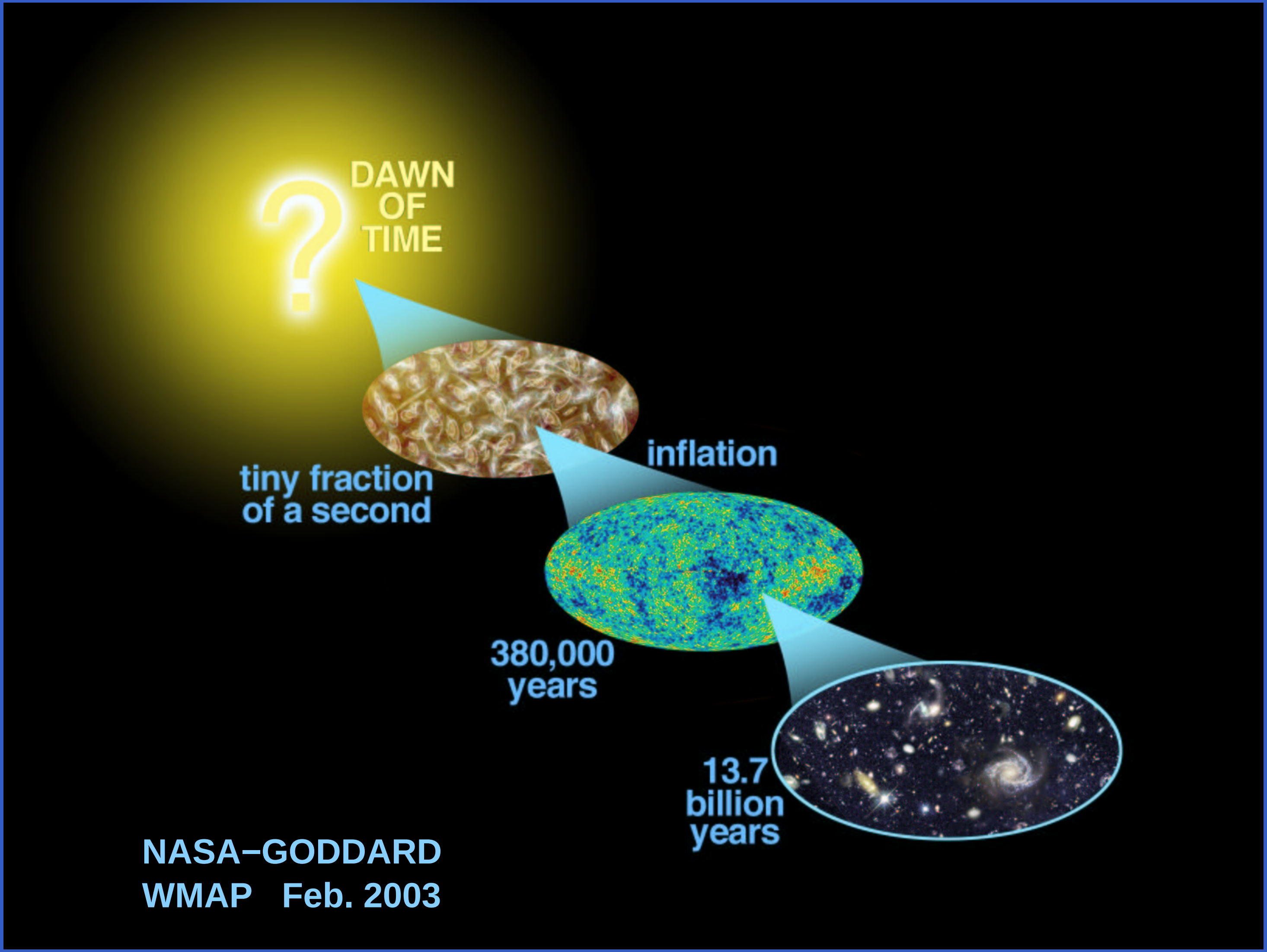
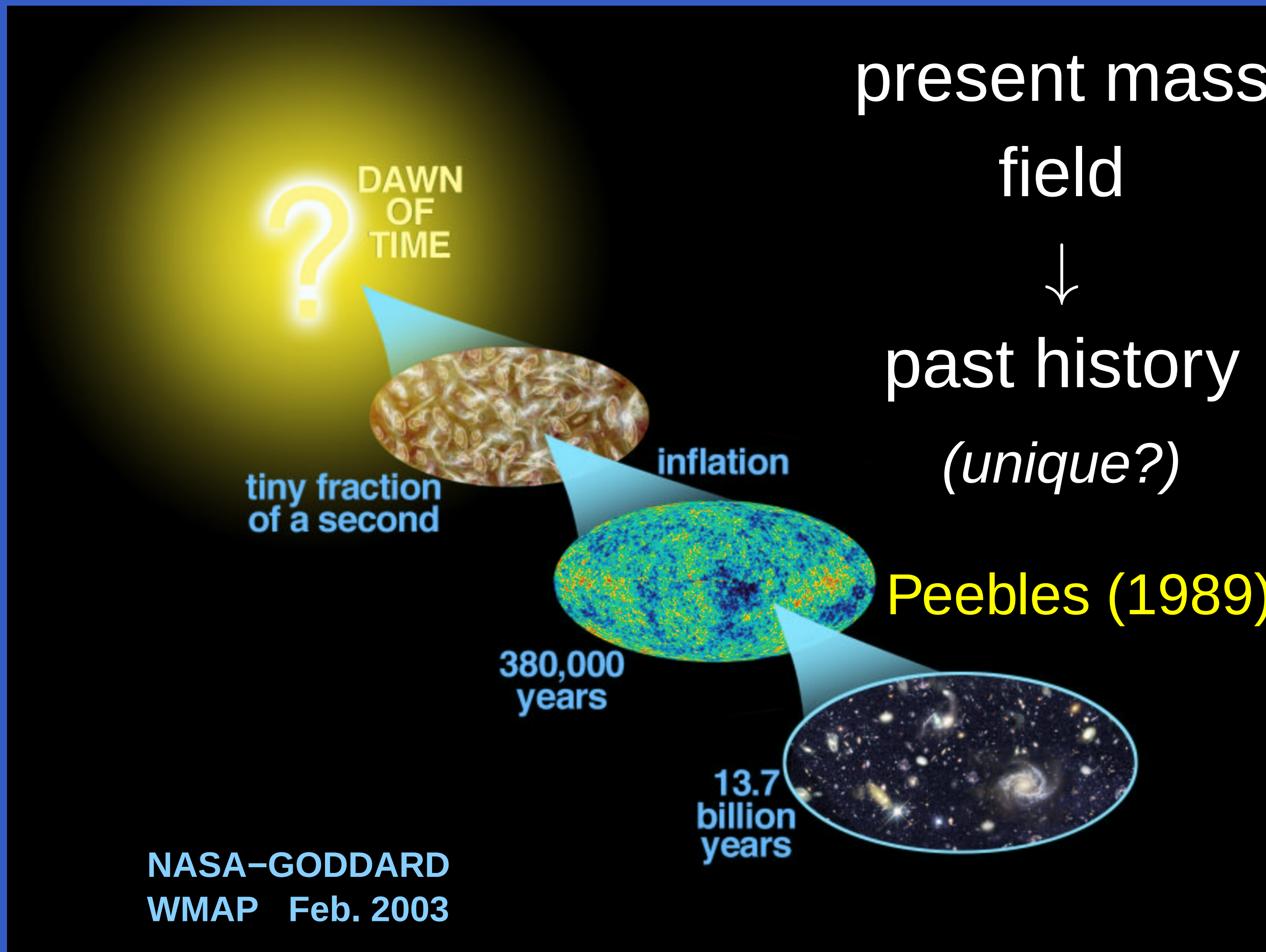
- Y. Brenier, U. Frisch, M. Hénon, G. Loeper, S. Matarrese, R. Mohayaee, A. Sobolevskiĭ *Mon. Not. R. Astron. Soc.* (2003, submitted), **astro-ph/0304214**
- U. Frisch, S. Matarrese, R. Mohayaee, A. Sobolevski *Nature* **417** (2002) 260–262

