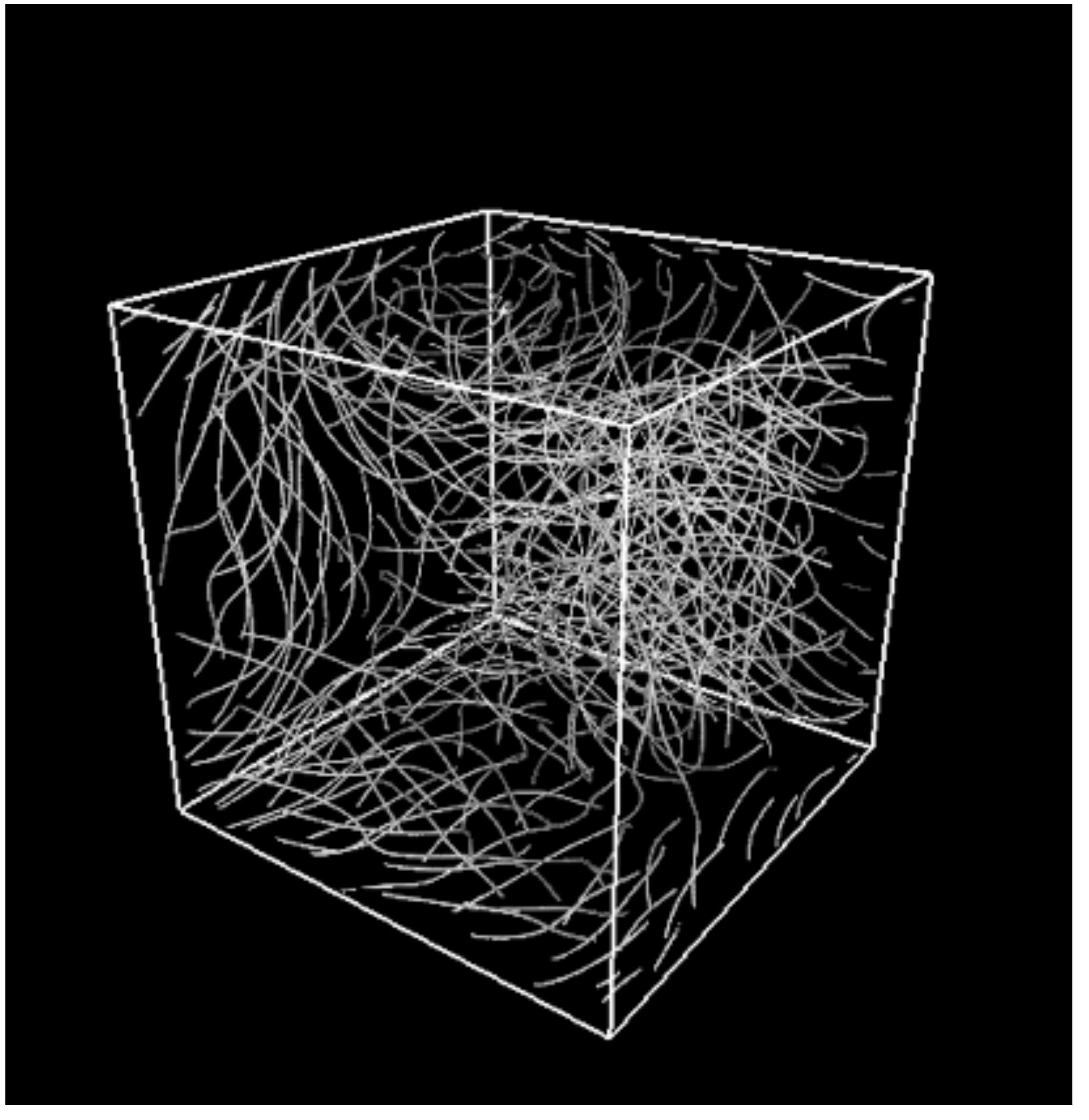
Quantum Turbulence
and
the Gross-Pitaevskii Equation



Marc-Etienne Brachet
LPS/ENS

Summer School Cargèse, Corsica
26th July -5th August 2016
ADVANCES IN GEOPHYSICAL AND ASTROPHYSICAL TURBULENCE

2 Lectures

- This morning: general introduction to quantum turbulence
- This afternoon: helicity in quantum flows

Work done in collaboration with

- Caroline Nore
- Giorgio Krstulovic
- Rahul Pandit
- Vishwanath Shukla
- Pablo Mininni
- Patricio Di Leoni

Plan of Talk

- Perfect fluids: Euler equation and its variational formulations...
- Superfluids: Gross-Pitaevskii Equation
- Coherence length and Quantum vortices
- Classical and Quantum turbulence
- Finite temperature effects in the GPE

What is a perfect fluid?

- Real classical fluids are viscous and conduct heat
- Perfect fluids are idealized models in which these mechanisms are neglected
- Perfect fluids have zero shear stresses, viscosities, and heat conduction
- Good approximation in some physical cases

