

Novel approach to turbulence and mixing

Research Report

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I. INTRODUCTION

Turbulence is part of daily experience: no microscopes or telescopes are needed to notice the meanders of cigarette smoke, the gracious arabesques of cream poured into coffee or turbulent whirls in a mountain torrent. The word “turbulence” indicated first the incoherent movements of the crowd (Latin: turba), then the whirls of leaves or dust. Since Leonardo da Vinci (around the year 1500) the term took its modern meaning of “incoherent and disordered movements of air or water”.

Turbulence appears everywhere: formation of galaxies in the primordial Universe, release of heat produced by nuclear reactions in the heart of the Sun, atmospheric and oceanic motions, flow around cars, boats and airplanes, flow of blood in arteries and the heart, and so forth. Turbulence is simultaneously a very fundamental subject, of interest to mathematicians, physicists, astronomers and geophysicists alike, and a subject with numerous practical ramifications in astrophysics, meteorology, engineering, and medicine. In the sense that techniques of analysis and models used to analyze turbulence can be carried over to the study of a number of other fluctuating phenomena such as stock market value, the subject is of relevance even beyond hard sciences. Moreover, turbulence is a paradigm of far-from-equilibrium phenomena which resisted so far description within any general framework.

The familiarity of turbulence as a phenomenon should not be misunderstood to mean that the subject is pedestrian. It is difficult to quantify precisely and analyze theoretically. Many famous

mathematicians, physicists and mechanics attempted to develop the true theory of turbulence, one that would allow better control, or even the suppression of turbulence altogether (as apparently done instinctively by dolphins). The basic equations have been known for two centuries, and it is ironic that progress in the understanding and description of turbulence has been very slow in spite of the efforts of numerous theoreticians and experimentalists. It is not yet clear how one may study all the features of turbulence from the equations alone.

An important step in our understanding of turbulence was the development of an approximate theory, resting on powerful intuition and dimensional analysis, proposed by the Russian mathematician Andrei Kolmogorov in the forties. Data from experiments and numerical simulations demonstrate substantial deviations from Kolmogorov's predictions which manifest themselves geometrically through the phenomenon of *intermittency*: small whirls are relatively much rarer in space than large ones, and come in "fractal" packets, somewhat similar to the branch-structure of broccoli. It also manifests itself through the presence of *anomalous scaling*: various statistical quantities follow power laws as functions of the scale being analyzed. Some exponents of those power laws seem universal yet cannot be predicted by mere dimensional analysis. Each exponent needs a separate theory.

After eluding theoretical description for forty years, important instances of intermittency and anomalous scaling have been explained for the first time in the last few years, thanks to a fruitful collaboration of Finnish, French, Israeli, Italian and Russian scientists. This advance was made possible by the development of new theoretical tools, predictions of which have been validated by experiments and numerical simulations of a new type.

This line of work is based on the analysis of random walks and is thus a direct continuation of Einstein's 1905 work on Brownian motion. We owe to Einstein even more: in the early fifties he encouraged one of his later-year assistants, Robert Kraichnan, towards studying turbulence. During the sixties, Kraichnan formulated his model of passive scalar turbulence. In 1995, the first analytical solutions of the Kraichnan model were found by the present collaboration, which opened a breakthrough in turbulence studies. In contrast to Brownian motion and similar near-equilibrium problems, the turbulent velocities of different particles are correlated. This property dictated the use of a novel approach that is based on the consideration of multipoint configurations and statistical geometry. The results obtained allowed scientists a glimpse of the intimate nature and some of the most fundamental aspects of turbulence and other phenomena that are far from equilibrium. In particular, a novel mechanism of symmetry breaking via the statistical integrals of motion has been discovered. For so-called direct turbulent cascades (that proceed by fragmentation towards small scales) the breakdown of scale invariance has been thus described. On the contrary, more symmetries than expected were discovered in the statistics of the inverse cascade (in two-dimensional turbulence), including conformal symmetry, which allowed to link turbulence with critical phenomena.

The new tools allowed much better description of mixing by different types of random flows. Further, the development of new quantitative methods of description allowed the prediction of the probabilities of rare events in different non-equilibrium systems and the design of new numerical methods that dramatically improved the performance of statistical evaluations for turbulent flows.

Armed with the new understanding of turbulence and with the new tools thus developed, scientists went to study flows of complex fluids, which include particles and polymers. Serious progress has been achieved this way in the statistical description of water droplets in turbulent clouds, particularly their collision rate, thus paving the way for more precise predictions of the initiation of rain. In fluid flow with polymers, the remarkable discovery of elastic turbulence was made; this discovery has thoroughly changed our perspective of the basics of fluid mechanics and non-equilibrium physics and has opened practical possibilities unimagined before.

II. SCIENTIFIC BACKGROUND

Turbulence studies span five centuries, from Leonardo da Vinci to the present day [1, 2] when they are part both of fluid mechanics and of statistical physics. The aim is to understand the special features of turbulent flows as well as general common features of systems with many strongly interacting degrees of freedom far from equilibrium.

Turbulent motion is irregular in both time and space. For example, the motion of a jet of water is turbulent when the Reynolds number $Re = VL/\nu$ —based on the jet velocity V , the orifice diameter L and fluid viscosity ν —becomes “large”. In this case, flow perturbations produced on scales of the order L are dominated by nonlinear effects. This nonlinearity produces motions of increasingly smaller scales until viscous dissipation truncates the process at a characteristic scale η that is much smaller than L . The scales between L and η , which, roughly speaking, remain unaffected directly by (boundary-induced) forcing as well as viscosity, form the so-called inertial range. According to a suggestion made in the thirties by the British meteorologist Lewis Fry Richardson [3], the kinetic energy that is injected at scales of the order L flows through the inertial range in a cascade-like process, becoming faster at each successive step, and is dissipated eventually at scales of the order η . Indeed, nonlinear energy transfer process is a dominant feature in the inertial range.

Because of this special property of the inertial range, it is natural to ask if some attributes of the inertial range are universal—that is, independent of the details of external forcing and viscosity [4]. Another question of the universality problem is what features are common to different turbulent systems. A quest for universality is motivated by the hope to distinguish some general principles that govern far-from-equilibrium systems in the spirit of variational principles that govern thermal equilibrium. Since all dynamical features of turbulence are irregular, the questions can be posed and answered only in terms of statistical averages—computed for appropriate regions of space, intervals of time, or suitably defined ensembles.

Conservation laws impose constraints on the dynamics, and so conserved quantities must play an essential role in addressing questions of universality. Conservation laws are broken by the large-scale forcing (usually due to boundary conditions) as well as the dissipation at small scales (usually through fluid viscosity), but—as already remarked—neither feature dominates the inertial range. The notion of the energy flux across scales explains the basic macroscopic manifestation of turbulence in the inertial range. On the average, this flux of energy equals the dissipation of the turbulent kinetic energy, which is an integral of motion in the absence of viscosity. A characteristic feature of turbulence is that the average rate of energy dissipation per unit mass $\langle\epsilon\rangle$ has a finite limit as the viscosity coefficient tends to zero, or the Reynolds number tends to infinity [5]. This is probably the first example of what is called “anomaly” in modern field-theoretical language: a symmetry of the inviscid equation (here, time-reversal invariance) remains broken even as the symmetry-breaking factor (viscosity) vanishes. A direct analogy between dissipative anomalies in turbulence and axial anomaly (breakdown of chirality conservation) in quantum field theory was noticed in [6].