Brief history of Universe



History of Universe reconstructed



present mass field

$\downarrow$

past history

(unique?)

Peebles (1989)

Fluid dynamics in expanding universe

Euler:

$$\partial_\tau \mathbf{v} + (\mathbf{v} \cdot \nabla_{\mathbf{x}})\mathbf{v} = -\frac{3}{2\tau}\left(\mathbf{v} + \nabla_{\mathbf{x}}\varphi_{\mathrm{g}}\right)$$

Mass conservation:

$$\partial_\tau \rho + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{v}) = 0$$

Poisson:

$$\nabla_{\mathbf{x}}{}^2 \varphi_{\mathrm{g}} = \frac{\rho - 1}{\tau}$$

Fluid dynamics in expanding universe

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Mass conservation:

$$\partial_\tau \rho + \nabla_{\mathbf{x}}\cdot(\rho\mathbf{v}) = 0$$

Poisson:

$$\nabla_{\mathbf{x}}{}^2\varphi_{\mathrm{g}} = \frac{\rho - 1}{\tau}$$

“Slaving” as $\tau \to 0$:

$$\mathbf{v}_{\mathrm{in}} = -\nabla_{\mathbf{x}}\varphi_{\mathrm{g}}^{(\mathrm{in})}, \qquad \rho_{\mathrm{in}} = 1$$

Fluid dynamics in expanding universe

Euler:

$$\partial_\tau \mathbf{v} + (\mathbf{v} \cdot \nabla_{\mathbf{x}})\mathbf{v} = -\frac{3}{2\tau}(\mathbf{v} + \nabla_{\mathbf{x}}\varphi_{\mathrm{g}})$$

Mass conservation:

$$\partial_\tau \rho + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{v}) = 0$$

Poisson:

$$\nabla_{\mathbf{x}}{}^2 \varphi_{\mathrm{g}} = \frac{\rho - 1}{\tau}$$

Zeldovich approximation:

$$\mathbf{v} \equiv -\nabla_{\mathbf{x}}\varphi_{\mathrm{g}}$$

late 1960s

Potential Lagrangian map

“Zeldovich” equation:

$$\partial_\tau \mathbf{v} + \mathbf{v} \cdot \nabla_{\mathbf{x}} \mathbf{v} = 0$$