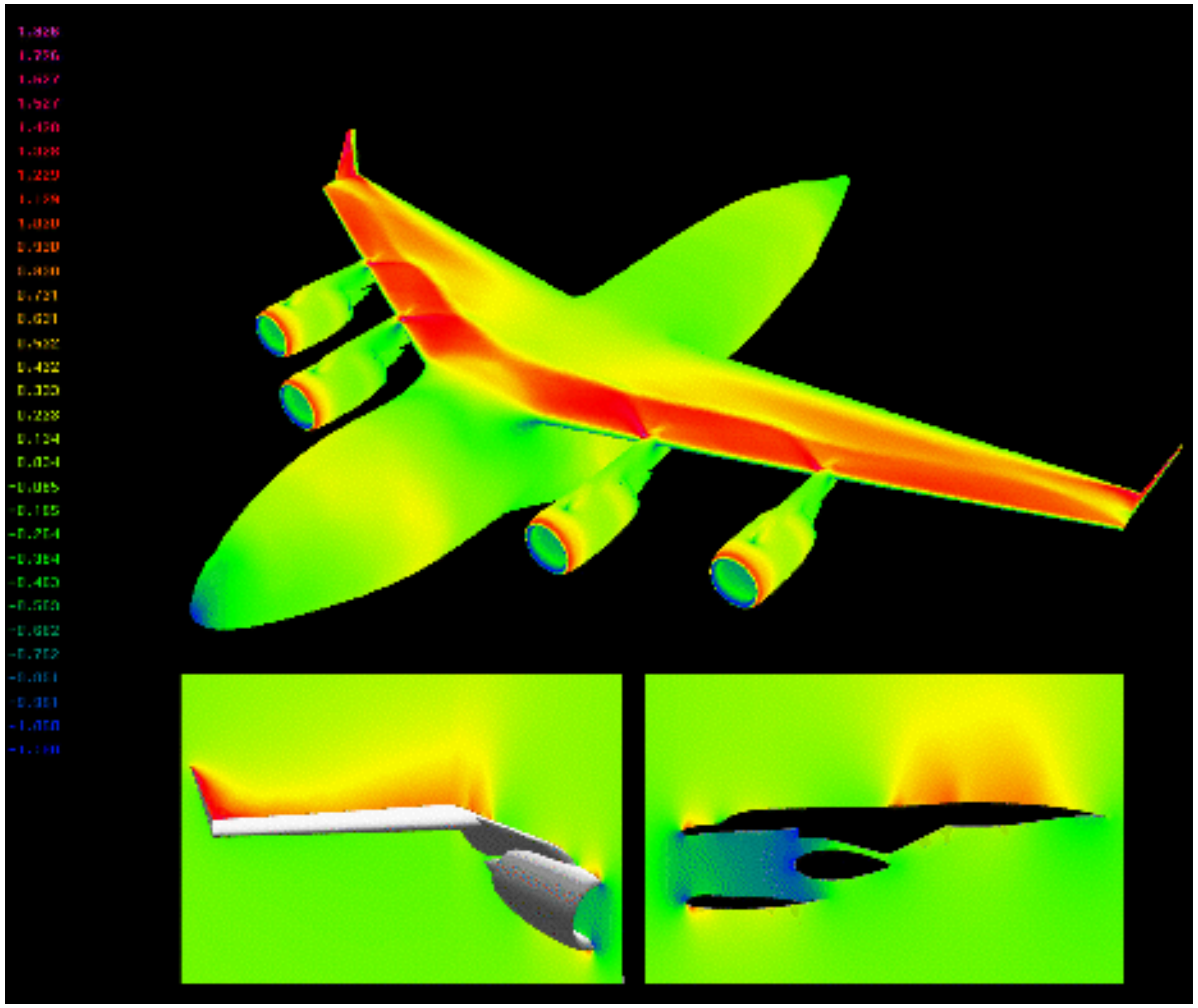
Physical quasi-perfect flows

Next slide is extracted from :

Applied Aerodynamics: A Digital Textbook

http://docs.desktop.aero/appliedaero/preface/welcome.html

Euler Equations



The Euler equations with the equations of energy and continuity are often solved by finite differences whereby the values of each velocity component, the density, and the internal energy are computed at each point. From these quantities constitutive relations (perfect gas law or isentropic pressure relation) are used to find pressure.

Since Euler equations permit rotational flow and enthalpy losses (through shock waves), they are very useful in solving transonic flow problems, propeller or rotor aerodynamics, and flows with vortical structures in the field.

Euler Equations

- A perfect fluid can be completely characterized by its velocity and two independent thermodynamic variables.
- If only one thermodynamic variable exists (e.g. isentropic perfect fluid) the fluid is barotropic.
- The density of a barotropic fluid is a function of pressure only.

Barotropic Euler equations

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla p$$
$$\partial_t \rho + \nabla(\rho \mathbf{v}) = 0$$

Barotropic: $p(\mathbf{x}, t) = f(\rho(\mathbf{x}, t))$

Acoustic propagation: $c = \sqrt{\frac{\partial p}{\partial \rho}}$

Note that the system is time-reversible:

$$t \to -t \; ; \mathbf{v} \to -\mathbf{v} \; ; \rho \to \rho \; ; p \to p$$

Two useful limits

1. incompressible:

$$\rho = cte \qquad c \to \infty$$

$$\nabla \mathbf{v} = 0$$

There is no equation of state and p is determined by maintaining the incompressibility

2. irrotational:

$$\nabla \times \mathbf{v} = 0$$

$$c = \sqrt{\frac{\partial p}{\partial \rho}}$$

$$\mathbf{v} = \nabla \phi$$

Only compressible modes...

Variational approach

- For the general case see e.g. : R. L. Seliger and G. B. Whitham, Variational Principles in Continuum Mechanics, Proc. R. Soc. Lond. A. 1968 305 1-25.
- Here I'll show how to deal only with the compressible irrotational case..

Irrotational case

$$\mathcal{L} = \rho\phi_t + \frac{\rho(\nabla\phi)^2}{2} + g(\rho)$$

$$\frac{\delta\mathcal{L}}{\delta\phi} = 0 \to \rho_t + \nabla(\rho(\nabla\phi)) = 0$$

$$\frac{\delta\mathcal{L}}{\delta\rho} = 0 \to \phi_t + \frac{(\nabla\phi)^2}{2} + g' = 0$$

define:

$$\mathbf{v} = \nabla\phi$$

$$\rho g'' = p'$$

taking the gradient of the last equation:

$$\mathbf{v}_t + \mathbf{v}\cdot\nabla\mathbf{v} = -\nabla g' \quad = -\frac{\nabla p}{\rho}$$

What is a superfluid?
Is it just an Eulerian perfect fluid?
No! Superfluids obey the Gross-Pitaevskii equation (GPE)

The quantum nature of the GPE does disturb some classical traditions of fluid mechanics. This often makes it unpopular…
One should fight this attitude!