An exact relation between the dissipative anomaly and a constant flux has been derived by Kolmogorov [7]. One characterizes the motion at the scale r by the velocity difference δv_r measured between two points separated by the distance r . The energy flux across the scale r in the inertial interval is proportional to $(\delta v_r)^3/r$. Kolmogorov has shown that when $L \gg r \gg \eta$ the flux constancy relation reads $S_3 \equiv \langle(\delta v_r)^3\rangle = -4\langle\epsilon\rangle r/5$. This so-called the 4/5-ths law describes the third-order moment of velocity difference, S_3 , which is called the third-order structure function. To describe completely the single-time statistics of the velocity difference at a given scale, one ought to know the structure functions of other orders n that are defined similarly, $S_n \equiv \langle(\delta v_r)^n\rangle$. One needs other ideas in addition to energy flux to describe statistics of all orders. Here, the most important question is whether the scale-invariance, broken by large-scale forcing, is restored when the measurement scale

becomes small compared to the scale of forcing. Kolmogorov assumed that this would be so—in particular, he proposed that the probability density function (pdf) of the velocity difference, $p(\delta v_r, r)$, would be scale-invariant, or a function of the single argument $\delta v_r/(\langle \epsilon \rangle r)^{1/3}$ rather than of two, δv_r and r [8]. This reasoning implies that the structure functions are power laws with exponents that depend linearly on the moment order, or $S_n \propto r^{\zeta_n}$ with $\zeta_n = n/3$, as also predicted by dimensional analysis.

Experiments show unambiguously that the scaling exponents of structure functions depart from Kolmogorov’s scaling. This breakdown (or non-restoration) of scale invariance, now called anomalous or multifractal scaling, takes place not only for (vector) velocity field but also for scalar fields (like temperature, salinity or the concentrations of reagents in the chemical reactor) carried by a turbulent flow. The anomalous scaling is arguably an important feature of turbulence, setting it apart from the usual critical phenomena: one needs to work out the behavior of each order moment independently without succumbing to dimensional analysis. Since ζ_n is convex then S_{2n}/S_2^n increases as $r \rightarrow 0$ that is the statistics of fluctuations is getting more non-Gaussian as one goes along the cascade. Apart from the fundamental aspect of broken symmetry, it is also of much practical importance. It is still far from practical to fully compute almost any realistic turbulent flow (water flow in kitchen pipes has about 10^9 excited degrees of freedom while cloud air flow has about 10^{18}). The life of physicists and engineers would be much easier if turbulence statistics would possess scale invariance so that one could model unresolved small scale modes simply by re-scaling the statistics of the resolved ones. The multifractality thus speaks to the limitations on realistically modelling small scales in computer simulations.

According to the 4/5-ths law, the third-order moment of turbulent velocity increments is determined completely by $\langle \epsilon \rangle$. Just as the finiteness of the energy dissipation is a consequence of the breakdown of time-reversal symmetry in the inertial range, are there other broken symmetries that yield other statistics? Are moments of other orders set by other integrals of motion? This is a fundamental question of turbulence—indeed of modern statistical physics.

III. DISCOVERY OF ZERO MODES AND THE DECADE OF RAPID PROGRESS

The best way to solve fundamental questions is to have a model which is rich enough to contain all the essential ingredients and simple enough to be solvable. Such models in physics are few and their discovery occurs rarely. Since the most mysterious part of turbulence problem is the high-order moments of turbulent fields, one needs a model that can treat multi-point configurations. Kraichnan’s profound insight was that spatial rather than temporal correlations of the velocity field matter for anomalous scaling [9, 10]. The Kraichnan model presumes the velocity to be fluctuating infinitely rapidly in time; this property makes the model amenable to analytical solution, while preserving anomalous scaling of spatial correlations, as has been shown by the present collaboration.

About thirty years elapsed between the introduction of the Kraichnan model and the discovery of its anomalous scaling. Short temporal correlations allow one to derive closed equations for the multi-point correlation functions of the passive scalar (or, equivalently, for the joint probability distribution that describes the simultaneous motion of several fluid particles). For two points, such an equation was solved by Kraichnan himself, reproducing within his model the Richardson law of relative pair diffusion [9]. Even for four points, one needs to solve a (partial differential) equation in six-dimensional space (six distances between particles). To overcome serious technical difficulties, a collaboration of fluid mechanicians, field theorists and statisticians was needed.

This has been accomplished simultaneously by two separate international collaborations, French-Finnish and Israeli-Russian, which later came to work together. The French-Finnish collaboration

chose as the point of departure Einstein's case of Brownian diffusion. Assuming fluid velocities at different points in space to be only weakly correlated, Gawędzki and Kupiainen, developed a perturbation theory which used Brownian motion as the zeroth approximation and in the first order gave anomalous scaling for the four-point correlation function of the passive scalar [11]. The Israeli-Russian collaboration considered the Kraichnan model in a space of very large dimensionality. While previous attempts to treat velocity field in infinite dimensionality failed, the new idea to treat passive scalar in this way succeeded. Distances between fluid particles fluctuate only weakly in a space of large dimensionality; this allowed Chertkov, Falkovich, Kolokolov and Lebedev to derive the anomalous exponent of the four-point correlation function of the passive scalar [12]. The two teams came into contact, their formulas were compared (in the limit of weak correlations in the space of large dimensionality where both approximations work) and were found to be in agreement; parallel publications followed in 1995 [11, 12]. The novelty of [11, 12], first and foremost, was the discovery of the so-called zero modes of turbulent diffusion that can be interpreted in terms of a qualitatively new type of integrals of motion which are conserved only on the average, yet determine the statistical properties of different systems. Simultaneously similar zero modes were discovered for a different model by Shraiman and Siggia [13] who were not involved in the present collaboration (and have since left the field).

The next step was undertaken by Chertkov and Falkovich who learnt how to generalize the results beyond four points [14]. Their work on the multi-point zero modes for large dimensionality was immediately followed by the work of Bernard, Gawędzki and Kupiainen for near-Brownian case [15]. That very year (1996), Pumir derived the three-point zero mode for the case of broken isotropy (external gradient of passive scalar) [16].

The solutions of the Kraichnan model [11–16] have played much the same role in turbulence as Onsager's solution of the Ising model in critical phenomena. The latter convinced physicists that the mean-field theory of Landau must be replaced, and the former convincingly demonstrated the inadequacy of Kolmogorov-like dimensional reasoning or simple closures.