Potential Lagrangian map

"Zeldovich" equation:

$$\partial_\tau \mathbf{v} + \mathbf{v} \cdot \nabla_{\mathbf{x}} \mathbf{v} = 0$$

$$\mathbf{v}_{\rm in} = -\nabla_{\mathbf{x}} \varphi_{\rm g}^{(\rm in)} \Rightarrow \mathbf{v} = -\nabla_{\mathbf{x}} \varphi_{\rm v}$$

Potential Lagrangian map

"Zeldovich"/Burgers equation:

$$\partial_\tau \mathbf{v} + \mathbf{v} \cdot \nabla_{\mathbf{x}} \mathbf{v} = 0$$

$$\mathbf{v}_{\mathrm{in}} = -\nabla_{\mathbf{x}} \varphi_{\mathrm{g}}^{(\mathrm{in})} \Rightarrow \mathbf{v} = -\nabla_{\mathbf{x}} \varphi_{\mathrm{v}}$$

Potential Lagrangian map

"Zeldovich"/Burgers equation:

$$\partial_\tau \mathbf{v} + \mathbf{v} \cdot \nabla_{\mathbf{x}} \mathbf{v} = 0$$

$$\mathbf{v}_{\mathrm{in}} = -\nabla_{\mathbf{x}} \varphi_{\mathrm{g}}^{(\mathrm{in})} \Rightarrow \mathbf{v} = -\nabla_{\mathbf{x}} \varphi_{\mathrm{v}}$$

$$\mathbf{x} = \mathbf{q} + \tau \mathbf{v}_{\mathrm{in}}(\mathbf{q}) = \mathbf{q} - \tau \nabla \varphi_{\mathrm{g}}^{(\mathrm{in})}(\mathbf{q})$$

$$= \nabla \left[\frac{|\mathbf{q}|^2}{2} - \tau \varphi_{\mathrm{g}}^{(\mathrm{in})}(\mathbf{q}) \right]$$

$$= \nabla \Phi(\mathbf{q})$$

Bertschinger–Dekel (1989)

Potential Lagrangian map

"Zeldovich"/Burgers equation:

$$\partial_\tau \mathbf{v} + \mathbf{v} \cdot \nabla_{\mathbf{x}} \mathbf{v} = 0$$

$$\mathbf{v}_{\rm in} = -\nabla_{\mathbf{x}} \varphi_{\rm g}^{(\rm in)} \Rightarrow \mathbf{v} = -\nabla_{\mathbf{x}} \varphi_{\rm v}$$

$$\mathbf{x} = \mathbf{q} + \tau \mathbf{v}_{\rm in}(\mathbf{q}) = \mathbf{q} - \tau \nabla \varphi_{\rm g}^{(\rm in)}(\mathbf{q})$$

$$= \nabla \left[\frac{|\mathbf{q}|^2}{2} - \tau \varphi_{\rm g}^{(\rm in)}(\mathbf{q}) \right]$$

$= \nabla \Phi(\mathbf{q})$ . . . (graph) invertible if Φ is convex

Bertschinger–Dekel (1989)

The Monge–Ampère equation

Mass conservation:

$$\det \nabla_{\mathbf{q}} \mathbf{x} = \frac{\rho_{\mathrm{in}}(\mathbf{q})}{\rho_0(\mathbf{x}(\mathbf{q}))} = \frac{\text{initial density}}{\text{present density}}$$

The Monge–Ampère equation

Mass conservation, $\rho_{\text{in}} = 1$:

$$\det \nabla_{\mathbf{q}} \mathbf{x} = \frac{1}{\rho_0(\mathbf{x}(\mathbf{q}))} = \frac{\text{initial density}}{\text{present density}}$$

The Monge–Ampère equation

Mass conservation, $\rho_{\text{in}} = 1$:

$$\det \nabla_{\mathbf{q}} \mathbf{x} = \frac{1}{\rho_0(\mathbf{x}(\mathbf{q}))} = \frac{\text{initial density}}{\text{present density}}$$

As $\mathbf{x}(\mathbf{q}) = \nabla_{\mathbf{q}} \Phi(\mathbf{q})$:

$$\det(\nabla_{q_i} \nabla_{q_j} \Phi(\mathbf{q})) = \frac{1}{\rho_0\big(\nabla_{\mathbf{q}} \Phi(\mathbf{q})\big)}$$

The Monge–Ampère equation

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Legendre–Fenchel transform:

$$\Theta(\mathbf{x}) = \max_{\mathbf{q}} \big(\mathbf{x} \cdot \mathbf{q} - \Phi(\mathbf{q})\big)$$