Say no to Superphobia!

superphobia
noun
unreasoning hostility, aversion, etc., toward superfluid flows.
Origin of superphobia
super(fluidity) + phobia

The Gross-Pitaeveski Equation (GPE)

$$i\hbar\partial_t\Psi = -\frac{\hbar^2}{2m}\nabla^2\Psi + g|\Psi|^2\Psi$$

$$\Psi = \sqrt{\rho/m}\exp i\frac{m}{\hbar}\Phi$$

- Describes a superfluid Bose-Einstein condensate at zero temperature
- Applies to a complex field
- Madelung's transformation gives hydrodynamical form
- Contains quantum vortices with quantized velocity circulation h/m

Variatitional formulation of the GPE

$$\mathcal{L} = -i\hbar\bar{\Psi}\partial_t\Psi + \frac{\hbar^2|\nabla\Psi|^2}{2m} + \frac{g|\Psi|^4}{2}$$

$$\Psi = \sqrt{\rho/m}\exp i\frac{m}{\hbar}\Phi$$

$$\mathcal{L} = \rho\partial_t\Phi + \frac{\rho\nabla\Phi^2}{2} + \frac{g\rho^2}{2m^2} + \frac{\hbar^2(\nabla\sqrt{\rho})^2}{2m^2}$$

Contrast and compare with Euler Equation Lagrangian:

$$\mathcal{L} = \rho\phi_t + \frac{\rho(\nabla\phi)^2}{2} + g(\rho)$$

GPE and Madelung

$$i\hbar\frac{\partial\psi}{\partial t} = \mathcal{P}_G\left[-\frac{\hbar^2}{2m}\nabla^2\psi + g\mathcal{P}_G[|\psi|^2]\psi\right],$$

$$\psi(\mathbf{x},t) = \sqrt{\frac{\rho(\mathbf{x},t)}{m}}\exp\left[i\frac{m}{\hbar}\phi(\mathbf{x},t)\right], \qquad \mathbf{v} = \boldsymbol{\nabla}\phi$$

Speed of sound $c = \sqrt{g|A_0|^2/m}$

Coherence length $\xi = \sqrt{\hbar^2/2m|A_\mathbf{0}|^2 g}.$

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{\nabla}\phi) = 0, \qquad \frac{\partial\phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2 = c^2(1-\rho) + c^2\xi^2\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}.$$

Continuity and Bernoulli equations for a compressible fluid

Irrotational fluid, except near nodal lines of ψ = superfluid vortices, with quantum of circulation $\Gamma = 4\pi c\xi/\sqrt{2}$, which can naturally reconnect in this model.

Superfluid Helium

Experiments

- Superfluidity exists in actual experiments
- There is a «Quantum turbulence» community actually planning and performing experiments
- The next slides are about a few of these experiments…

Experimental superfluids

Superfluid Helium

- 1930 Kapitsa-Allen-Misener
- Landau and Tisza two-fluid model.
- 1950 Mutual friction. Hall and Vinen
- 1980 Schwartz model

Recent experiments: visualizations using hydrogen solid particles

BEC

- 1925 Bose-Einstein
- Bogoliubov, Gross and Pitaevskii theories.
- 1995 Cornell-Weiman
- 1995 Finite-temperature theories

Recent experiments: emergence of turbulence in oscillating BEC

Experiments in oscilating BEC

Selected for a Viewpoint in *Physics*
PRL **103,** 045301 (2009) PHYSICAL REVIEW LETTERS week ending 24 JULY 2009

Emergence of Turbulence in an Oscillating Bose-Einstein Condensate

E. A. L. Henn,[1,*] J. A. Seman,[1] G. Roati,[2] K. M. F. Magalhães,[1] and V. S. Bagnato[1]
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(Received 23 April 2009; revised manuscript received 10 June 2009; published 20 July 2009)