IV. SIMULATIONS AND THE TEST OF THE THEORY

After the near-Brownian and the large-dimension expansion methods were introduced, a hefty debate took place. In both of these methods the scaling anomaly, that is the discrepancy between the actual scaling exponent and what is predicted by mere dimensional arguments, vanishes with the perturbation parameter. However, in the paper in which anomalous scaling was first predicted for the Kraichnan model [10], this vanishing did not take place. Furthermore, predictions for the scaling exponent from Ref. [10] seemed to be supported by the best available numerical simulations performed on large computers at Los Alamos. In the opinion of Uriel Frisch and his collaborators, the traditional methods of simulation, using spectral methods applied to the advection diffusion equation in Eulerian coordinates, did not give sufficiently clean scaling to settle the debate. After trying various methods for a few years, they discovered a new Lagrangian method in which the problem was rewritten in terms of fluid particles. This reduces the problem to solving ordinary stochastic differential equations by a Monte Carlo method and led to exquisite scaling laws, thereby bringing conclusive evidence in favour of the perturbation theory [17–19]. Similar ideas were used independently in the group of Itamar Procaccia [20].

Further numerical confirmation of the theoretical results obtained by perturbative techniques came from the direct analysis of the large dimensionality limit. While this remains numerically unattainable by means of an Eulerian (fixed-space) approach, resorting to the Lagrangian (particle-tracking) method allows to investigate scalar transport up to a thirty-dimensional configuration space [21].

The Lagrangian approach developed by Antonio Celani and co-authors gave a new boost to the study of turbulence and proved fundamental for settling several unsolved problems, in particular the frontogenesis and the statistics of rare fluctuations [22] (see Sec. VI) and organization of large-scale structures starting from a small-scale incoherent driving force [23, 24] (see Sect. IX).

Parallel to Lagrangian particle simulations, a direct solution of the dynamical equations governing turbulent flows was brought into a new level. Such computations had to face two problems until recently. First, the available computing power was relatively meager so that they could not make serious connection to theory, which invariably invoked very high Reynolds number and very high Schmidt number. Lagrangian properties approach their asymptotic state more slowly than Eulerian ones; hence, theoretical ideas are more difficult to test for the former. The second problem was that the versatility of the theory itself was modest. Tremendous advances in supercomputing power are taking place, as evidenced by machines like the Earth Simulator in Japan and the IBM Blue Gene in the US. This, together with the recent advances in theory described in other sections, make the present time ideal for creative synergy between theory and simulations. One can indeed perform direct numerical simulations (DNS) of the governing equations at high enough Reynolds number and with adequate enough grid resolution to be taken quite seriously. The numerical analysis in Eulerian coordinates is particularly effective for scalar transport in low-dimensional spaces. For instance, in two-dimensional turbulent advection by realistic Navier-Stokes turbulence, it has been shown that the anomalous scaling exponents are universal with respect to the choice of the pumping mechanism, this being a fundamental consequence of the existence of zero modes [25] (see Sec. V D).

For a much more difficult three-dimensional case, the research group of P.K. Yeung (in collaboration with Katepalli Sreenivasan, who has relocated from the US to Italy in the meantime) has accumulated over the last ten years a DNS database that has recently reached a Taylor-scale Reynolds number of 700 on a 2048^3 grid [26]. These latest data show relatively well-developed inertial range in the velocity field. In conjunction with this velocity field, the advection-diffusion equation for the passive scalar has also been solved on the same grid, this being the largest such computation to-date. In particular, the data show a clear evidence of an inertial-convective range in both the spectrum (predicted by Obukhov and Corrsin) and in structure functions (predicted by Yaglom). Yeung and Sreenivasan were also able to address another fundamental anomaly of turbulent mixing, namely, the anomalous scale dependencies of the departures from local isotropy. As we shall discuss in Sect. V C, those dependencies are also related to the zero modes. Yeung and Sreenivasan have shown that, in terms of normalized quantities, departures from isotropy are sustained at high Reynolds numbers (see Sec. V C), and that intermittency becomes progressively stronger. Further simulations of a similar size are also in progress for turbulent mixing at higher Schmidt number (the ratio of viscosity to diffusivity); as Schmidt number increases, there is a tendency for local isotropy to be restored and for intermittency to saturate [27]. These simulations are at high enough Reynolds numbers for even Lagrangian studies to be performed with confidence.

It should be emphasized that these databases, which consist of many tera-bytes of data stored on mass-storage archival systems of supercomputers in Italy and the US, provide great opportunities for analysis of predictions of the analytical theories. For example, appropriate software adopted for parallel computers has been developed to compute multi-fractal scaling properties, anomalous scaling of structure functions, acceleration statistics, Lagrangian statistics of fluid particles—singly, or in pairs or in groups of four—and conditional quantities along particle trajectories. Some of these quantities are of interest in practically important subjects such as combustion.

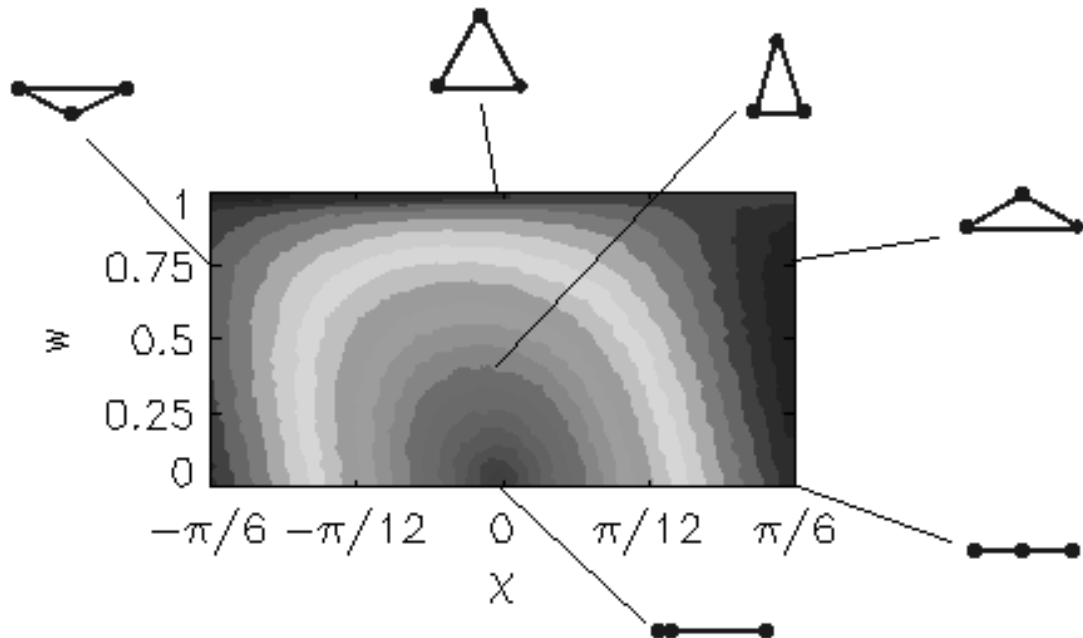


FIG. 1: The contour lines of the three-particle zero mode as a function of the shape of the triangle defined by particles