The Monge–Ampère equation

Mass conservation, $\rho_{\text{in}} = 1$:

$$\det \nabla_{\mathbf{x}} \mathbf{q} = \frac{\rho_0(\mathbf{x})}{1}$$

As $\mathbf{q}(\mathbf{x}) = \nabla_{\mathbf{x}} \Theta(\mathbf{x})$:

$$\det(\nabla_{x_i} \nabla_{x_j} \Theta(\mathbf{x})) = \rho_0(\mathbf{x})$$

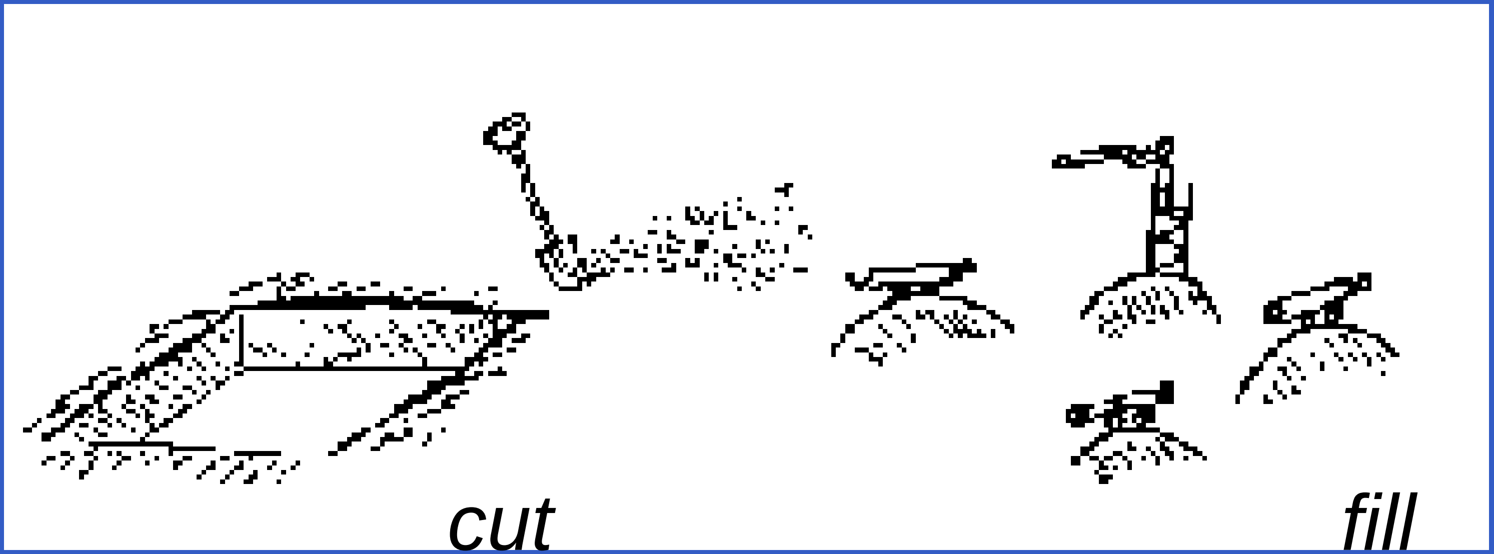
$\rho_0(\mathbf{x})$ prescribed

1820

Monge's mass transportation problem

*. . .Il n'est pas indifférent que telle molécule de déblai soit transportée dans tel ou tel autre endroit du remblai, mais qu'il y a une certaine distribution à faire des molécules du premier dans le second, d'après laquelle la somme de ces produits sera la moindre possible, et le prix du transport total sera un **minimum**.*

Monge’s mass transportation problem



. . .It is not indifferent that any given molecule of the cuts be transported to this or that place in the fills, but there ought to be a certain distribution of molecules of the former into the latter, according to which the sum of these products will be the least possible, and the cost of transportation will be a ***minimum****.*

Monge's mass transportation problem

For given $\rho_{\rm in}(\mathbf{q})$, $\rho_0(\mathbf{x})$ minimize

$$\int |\mathbf{x}(\mathbf{q}) - \mathbf{q}| \; \rho_{\rm in}(\mathbf{q})\, d\mathbf{q} = \int |\mathbf{x} - \mathbf{q}(\mathbf{x})| \; \rho_0(\mathbf{x})\, d\mathbf{x}$$

over all $(\mathbf{x}(\mathbf{q}), \mathbf{q}(\mathbf{x}))$ such that $\rho_{\rm in}(\mathbf{q})\, d\mathbf{q} = \rho_0(\mathbf{x})\, d\mathbf{x}$

Monge, Ampère, mass transportation

For given $\rho_{\text{in}}(\mathbf{q})$, $\rho_0(\mathbf{x})$ minimize

$$\int |\mathbf{x}(\mathbf{q}) - \mathbf{q}|^{\mathbf{2}} \rho_{\text{in}}(\mathbf{q})\, d\mathbf{q} = \int |\mathbf{x} - \mathbf{q}(\mathbf{x})|^{\mathbf{2}} \rho_0(\mathbf{x})\, d\mathbf{x}$$

over all $(\mathbf{x}(\mathbf{q}), \mathbf{q}(\mathbf{x}))$ such that $\rho_{\text{in}}(\mathbf{q})\, d\mathbf{q} = \rho_0(\mathbf{x})\, d\mathbf{x}$

Monge, Ampère, mass transportation

Theorem (Brenier 1987, 1991) *The minimizing maps are gradients of convex functions:*

$$\mathbf{x}(\mathbf{q}) = \nabla_{\mathbf{q}}\Phi(\mathbf{q}), \quad \mathbf{q}(\mathbf{x}) = \nabla_{\mathbf{x}}\Theta(\mathbf{x})$$