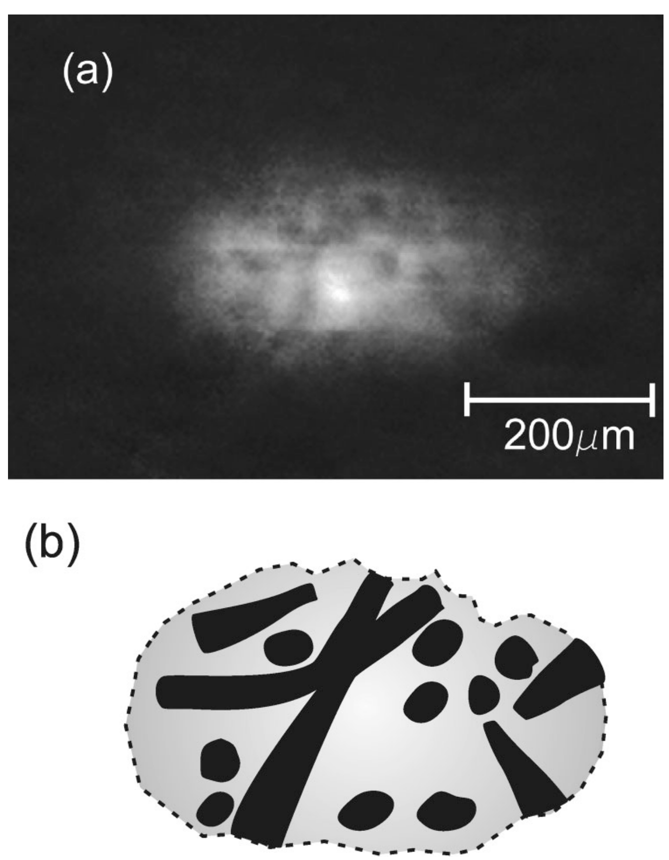
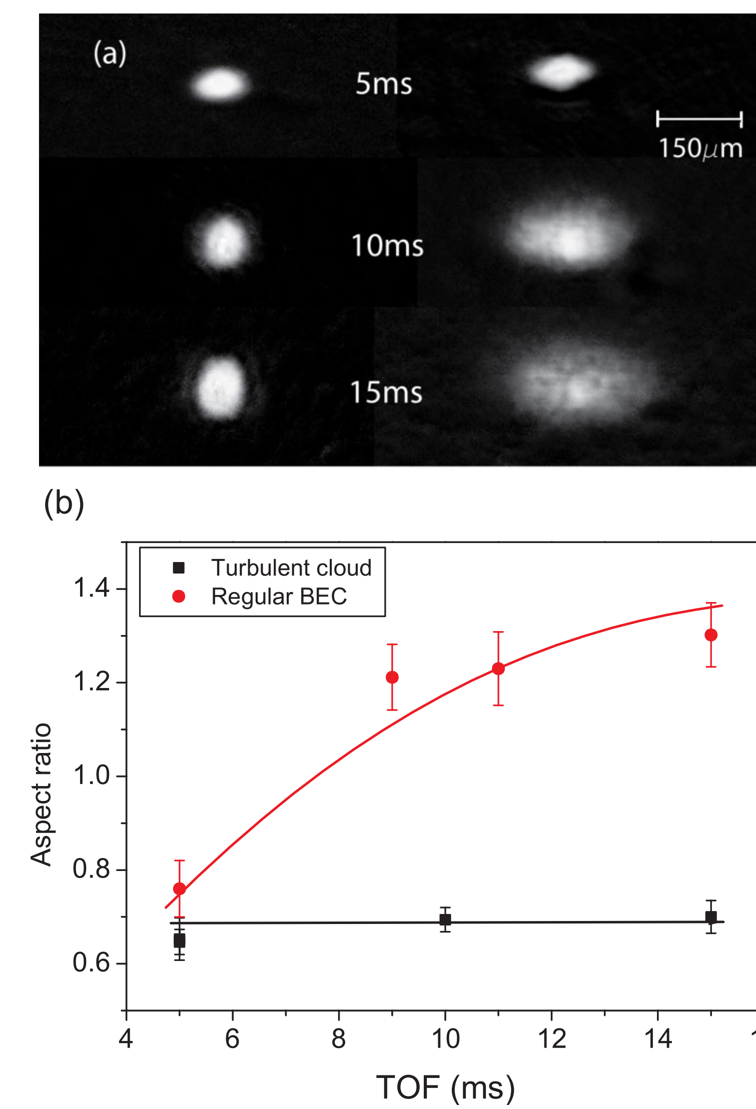
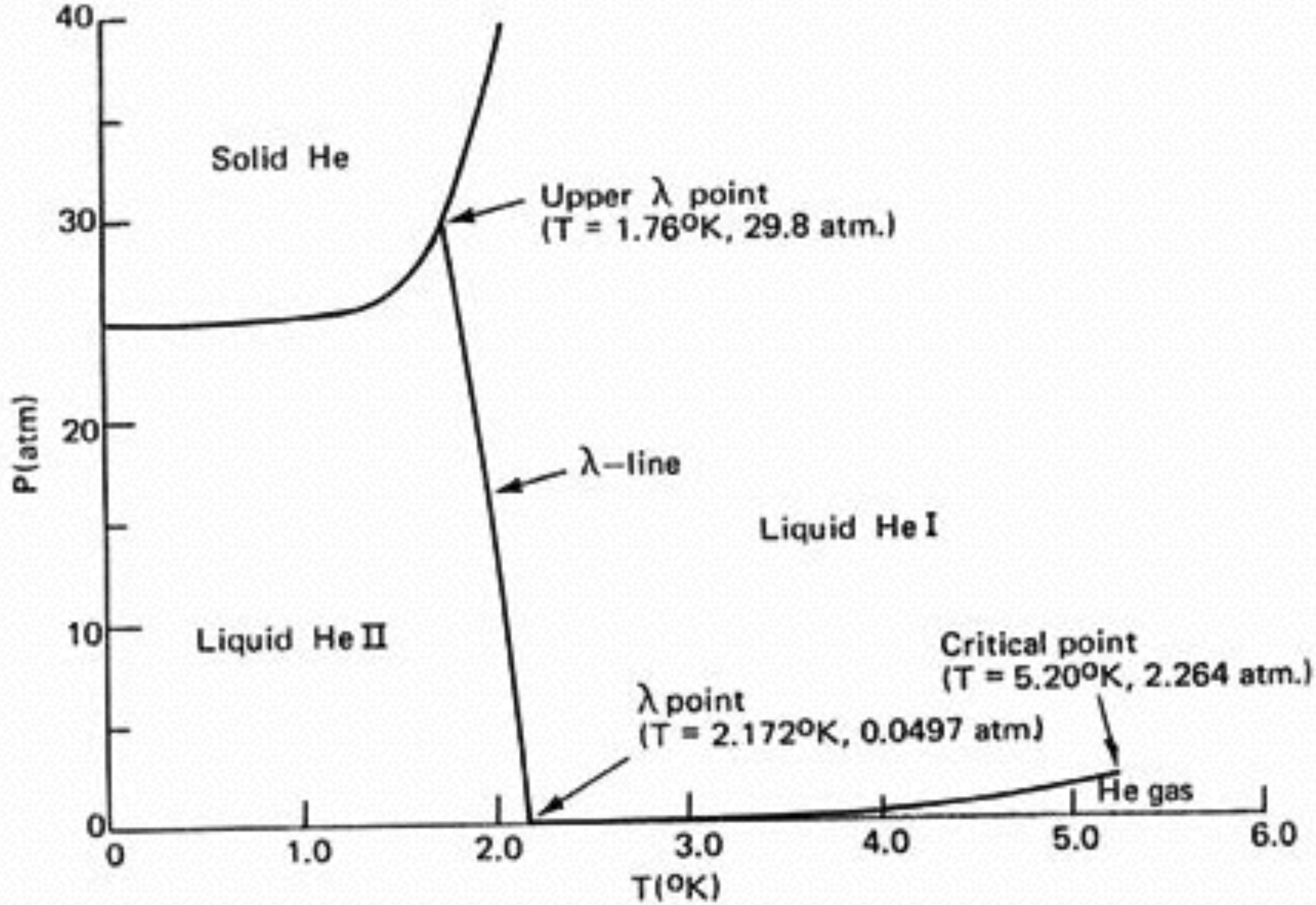


FIG. 2. (a) Atomic optical density after 15 ms of free expansion showing vortex structures spread all around the cloud resembling the vortex tangle regime proposed in Ref. [8], and (b) schematic diagram showing the inferred distribution of vortices as obtained from image shown in (a).