V. ZERO MODES AS STATISTICAL INTEGRALS AND BREAKDOWN OF SYMMETRIES

A. Understanding zero modes

From a mathematical viewpoint, zero modes are martingales, i.e. quantities conserved in the mean by stochastic processes [28]. To understand the physical mechanism of zero-mode conservation, one should look at the geometry of multi-particle configurations (or shapes). Consider n fluid particles and ask if some functions of inter-particle distances R_{ij} (between particles i and j) are preserved on the average, as the particles move in a random flow. There are two basic features of this evolution: the n -particle cloud grows in size while shape fluctuations of the cloud decrease in magnitude. If one can find such functions of size and shape that are conserved on the average? Consider first the simplest case of Brownian motion where the time derivative of the mean of any function of the distances separating particles is equal to Laplacian of this function. Harmonic polynomials turn Laplacian into zero (and can thus be called zero modes of Laplacian), therefore they are conserved on the average precisely because the growth of distances is compensated by the decrease of shape fluctuations (formally, the conservation occurs because in acting on a zero mode the radial part of the Laplacian exactly cancels the angular part). The same mechanism of conservation based on compensation must work for the turbulent velocity as well. A vivid and explicit example is provided by the case of three particles. The contour lines of the relevant zero mode as a function of the shape of the triangle are shown in Fig. 1 for a real turbulent velocity [29]. The function tends to decrease for configurations where all the inter-particle distances are comparable. It is then clear that the decrease in the shape average is simply due to particle evolving toward symmetrical configurations with aspect ratios of order unity and to increased cancellations between contributions of other shapes [29, 30].

For Brownian motion, the zero modes are polynomials in R_{ij}^2 , i.e. the average of a generic function

of the shape relaxes to a constant as an inverse integer power of R^2 or t . Particularly important are the irreducible zero modes which involve distances among all n particles. The lowest scaling dimension of the irreducible zero modes, $\zeta_n = n$ (for n even), linearly depends on n because particles move independently, and the average of its shape part decays as $t^{-n/2}$. This is not so when the velocity field itself has power-law correlations in space, as we believe happens in turbulence. When velocity fields are non-smooth in the inviscid limit, then $\delta v_R \propto R^\alpha$ with $0 < \alpha < 1$. As a result of velocity correlations, fluid particles undergo super-diffusion ($R_{ij}^{1-\alpha} \propto t$): the farther they are the faster they tend to move away from each other, as in Richardson's law of diffusion. This means that, statistically, the system behaves as if there was an attraction between particles that weakens with the distance, though, of course, there is no physical interaction among particles (but only mutual correlations because they are inside the correlation radius of the velocity field). As a result, shape fluctuations decay slower than $t^{-n/2}$ and the scaling exponents of the statistical integrals of motion, ζ_n , grow with n slower than linearly (very much like the energy of the cloud of attracting particles does not grow linearly with the number of particles). Let us stress that the statistical integrals of multi-particle evolution exist for velocity field with different spatial characteristics, from smooth ($\alpha = 1$) to extremely rough. Yet only velocity fields which are non-smooth and have power-law correlations between points ($0 < \alpha < 1$) provide for the integrals with the scaling exponents ζ_n depending on n in a nonlinear way.

The question now arises: How does nonlinear dependence of ζ_n on n translate to an anomalous scaling of fields mixed by turbulence? We shall see that the existence of zero modes leads to a breakdown (or more accurately, non-restoration in the small-scale limit) not only of the scale invariance but of other symmetries of field statistics as well, notably isotropy.

B. Zero modes and the anomalous scaling of passive scalars

Consider a passive scalar $\theta(\mathbf{r}, t)$ transported by a turbulent flow. In many situations one has scalar turbulence and a cascade very much like the energy cascade: forcing (or heating, if θ is the temperature) produces large-scale fluctuations of the scalar field that are distorted by velocity gradients, thus generating small scales that are then smeared out by diffusion. The correlation functions of θ are proportional to the times spent by the particles within the correlation scale of the forcing L_θ . The structure functions of θ are differences of correlation functions with different initial particle configurations as, for instance, $S_3(r_{12}) \equiv \langle [\theta(\mathbf{r}_1) - \theta(\mathbf{r}_2)]^3 \rangle = 3\langle \theta^2(\mathbf{r}_1)\theta(\mathbf{r}_2) - \theta(\mathbf{r}_1)\theta^2(\mathbf{r}_2) \rangle$. Imagining an equivalence between time lapse and space separation, any structure function can thus be regarded as proportional to the time difference for different initial configurations or, in other words, to the time required to forget the initial shape. For example, in calculating S_3 , we compare two histories: the first one with two particles initially close to the point $\mathbf{r}_1$ and one particle at $\mathbf{r}_2$, and the second one with one particle at $\mathbf{r}_1$ and two particles at $\mathbf{r}_2$. That is, S_3 is proportional to the time during which one can distinguish one history from another, or to the time needed for elongated triangles to relax to equilateral ones. The dependence of such times on distances r_{12} is precisely given by the laws governing the decay of shape fluctuations (as negative powers of t or r), so the scaling exponents of structure functions coincide with the scaling of the irreducible zero modes, $S_n(r) \propto r^{\zeta_n} L_\theta^{n(1-\alpha)-\zeta_n}$. Since ζ_n grows with n slower than linearly, we have $S_{2n}/S_2^n \propto (L_\theta/r)^{n\zeta_2-\zeta_{2n}} \gg 1$, so that the probability distributions of scalar differences become increasingly non-Gaussian as $r/L_\theta \rightarrow 0$.

We thus see that the existence of statistical conserved quantities breaks the scale invariance of statistics in the inertial range and explains why scalar turbulence “knows more” about forcing than just the value of the flux. A comprehensive description of the theory can be found in the review written by the French-Israeli collaboration [31] while popular description is in the review written

by Israeli-Italian collaboration [32]. Note that conceptual breakthrough in the theory of turbulent mixing also triggered new mathematical developments in the probability theory [33, 34].

C. Breakdown of isotropy

In essentially all situation of practical importance, the flow is set up at large scale by a very anisotropic procedure. That means that forcing breaks not only scale invariance but isotropy as well. One of the main assumptions of the Kolmogorov theory is that despite the very specific properties of the fluid motion at the largest scales, the motion at small scale tends to have universal features. In particular, the postulate is that the fluid motion becomes isotropic at small scales: the symmetry broken by forcing is restored after many steps of the turbulent cascade. Obviously, this is an asymptotic statement, which should become more and more true as the Reynolds number increases, or the range of scales becomes very large. The same postulate was generally made about passive scalars. However, strong experimental evidence suggests that the flow in fact does not restore the isotropy of the scalar field [1, 35]. This is the case, for example, over a heated turbulent boundary layer, where the scalar exhibits a very characteristic ‘ramp and cliff’ structure, suggesting a persistent anisotropy down to the very smallest scales.