Φ *and* Θ *solve suitable Monge–Ampère equations*

For given $\rho_{\mathrm{in}}(\mathbf{q})$, $\rho_0(\mathbf{x})$ minimize

$$\int |\mathbf{x}(\mathbf{q}) - \mathbf{q}|^2 \rho_{\mathrm{in}}(\mathbf{q})\, d\mathbf{q} = \int |\mathbf{x} - \mathbf{q}(\mathbf{x})|^2 \rho_0(\mathbf{x})\, d\mathbf{x}$$

over all $(\mathbf{x}(\mathbf{q}), \mathbf{q}(\mathbf{x}))$ such that $\rho_{\mathrm{in}}(\mathbf{q})\, d\mathbf{q} = \rho_0(\mathbf{x})\, d\mathbf{x}$

Kantorovich relaxation

For given $\rho_{\text{in}}(\mathbf{q})$, $\rho_0(\mathbf{x})$ minimize

$$\int |\mathbf{x} - \mathbf{q}|^2 \rho(\mathbf{q}, \mathbf{x})\, d\mathbf{q}\, d\mathbf{x}$$

over all $\rho(\mathbf{q}, \mathbf{x})$ such that

$$\int \rho(\mathbf{q}, \mathbf{x})\, d\mathbf{x} = \rho_{\text{in}}(\mathbf{q})$$
$$\int \rho(\mathbf{q}, \mathbf{x})\, d\mathbf{q} = \rho_0(\mathbf{x})$$

1942

Discretization and assignment

Discrete densities:

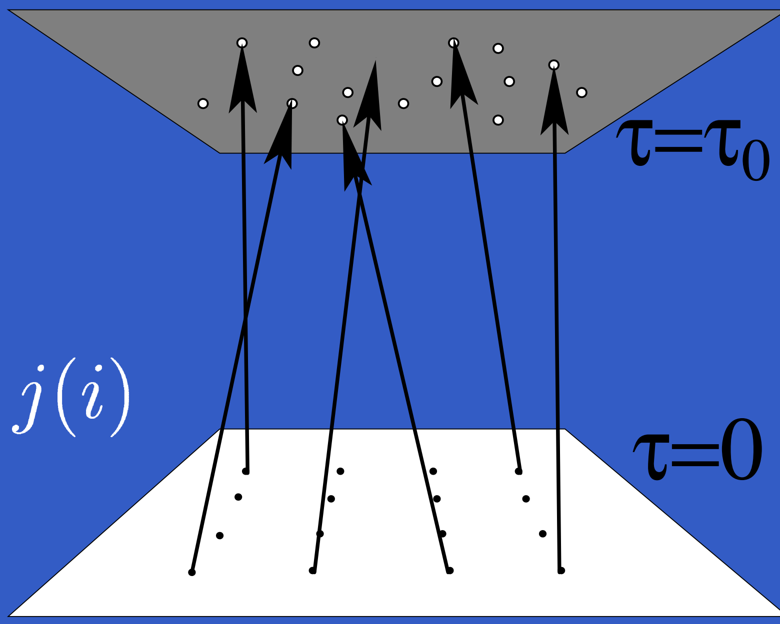
$$\rho_0(\mathbf{x}) = \sum_{i=1}^{N} \delta(\mathbf{x} - \mathbf{x}_i), \quad \rho_{\text{in}}(\mathbf{q}) = \sum_{j=1}^{N} \delta(\mathbf{q} - \mathbf{q}_j)$$

Minimize the cost

$$\sum_{i=1}^{N} |\mathbf{x}_i - \mathbf{q}_{j(i)}|^2$$

over all permutations $i \mapsto j(i)$

of $\{ 1, 2, \ldots, N \}$



Relaxation and the dual problem

Minimize $\sum_{i=1}^{N} |\mathbf{x}_i - \mathbf{q}_{j(i)}|^2$

over all permutations $i \mapsto j(i)$

Relaxation and the dual problem

Minimize $\displaystyle\sum_{i,j=1}^{N} |\mathbf{x}_i - \mathbf{q}_j|^2 f_{ij}$

over all bistochastic matrices:

$$f_{ij} \geq 0, \qquad \sum_{k=1}^{N} f_{kj} = \sum_{k=1}^{N} f_{ik} = 1$$

Relaxation and the dual problem

Minimize $\sum_{i,j=1}^{N} |\mathbf{x}_i - \mathbf{q}_j|^2 f_{ij}$

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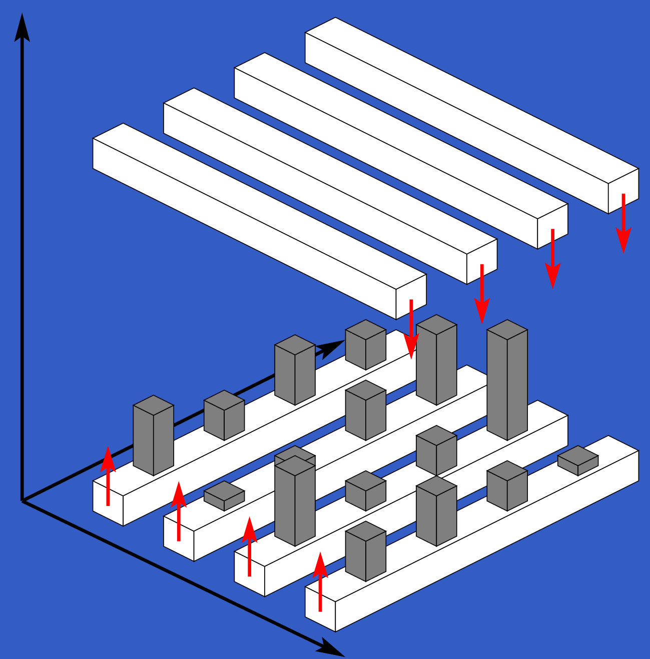
Dual problem:

Minimize $\sum_{i=1}^{N} \alpha_i - \sum_{j=1}^{N} \beta_j$

$$\alpha_i - \beta_j \geq C - |\mathbf{x}_i - \mathbf{q}_j|^2$$



Hénon 1950s–1990s

Relaxation and the dual problem

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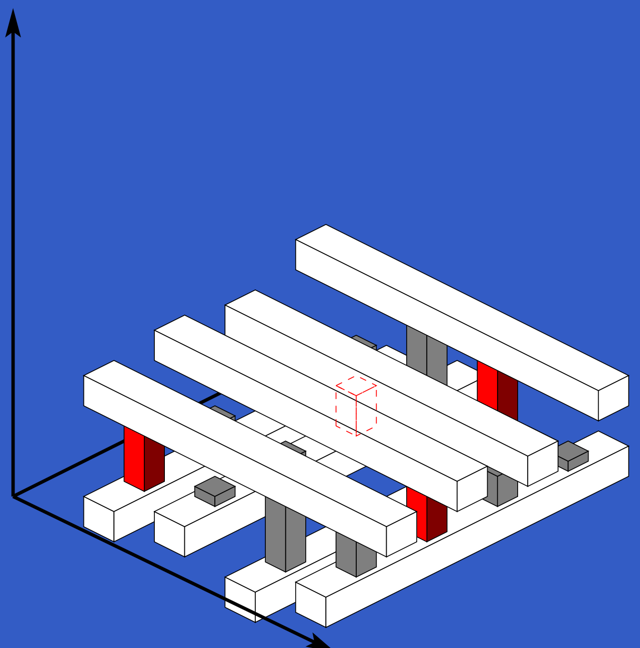