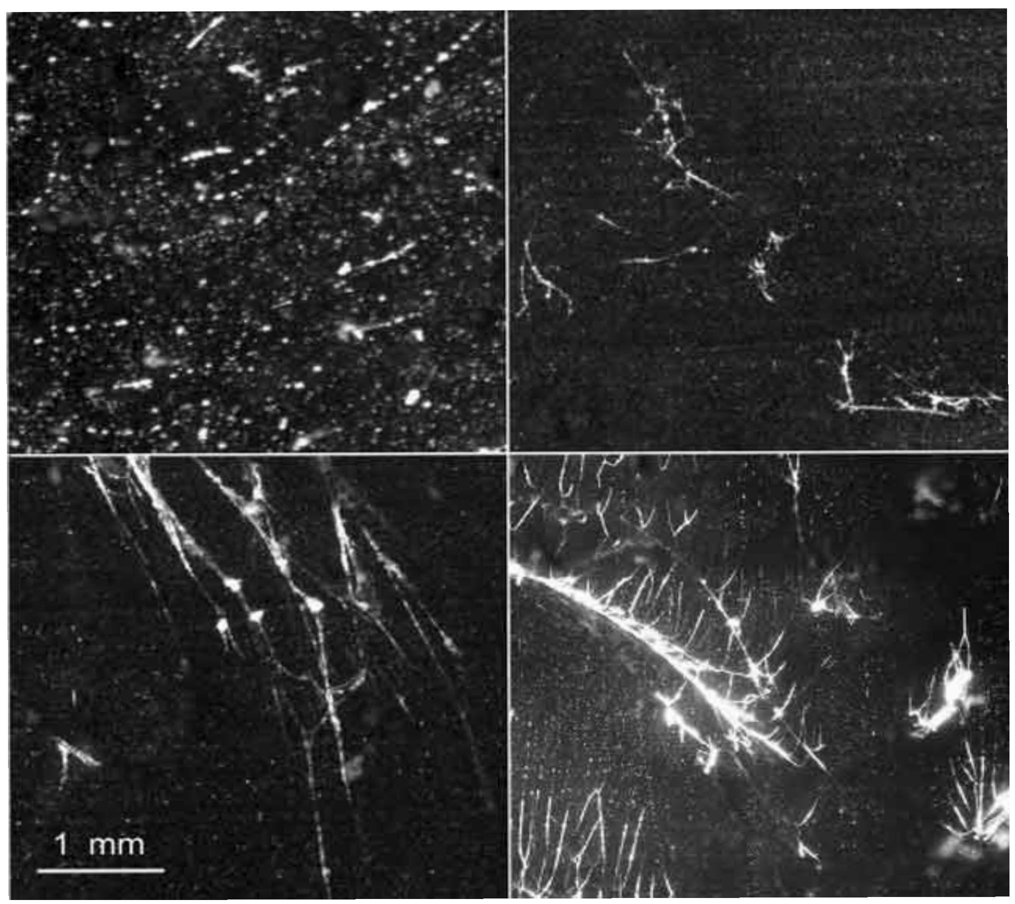
He Phase Diagram

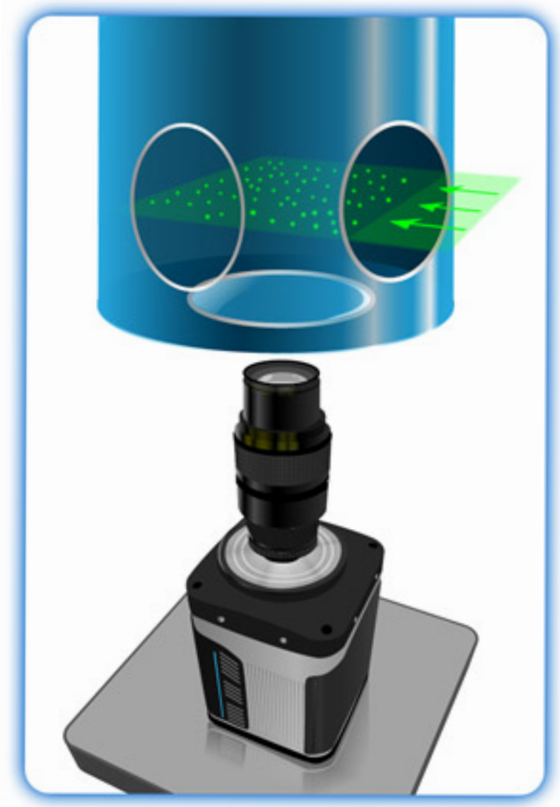


The phase diagram of He^4.