Elementary symmetry considerations show that for an isotropic field, odd order moments should vanish identically. Because of the large-scale anisotropic forcing, the third order moment, say, is nonzero at large scale. The question then is, how does such correlation function return to zero at small scales? Normal scaling considerations, based on Kolmogorov arguments, provide a simple, dimensional answer which indicates that anisotropy diminishes as a power law in the scale size. Using Kraichnan model, one can show that this normal scaling part is actually subdominant, and it is the appropriate zero mode which dominates the triple correlation function [35]. The main lesson, from the physical point of view, is that the decay of anisotropy with decreasing scale is considerably slower than previously anticipated. Thus, somewhat unexpectedly, the theoretical understanding of intermittency of passive scalar in turbulent flows lead also to a new insight about isotropy (or lack thereof), which had remained a riddle for decades.

Therefore, if the forcing breaks isotropy, an anomalous scaling of zero modes is also responsible for the non-restoration of isotropy in the limit $r/L_\theta \rightarrow 0$. On the one hand, the higher angular harmonics in a suitable spherical harmonics expansion, characterized by the angular momentum j , say, decay faster with j . That means that the degree of anisotropy of every moment decreases as $r \rightarrow 0$. On the other hand, the quantities with no isotropic contribution, like the odd moments S_{2n+1} , could decay slower than $S_2^{n+1/2}$ if there exist $j \neq 0$ such that $\zeta_n^j < n\zeta_2/2$. This means that the pdf of the scalar differences could get *more* anisotropic as $r/L \rightarrow 0$ [36]. The study of anisotropy required both new measurement techniques and new way of data analysis. This was achieved via the interaction between the experimental group of Sreenivasan and the theoretical group of Procaccia (see e.g. [36]).

Let us stress that the symmetries of scale invariance and isotropy (possibly broken by forcing) and time reversibility (broken by damping) are not restored even when $r/L \rightarrow 0$ and $\eta/r \rightarrow 0$, respectively. The two anomalies are related intimately: the flux constancy imposes some scaling properties of the velocity field, which generally lead to super-diffusion and to anomalous scaling.

D. Beyond the Kraichnan model

The perfect duality between the description of passive scalar transport in terms of fields and particles allows us to generalize the results obtained in the framework of the Kraichnan model

to real turbulent mixing. The Lagrangian interpretation of intermittency in terms of statistically invariant functions lends itself to a straightforward generalization, since this concept is well defined for any statistical ensemble of fluid velocities. In a real world, instead of proving that some functions of the inter-particle distances are responsible for anomalous scaling, one can proceed in the reverse direction: first extracting from laboratory [37] or numerical experiments [29, 38] the anomalous part of the multi-point correlation of the scalar field, and then checking if this function is statistically preserved by the Lagrangian flow (Pumir, Celani and Procaccia in collaboration with others). In other words, one starts from considering the forced passive scalar turbulence and find the anomalous parts of the correlation functions. Considering then decaying turbulence without pumping, one projects initial conditions into those anomalous parts and finds that *the resulting projection does not decay* [29, 38].

A complete study can be carried out in the case of the three-point correlation function: this function is indeed preserved in the real flow of two-dimensional Navier-Stokes turbulence and corresponds to the zero mode depicted in Fig. 1 [29]. The connection between intermittency and turbulent diffusion leads to yet another interpretation in terms of cluster shapes mentioned in Sec. V [20]. Consider the evolution of the shape of a configuration of n particles transported by the turbulent flow. As the size of the particle cluster increases, the probability density function over the angular degrees of freedom approaches a stationary distribution. This distribution is interesting in itself, since it is sensitive to the local stretching of sets of points by the flow [39–41]. The convergence to self-similarity is controlled by the ratio of final to initial size of the cluster, R_f/R_i : the distance from the asymptotic distribution decays as $(R_f/R_i)^{-\zeta_n}$ where the dependence on the shape of the configuration is specified by the zero modes and the exponents ζ_n are the anomalous exponents of the passive scalar problem.

A direct consequence of the identification of the zero modes with the anomalous parts of the correlation functions is that the scaling exponents are *universal*, i.e. independent of the choice of the scalar pumping (or forcing), since the latter does not appear in the definition of the statistically preserved functions. The precise values of ζ_n , however, still depend on the statistics of the velocity field. The numerical prefactors that appear in front of the moments of scalar differences are nonuniversal and change according to the details of the pumping. This is a narrower universality than the one prospected by Kolmogorov, and is relevant to a wide class of turbulent transport problems [38]: whether this picture applies also to the velocity statistics in Navier-Stokes turbulence is an open question.

VI. RARE FLUCTUATIONS AND SCALAR FRONTS

The practical importance of describing statistics of large rare fluctuations is obvious if one considers, for example, the probability of a pollutant concentration exceeding some tolerable level. Large concentration fluctuations control the rate of slow (high order) chemical reactions, for example, in the process of atmospheric ozone destruction and many other phenomena. Rare fluctuations are described by the tails of the probability distributions or, equivalently, by high-order correlation functions. The present collaboration pioneered the systematic description of large fluctuations in turbulence. For that end, we adapted to turbulence the saddle-point analysis of path integrals, which was called the instanton method. The term “instanton” was introduced in the seventies in the quantum field theory for special time-dependent solutions of the classical field equations [42, 43]. In our framework, instanton describes the behavior of the rare fluctuation that gives the leading contribution to the high-order correlation function of the fluctuating field. Indeed, the correlation functions of velocity, temperature and other fields in highly non-equilibrium (turbulent) systems can be written as path integrals formally very close to those used in the quantum field theory [44, 45].

The saddle-point approximation in the path integral leads to the extremum equations whose solution is instanton. Usually, the saddle-point approximation is justified if one examines tails of the probability distribution functions or high moments of fluctuating variables. The method is similar to the optimal fluctuation method used at treating properties of a solid with quenched disorder [46] and to the treatment of high orders in perturbation theory [47].

The first such example was examined by the Russian-Israeli team [48] for the passive scalar statistics in the spatially smooth and temporally random two-dimensional flow (Batchelor case described in Sec. VII below). Exploiting a large time of stretching (from a given small scale to the pumping scale), one can write the instanton equations and find an explicit solution, which determines the complete probability distribution function of the small-scale passive scalar fluctuations. The function reproduces both the nearly Gaussian statistics of low moments of the passive scalar [49], and the exponential tail [49, 50], related to rare events when stretching is absent. These details have been verified for passive scalars advected by Navier-Stokes velocity fields at the scales less than the viscous scale of the turbulence [51].

To extend the considerations for the passive scalar in the inertial range of developed turbulence, the Russian-Israeli collaboration used the Kraichnan model [52, 53]. The model admits a path integral representation for the trajectories of the fluid particles, used in [52], and also a “field” representation, used in [53]. In both works, the emphasis was put on anomalous scaling of the passive scalar structure functions $S_n(r)$, which is manifested in a non-linear dependence on n of the exponents ζ_n , $S(r) \propto r^{\zeta_n}$. By using the instanton technique for large dimensionality of space, it was demonstrated that, starting from some n , ζ_n saturate (that is, cease to depend on n).