Hénon 1950s–1990s

Time complexity

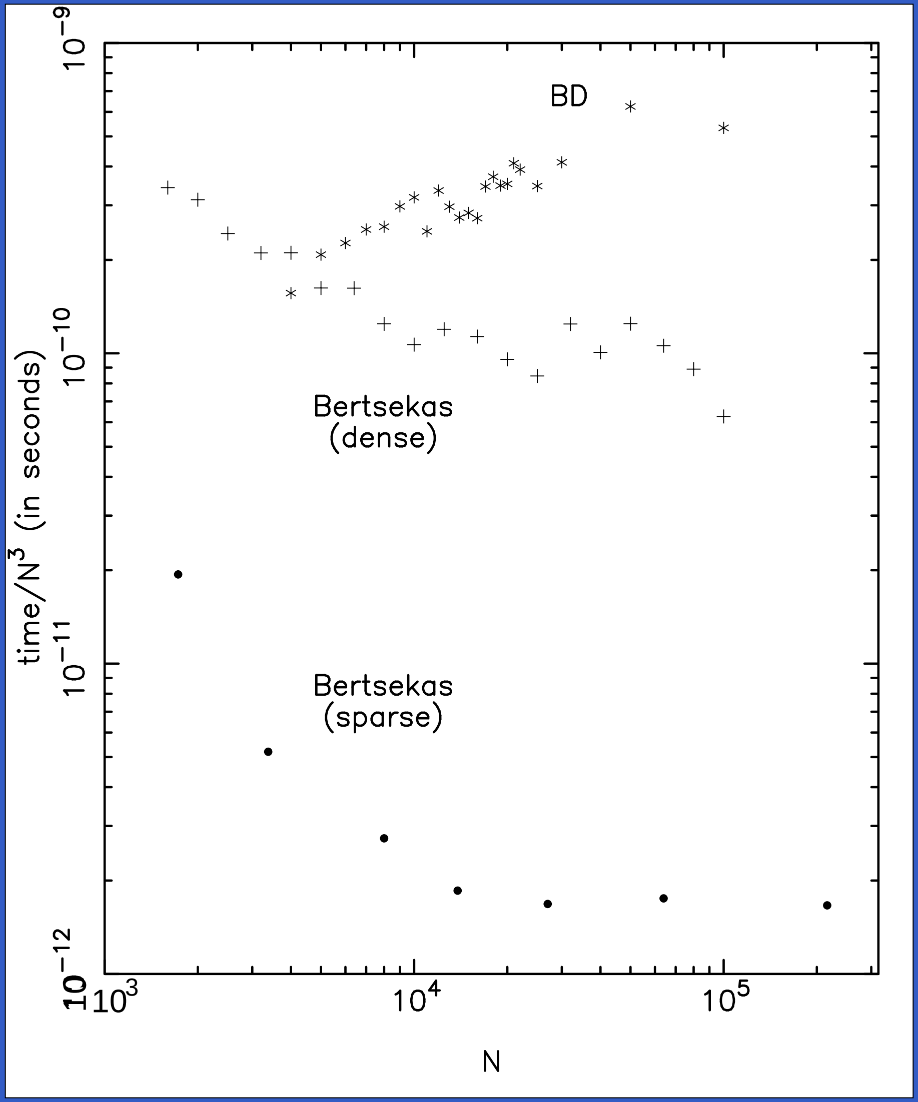
$*$ Burkard & Derigs 1980

Bertsekas 1979–2003

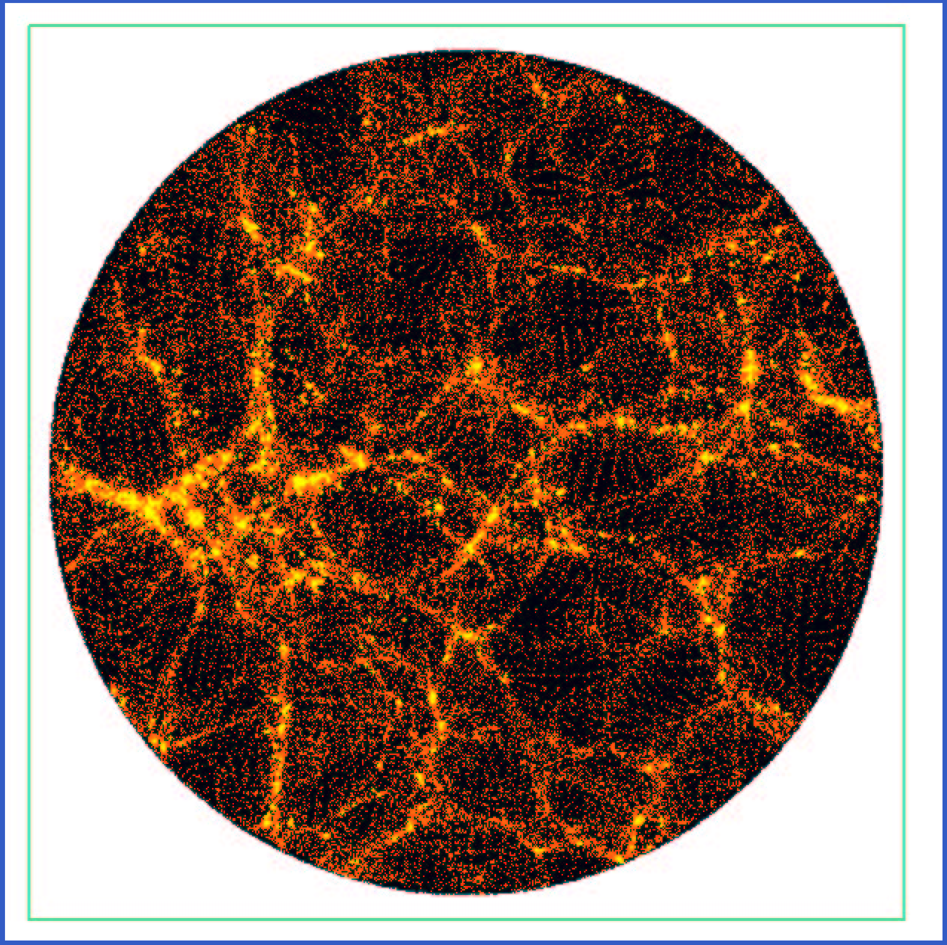
$+$ “dense” auction

• “sparse” auction

(time in seconds divided by N^3)