Experiments in superfluid He

Daniel P. Lathrop's
Nonlinear Dynamics Lab

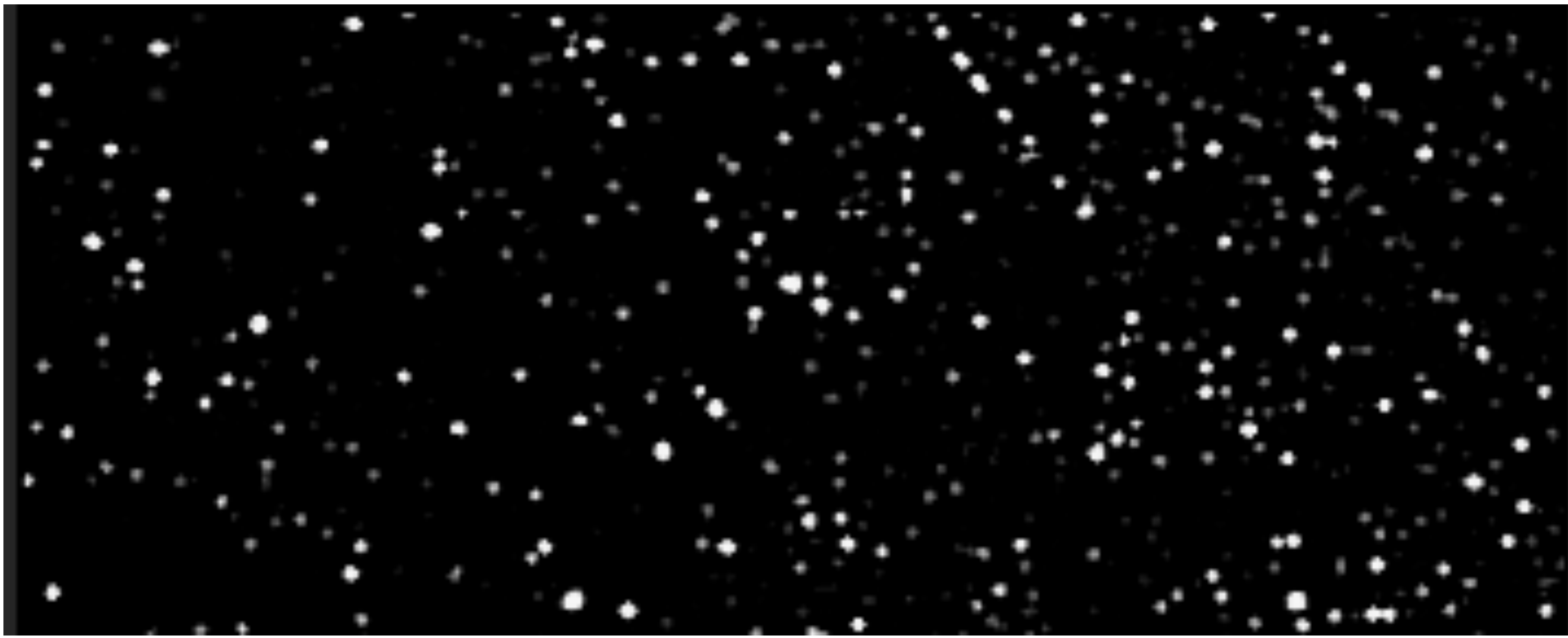




Technique

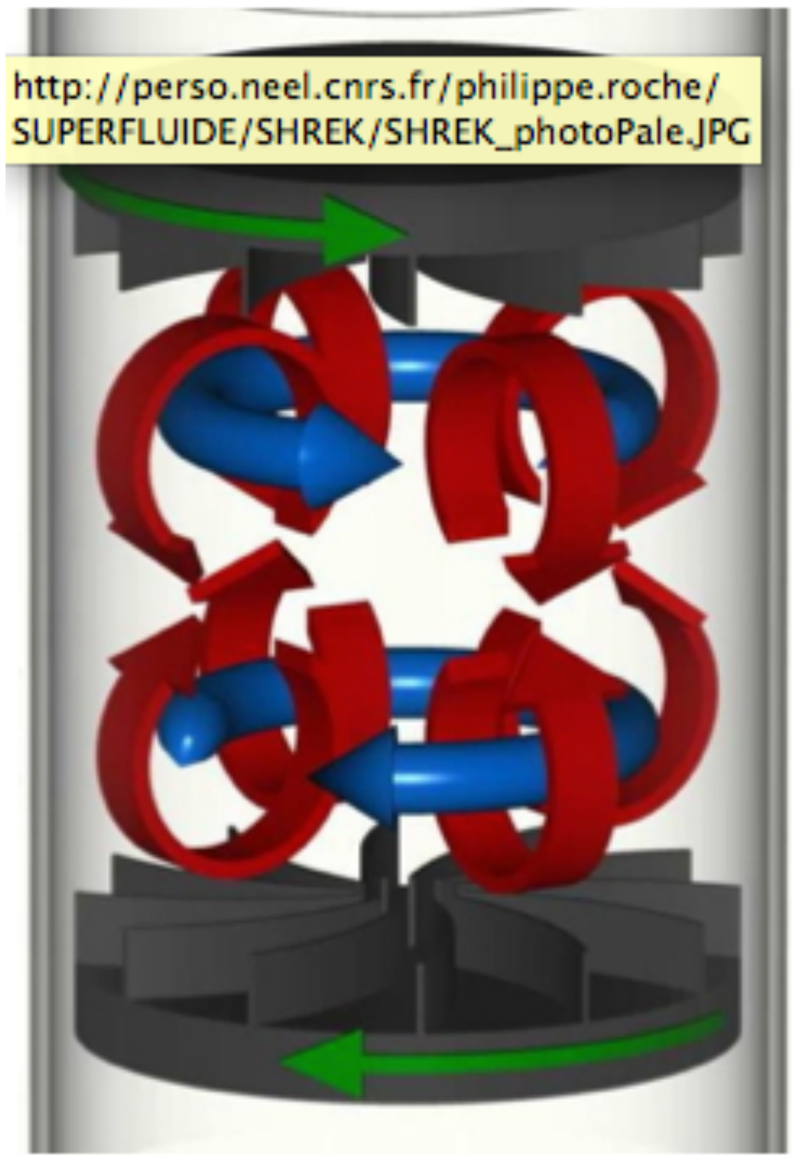
Our visualization technique begins with the injection of hydrogen gas into the liquid helium above the superfluid transition temperature. The hydrogen forms solid particles of sizes of order 1 micron. Evaporative cooling is used to lower the temperature of the liquid helium below the superfluid transition. The solid hydrogen is attracted to and trapped by the filaments of the vortices and may then be used to directly visualize the formation and dynamics of the line vortices in the bulk of the superfluid.

Experimental movie



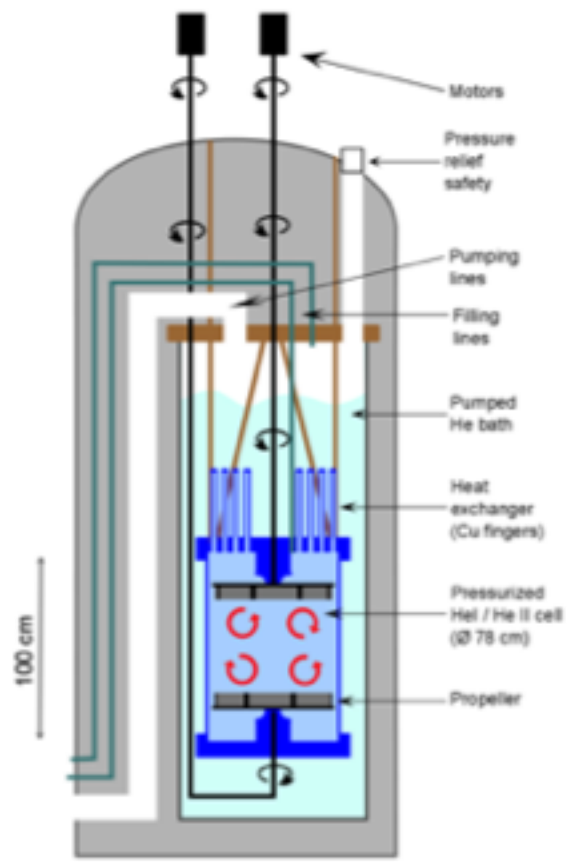
Daniel P. Lathrop's
Nonlinear Dynamics Lab