The physics behind the saturation of scaling exponents can be related to the existence of intense structures in the scalar field: the “cliffs”. The jump of the value of the scalar between the two sides of a cliff can be as large as ten times the typical fluctuation across a very small distance (see the review by Sreenivasan [1] and references therein). In Ref. [54] Celani et al. offer a simple geometrical interpretation that the value of the saturation exponent coincides with the fractal codimension of the set of points of the physical space that host the jumps (or fronts) of the scalar field. For the scalar field, we have $\zeta_\infty \approx 1.4$ both in two and in three dimensions: this yields a fractal dimension $D_F \approx 0.6$ in two dimensions (i.e. fronts live on a broken line) and $D_F \approx 1.6$ in three dimensions (jumps take place across broken surfaces). By letting the number of points go to infinity we were thus able to calculate analytically the fractal properties of lines in turbulence and compare this with measurements.

A snapshot of the scalar field advected by the two-dimensional real flow is shown in Fig. 2. A clear feature is that strong scalar gradients tend to concentrate in sharp fronts separated by large regions where the variations are weak. The cliffs are present even in synthetic velocity fields generated in computer simulations of passive scalar transport [55, 56]. This suggested the possibility that qualitative features of passive scalar can be studied well within the Kraichnan model and the results obtained by Chertkov and by Balkovsky and Lebedev [52, 53] can be applied to less idealized flows. This aspect has been demonstrated by Celani et al. for the passive scalar problem in Refs. [25, 54] and for thermal convection in Ref. [57] in two-dimensional turbulence. This finding elicited an experimental quest for saturation in turbulent mixing experiments [58–60] as well as in Rayleigh-Bénard turbulent convection [61].

Note that the instanton (saddle-point) technique developed for the turbulent problems can be applied to different physical problems where the role of noise is relevant. For example the Russian-Israeli and Russian-American teams used the technique to study the statistics of the bit-error rate in modern optical communication lines [62, 63].

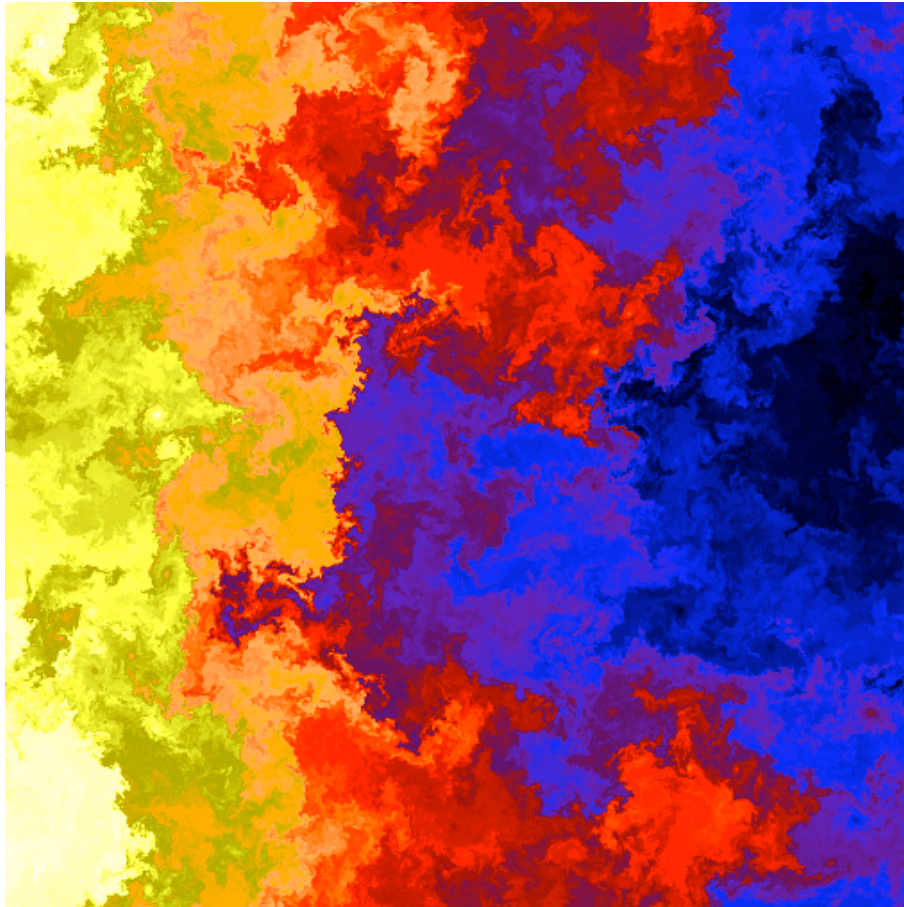


FIG. 2: A typical snapshot of a scalar field transported by a turbulent flow, from [54]

VII. MIXING IN SMOOTH FLOWS

The problems and advances discussed in the previous sections were mostly related to non-smooth flows (i.e. turbulent flows considered at the scales larger than the viscous scale). At the same time, random flows that are spatially smooth are ubiquitous in nature and industry. This is the case either for turbulent flows for scales smaller than the viscous scale or very viscous fluids driven randomly. To the same class belongs the so-called Lagrangian chaos when the streamlines of the flow may become chaotic with the help of a specially arranged geometry or unsteadiness. Another fascinating example of spatially smooth and temporally random flow is elastic turbulence discovered by the present collaboration and described in Sect. VIII below. Mixing in such flows was considered first by Batchelor [64] and it is thus customary to call it the Batchelor case. The theoretical methods developed by the present collaboration made the theory of mixing in the Batchelor case the most developed and well-understood among the large family of problems studied in statistical hydrodynamics: the Kraichnan model was exactly solved by the French-Finnish team [28] by discovering a surprising map to a soluble problem of many-body quantum mechanics, while a finite-correlated case was studied by the Russian-Israeli-American team using the Lagrangian path integral and the instanton method [49, 65, 66]. Close collaboration of Russian, Israeli and American theoreticians and Israeli experimentalists led to the major theoretical predictions being confirmed experimentally and new experimental findings being explained theoretically.

In a spatially smooth flow, the velocity difference is proportional to the distance between two fluid particles. As a result, particles separate or approach exponentially in time. The statistics of stretching and contraction of the fluid element can be described by the formalism of dynamical systems using the notions of Lyapunov exponents which are the exponential rates. The Lyapunov exponents are equal in number to space dimensionality, and are ordered such that the first one is the largest and describes the separation rate of two fluid particles. Using classical results of mathematicians Oseledec, Lyapunov and Furstenberg, the Russian-Israeli team has shown that the statistics of stretching in quite arbitrary random flows allows for universal description in terms of the Large Deviation Theory [49, 66]. That collaboration also predicted the so-called collinear anomaly (first, within the Kraichnan model and then for a general case [65, 66]): spatially smooth flow locally preserves straight lines. This anomaly translated into the passive scalar spatial distribution suggests that parallel stripes are the major structural elements in the Batchelor case. This simple yet fundamental feature was observed experimentally by the Israeli team for passive scalar in the elastic turbulence [67, 68]. The collinear anomaly is also the source of strong intermittency and the absence of scale invariance observed at the scales larger than the forcing scale [69]. It is interesting that adding linear damping brings anomalous scaling of the scalar turbulence in a smooth random flow [70].