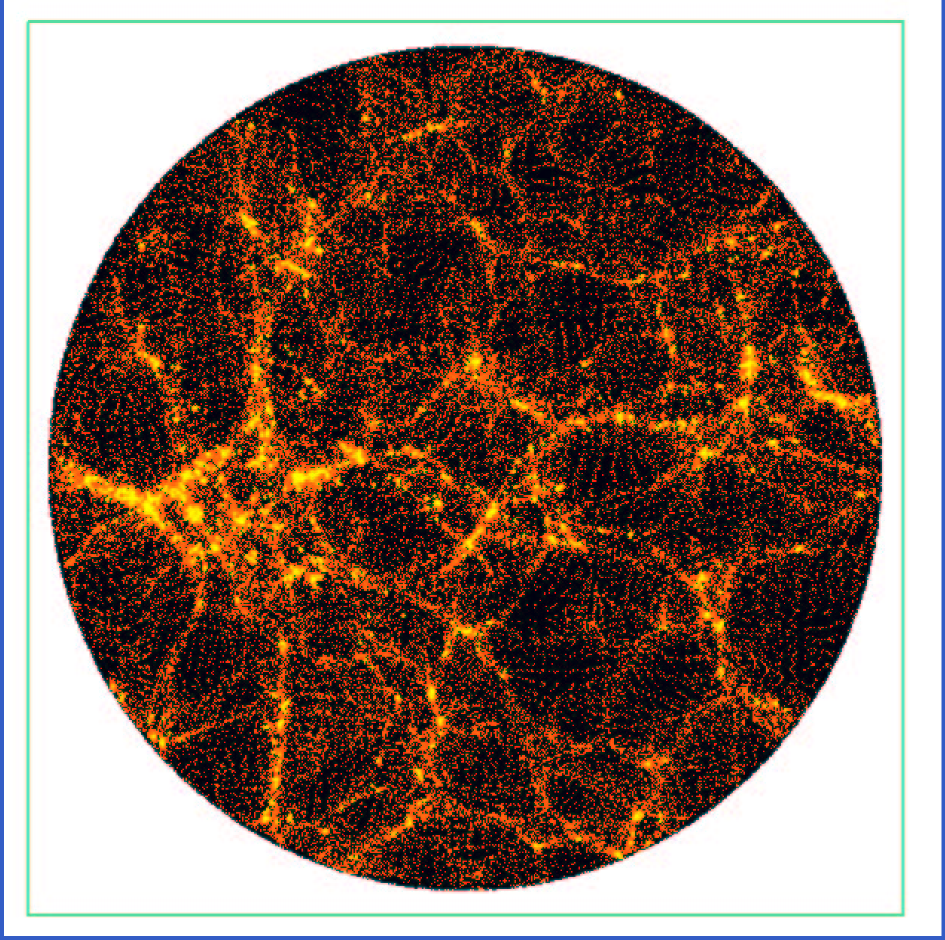


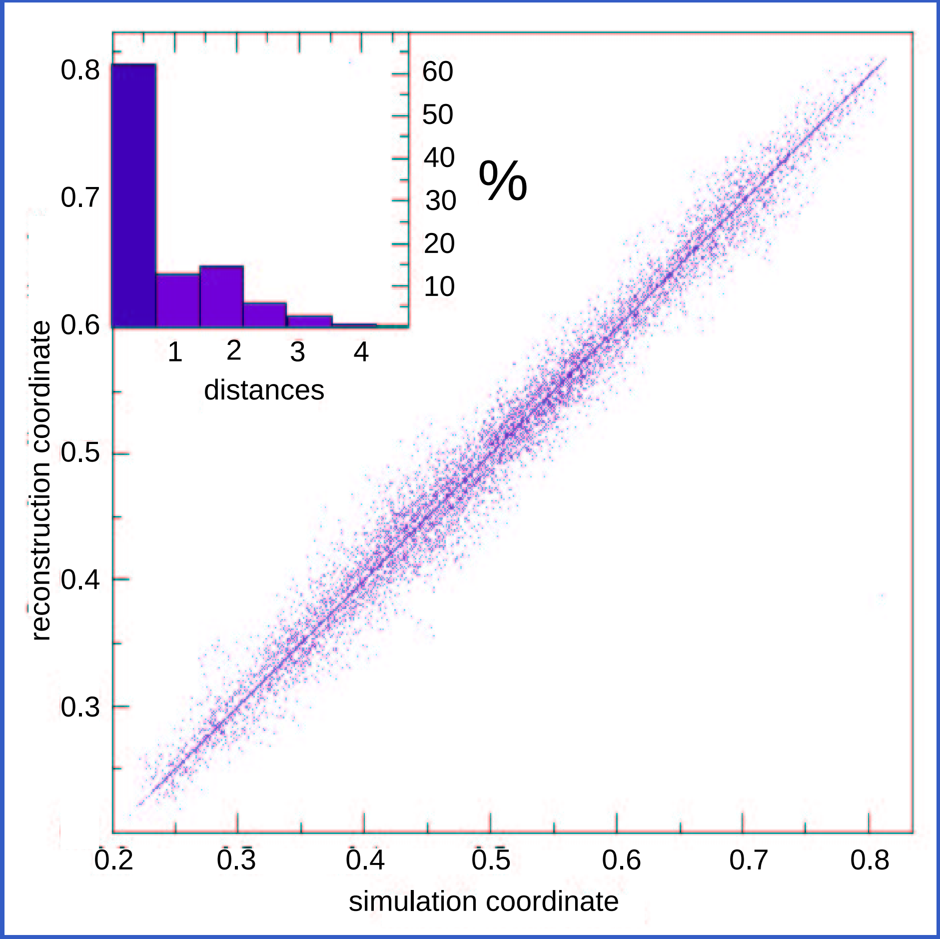
Testing against N-body simulation



128^3 points in a box of $200\,h^{-1}$ Mpc size (middle 10% slice)

Testing against N-body simulation





Euler–Poisson variational problem

Minimize $\displaystyle\int_0^{\tau_0} \mathrm{d}\tau \int \mathrm{d}^3\mathbf{x}\ \tau^{3/2} \left(\rho|\mathbf{v}|^2 + \frac{3}{2}|\nabla_{\mathbf{x}}\varphi_{\mathrm{g}}|^2 \right)$

- Mass conservation: $\partial_\tau \rho + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{v}) = 0$
- Poisson equation: $\nabla_{\mathbf{x}}{}^2 \varphi_{\mathrm{g}} = \dfrac{\rho - 1}{\tau}$
- $\rho(\mathbf{x}, 0) = \rho_{\mathrm{in}}(\mathbf{x})$, $\rho(\mathbf{x}, \tau_0) = \rho_0(\mathbf{x})$

Theorem (Loeper 2003) *Up to a change of variables this is a convex minimization problem with a unique solution* $(\rho, \mathbf{v}, \varphi_{\mathrm{g}})$.