SHREK experiment



Large scale flow in the cell

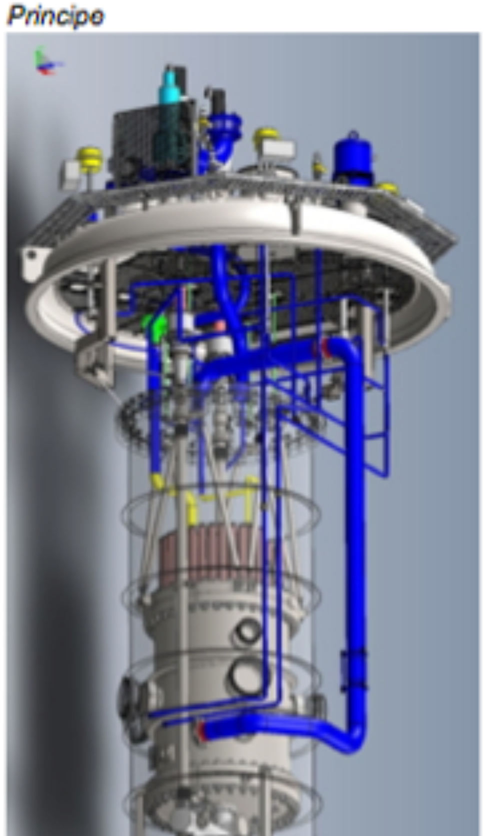
Picture of the propellers



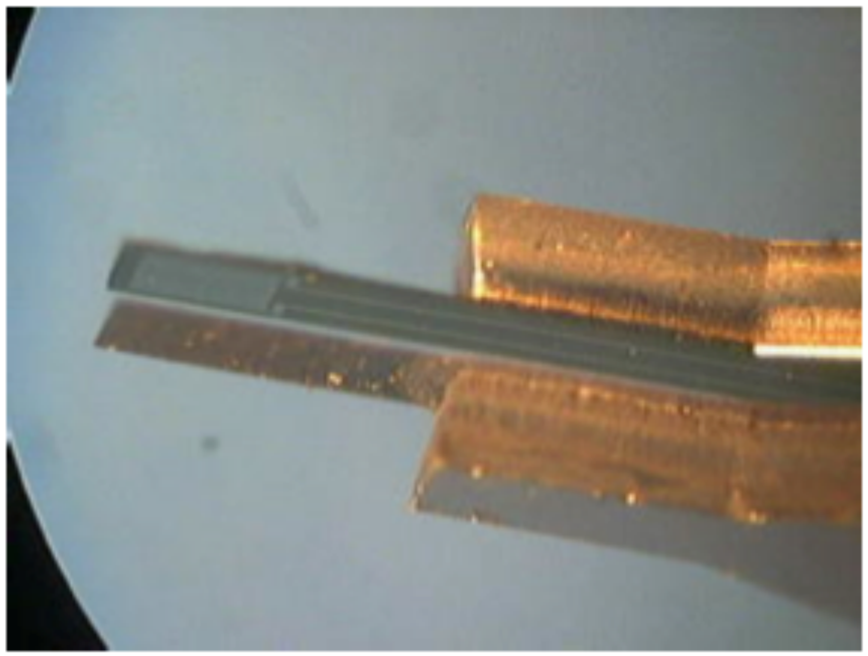
Principe

SHREK cell

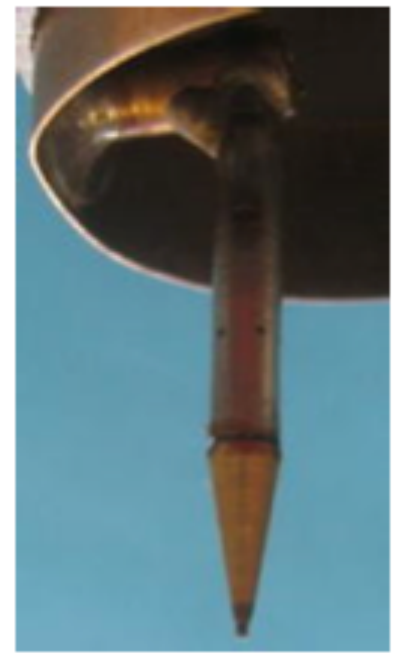
Cryostat (before floor construction)