The schematic description of how numerous theoretical results were obtained can be summarized as follows: recast n -th order correlation function of a scalar or scalar derivative in terms of a suitable average over the set of n simultaneously evolving Lagrangian particles and then reduce the total number of dynamical degrees of freedom to the $d - 1$ Lyapunov exponents [49, 50, 66, 71–73]. The Lagrangian technique has allowed one (a) to predict the exponential tails of the passive scalar distribution function in the steady regime [49, 50], as was later confirmed experimentally and numerically; (b) to account for a complex interplay between Lagrangian stretching and diffusion, thus showing that the probability density of the scalar dissipation has a stretched-exponential tail [72]; (c) to explain strong intermittency in the scalar decay problem [66, 71], with the decay rate being exponential in time and the exponent saturating to a constant for high-order moments. While Israeli experiment [67] confirmed the exponent saturation, it contradicted some other conclusions of the theory [66, 71] developed for infinite domains. That motivated Russian-American collaboration to develop a new (and much-needed) theory of mixing in a finite vessel [73]. This collaboration explained that long-time scalar decay is determined by the stagnation regions in the flow, in particular boundary domains. The theory predicted a nontrivial dependence of the decay rate on diffusivity which was remarkably confirmed by the elastic turbulence experiment on mixing [68].

VIII. ELASTIC TURBULENCE

All fluids have internal friction called viscosity whose relative role with respect to the inertia of the moving fluid is characterized by the Reynolds number. When the Reynolds number is low, friction dominates and the motion is expected to be laminar i.e. smooth and regular. This used to be the cornerstone of basic fluid mechanics. A practical consequence of flow regularity that was widely recognized—and lamented upon—is the inability to mix viscous fluids. Indeed, to mix fluids effectively one needs irregular motion which brings into near contact distant portions of the fluids.

The experimental discovery of the Steinberg group from the Weizmann Institute have shown the way to produce an irregular motion of very viscous fluids [74]. They studied the consequence of adding to a liquid a minute quantity of polymer molecules (typically less than a fraction of percent by weight). This is like adding some sugar to water. Long polymer molecules make the liquid elastic with stresses depending on the history of deformation, thereby giving the medium a memory. As was

brilliantly demonstrated, elastic memory can overcome viscous dissipation and sustain complicated and irregular flow patterns, called elastic turbulence. It has been shown that with enough elasticity one can excite turbulent motion at arbitrary low velocities and in arbitrarily small vessels [74–76].

A phenomenon behind the unusual macroscopic properties of dilute polymer solutions is the so-called coil-stretch transition discussed by Lumley [77] and then elaborated by de Gennes [78]. The essence of the coil-stretch transition is as follows. If there is no flow gradient then all the polymers are in the coiled state, constituting a compact and nearly spherical tangles with radii much smaller than the total polymer length. In the presence of the random flow gradients, the polymers are deformed into ellipsoid-like tangles. The mean deformation is moderate if the stretching rate of the flow is smaller than the inverse polymer relaxation time. When the stretching rate exceeds the inverse relaxation time, most of the polymers are strongly stretched up to the order of the total polymer length (since the stretching is stopped only by non-linear effects). That abrupt change in the polymer conformation occurring at large stretching rates is called the coil-stretch transition.

The coil-stretch transition leads to remarkable macroscopic consequences. In the coiled state, the influence of polymers on the flow is negligible and the polymer solutions can be treated as Newtonian fluids (without memory). However, in the stretched state polymers produce an essential feedback on the flow, and properties of the solutions become strongly different from those of Newtonian fluids. Let us stress that the coil-stretch transition can occur even at small Reynolds numbers, or at low level of hydrodynamic nonlinearity. The elasticity of the polymers itself (which manifests itself above the coil-stretch transition) is a source of nonlinearity. This explains how a chaotic state, which requires a high level of nonlinearity, can be set up in elastic turbulence.

An appropriate language to describe the coil-stretch transition is the probability distribution of polymer elongations. Below the coil-stretch transition the distribution is peaked near the equilibrium tangle size determined by thermal fluctuations. In a random flow, the distribution has an extended (power) tail related to stretching events that overcome polymer relaxation, thus leading to an essential extension of polymer molecules. The character of the tail was theoretically examined in the works by Russian-Israeli-American team [79, 80]. Above the coil-stretch transition, the law of the polymer size distribution is changed, so that it is now peaked at an elongation of the order of the total polymer length, the value of which is determined by an interplay of the stretching, the polymer nonlinearity and the feedback of the polymers on the flow [79, 81].

The predictions of the theory were checked in the experiment [67] where the size distribution of the DNA molecules in a random flow of elastic turbulence was directly measured by optical methods. A good agreement between the theory and experiment was found concerning both the general form of the polymer size distribution and its tails.

The next question is in character of the flow excited in the elastic turbulence state. The power spectrum of velocity fluctuations measured in the experiments [74–76] is like a power-law of the type k^{-x} with $x > 3$. Therefore, the fluctuations of velocity and velocity gradients are both determined by the integral scale (the size of the vessel), which explains why the elastic turbulence is a chaotic state formed by a strong nonlinear interaction of a few large-scale modes. A theoretical explanation of the power velocity spectrum was provided by the Russian-Israeli collaboration [82].

The spectrum k^{-x} with $x > 3$ implies that the mixing in elastic turbulence can be considered as an example of the Batchelor case because this temporally random flow is spatially smooth. Such a Batchelor flow is an efficient mixer because of exponential divergence of Lagrangian trajectories. This property was directly checked experimentally in curvilinear flows of polymer solutions where elastic turbulence was excited [76]. The passive scalar (dye) mixing has an exponential character along the channel. Moreover, a theoretical prediction that the decay of the passive scalar along the channel is related to boundary regions [73, 83] was verified [68].

These experimental discoveries and subsequent theoretical developments show the need to readjust

the thinking in fundamental fluid mechanics (that mixing of highly viscous fluids is difficult) but also open up a whole new world of practical applications. Elastic turbulence gives the possibility to increase dramatically mixing rates (and, correspondingly, rates of chemical reactions) of polymer melts in industrial reactors and in small-scale flows (via capillaries, for instance). That way of mixing begins to find applications in microfluidics and biophysics. It is likely that Nature itself uses the trick of small-scale mixing by elastic turbulence, which means that biological discoveries and applications along this direction are likely to follow.

IX. COMPRESSIBLE FLOWS AND RAIN

Consistent application of the Lagrangian approach led also to significant progress in the theory of compressible and multi-phase random flows, thus discovering new turbulent regimes and opening new meteorological applications of turbulence theory.

Compressibility allows for more flexible flow geometries with regions of ongoing compression effectively trapping particles for long times and counteracting their tendency to separate. From the viewpoint of passive scalar in a random flow, one expects some mixing, which makes the concentration more uniform. On the other hand, when the flow is compressible, it creates fluctuations of concentration. A central theme is thus the competition between the two tendencies. The main questions are: (i) Is there a statistically steady state of concentration fluctuations in which the production of fluctuations is balanced by mixing? (ii) Are there qualitative changes of the regime (phase transitions) when the degrees of compressibility or non-smoothness of the flow change?

Again, analytic solution of Kraichnan model allows to answer the questions and bring new results. Chertkov (meanwhile relocated to USA), Kolokolov (Russia) and Vergassola (France) extended the results of the French mathematician Le Jan [84] on the Lyapunov exponents in a random smooth compressible flow and applied them to the description of passive scalar in such a flow [85]. They have discovered that the direction of the scalar cascade changes when the compressibility degree exceeds some threshold. The next effort of extending the analysis to non-smooth flows was accomplished by Gawędzki and Vergassola who discovered a sequence of phase transitions that take place when the degrees of compressibility or non-smoothness of the flow change [23], see also [34, 86] for further developments.

For the passive scalar, those changes correspond to clustering, rather than mixing, as the leading effect. The general theory finds applications (developed by the collaboration of Israeli and American teams [87]) in describing nonuniform distributions that one can observe in Nature (such as “patchiness” in the distribution of leaves and litter on the surface of lakes and pools and of oil slicks and seaweeds on the sea surface). As shown by the French team, those extremely nonuniform distributions are actually multifractal [88]. They are realized not only in spatial distribution of fluid flows but also in the phase space of dynamical systems far from equilibrium [89], thus linking our theory with the most fundamental problems of modern statistical physics

On the other hand, the theory of clustering in compressible flows has found an important application in the description of multiphase flows like that of the droplet-laden air in clouds or bubbling champagne. Different strengths of inertia of particles in the second phase make their flow compressible even when the motion of the ambient fluid is incompressible. Indeed, in every vortex in the flow, centrifugal force throws heavier-than-air water droplets outwards. On the other hand, bubbles are sucked into the whirls (as one can observe by opening the sink in the bathtub filled by champagne). The phenomenon is long-known yet the consistent statistical theory of concentration fluctuations in random flows has been developed only recently by the collaboration of French, Israeli and American teams. This collaboration has described the statistics of droplet clusters within the Kraichnan model

of the ambient fluid flow [88, 90], generalized it for realistic flows (as in high-Reynolds-number cloud turbulence) [91] and carried out numerical simulations to check the theory [92]. Our main practical achievement is the quantitative scheme to calculate the collision rate of droplets in a turbulent flow. Since coalescence of droplets is a primary mechanism of droplet growth (from few microns to raindrop sizes) in warm clouds, a knowledge of the collision rate is needed for accurate forecast of precipitation. Our study shows that turbulence can substantially increase the collision rate, both by clustering [91, 92] and by the so-called sling effect (which is the effect of collisions due to jets of droplets detached from the air flow predicted by us [91]). Enhancement of collisions by turbulence is determined by the statistics of velocity gradients which, due to intermittency and anomalous scaling described above, depends on the Reynolds number i.e. on the macro-scale flow. This property links cloud scales with the microphysics of clouds—a link that we believe will be an integral part of future meteorological tools. The statistical theory of aerosol distribution and clustering in turbulent flows finds other applications that range from fuel droplets in internal combustion engines (where clustering makes for nonuniform burning which decreases efficiency), environmental norms of air quality, interstellar dust and appearance of planetesimals in the formation of the Solar System and many other instances of sprays and particle-laden flows. As the awareness grows that atmospheric aerosols (along with greenhouse gases) are responsible for global warming, our results on aerosol statistics find their place in climate research.

X. CONFORMAL INVARIANCE IN 2D TURBULENCE

Simplicity of fundamental physical laws manifests itself in fundamental symmetries. While systems with an infinity of strongly interacting degrees of freedom (in particle physics and critical phenomena) are hard to describe, they often demonstrate symmetries, in particular scale invariance. In two dimensions (2d) locality often promotes scale invariance to a wider class of conformal transformations which allow for nonuniform re-scaling. Conformal invariance allows a thorough classification of universality classes of critical phenomena in 2d. Is there conformal invariance in 2d turbulence, a paradigmatic example of strongly-interacting non-equilibrium system [6]? French-Israeli collaboration [93], using numerical experiment, has shown that some features of 2d inverse turbulent cascade display conformal invariance. They observed that the statistics of vorticity clusters is remarkably close to that of critical percolation, one of the simplest universality classes of critical phenomena.

The novelty of their approach was in analyzing the inverse cascade in 2d by describing the large-scale statistics of the boundaries of vorticity clusters, i.e. large-scale zero-vorticity lines. In equilibrium critical phenomena, cluster boundaries in the continuous limit of vanishingly small lattice size and at criticality were recently found to belong to a remarkable class of curves that can be mapped into Brownian walk (called Stochastic Loewner Evolution or SLE curves [94, 95]). In simple words, the locality in time of the Brownian walk translates into the local scale-invariance of SLE curves, i.e. conformal invariance. French-Israeli collaboration [93] has developed a SLE-based numerical tool for testing conformal invariance in physical systems, established this symmetry (within experimental accuracy) for 2d inverse cascade and used it as a powerful new tool in turbulence study which allowed it to make new quantitative predictions confirmed by the experiment. Exact analytic expressions from percolation and conformal field theory (of the type never before used in turbulence) were suggested for different probability distributions of vorticity clusters and iso-vorticity lines and were found to be in amazing agreement with turbulence data. That established a direct link between critical phenomena and turbulence and showed how conformal invariance spans the whole physics, from exalted subjects like string theory and quantum gravity, via statistical mechanics and condensed matter, down to earthly atmospheric turbulence.

XI. CONCLUDING REMARKS

In summary, we solved the fundamental problem of intermittency and anomalous scaling in turbulent mixing. In so doing we developed a new concept of the statistical integrals of motion which is believed to be of primary value for an emerging general approach to systems that are far from equilibrium. We have described the statistical integrals analytically and found them numerically and experimentally. We have elucidated the role of the integrals in breaking the symmetries of turbulent state and were able to tackle a number of problems that were demanding an explanation since long time (such as the persistence of anisotropy and an anomaly of scaling exponents); it has also been possible to explain the results of experiments dealing with new phenomena (such as elastic turbulence that we discovered experimentally). We have linked turbulence with critical phenomena by establishing conformal invariance in turbulence. This body of knowledge will form a permanent contribution in understanding the properties of turbulence, a long-standing problem of paramount importance, and also in understanding a variety of strongly non-equilibrium phenomena. This is at once at the forefront of the physics of nonlinear and non-equilibrium systems.

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