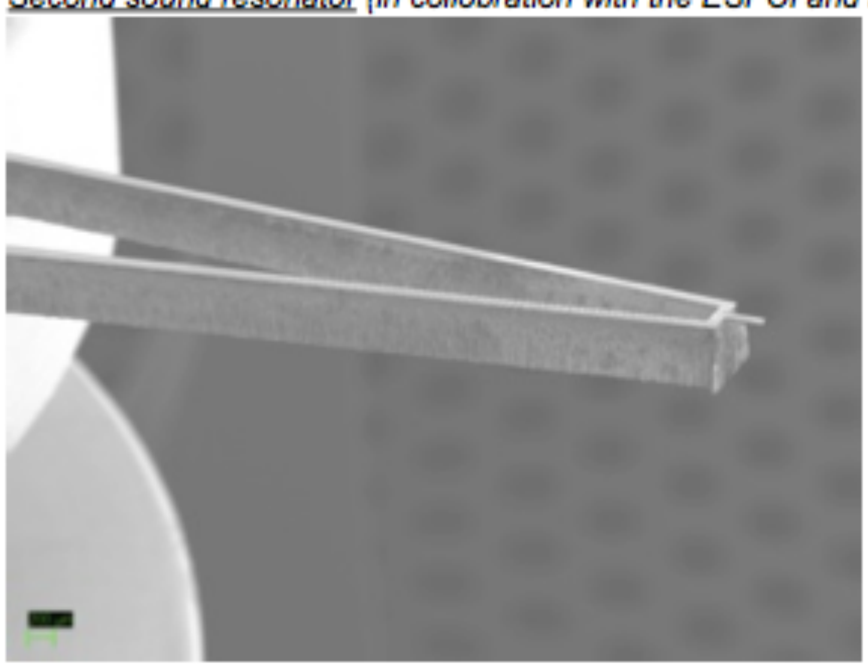
SHREK probes

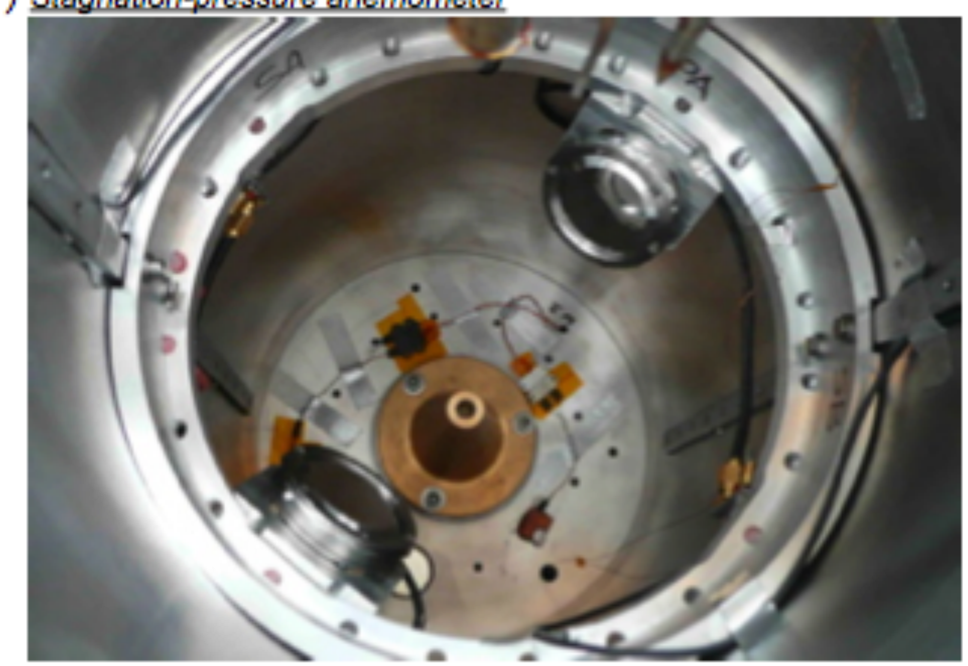


Second sound resonator (in collobration with the ESPCI and the IEF)



Stagnation-pressure anemometer





What is Quantum Turbulence?

- Finite dissipation in turbulence when viscosity vanishes...
- Mathematical problem: singularity in Euler?
- Physical problem: superfluids can flow without dissipation... [a macroscopic quantum effect]
- Turbulence in superfluids?

Two types of Quantum turbulence

- Co-flow [similar to classical]
- Counter-flow [only in superfluids]
- We studied co-flow GPE turbulence at zero temperature with C. Nore 18 years ago...
- How can one numerically study finite temperatures effects [and counter-flow] ?

- In Landau's two-fluid model, Helium is a mixture (in an amount depending on temperature) of a normal component and a superfluid component. Landau's model gives no description of the dynamics of quantum vortices and their interaction with the normal fluid.

- The Gross- Pitaevskii equation describes very well these vortices, but only at zero temperature!

- The traditional approach is to introduce quantum vortices in the two-fluid model postulating ad hoc rules to describe the motion of vortices, their reconnection and their interaction with the normal fluid.

- Idea: extend GPE to finite temperatures!

Truncated (or projected) Gross-Pitaeveskii

Description of BEC at finite temperature:
thermal fluctuations overwhelm quantum fluctuations

It is only one of the (many) models of finite-temperature effects in BEC

PHD TUTORIAL

Finite-temperature models of Bose–Einstein condensation

Nick P Proukakis and Brian Jackson[1]

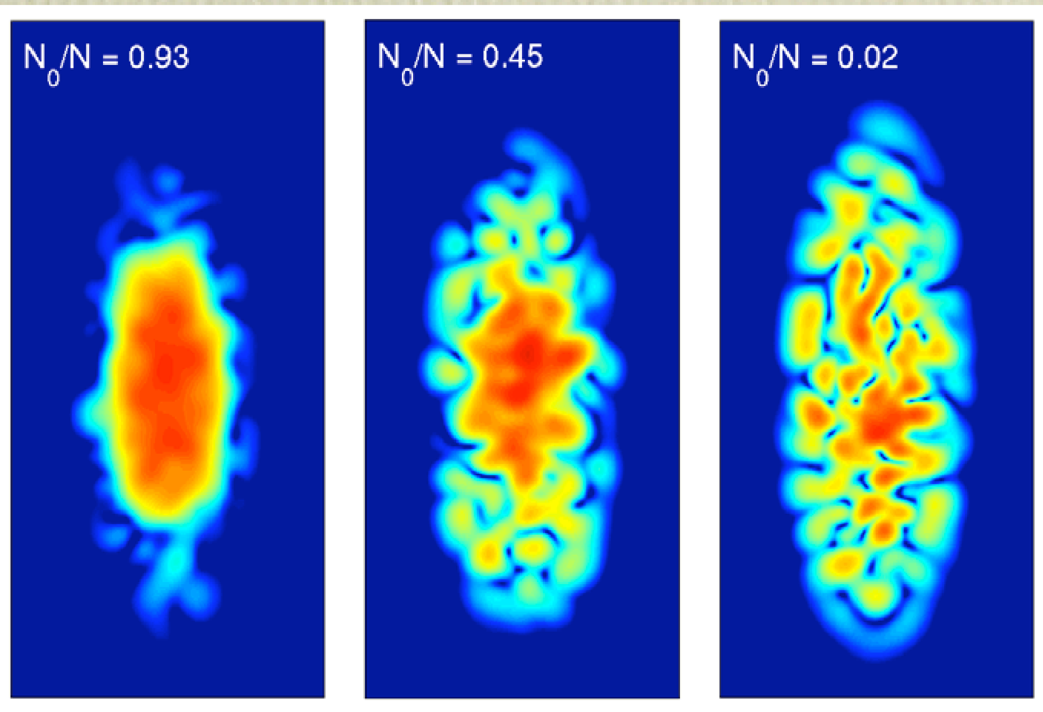


Figure 10. Typical thermalized (equilibrium) images of a classical field consisting of a fixed number of 2000 atoms at three different energies (i.e. temperatures). Plotted are the density profiles $|\phi(x, y, 0)|^2$ of an anisotropic ($\omega_\perp/\omega_z = \sqrt{8}$) trapped 3D Bose gas arising from a single run of the PGPE, with colour representing the atomic density (plotted on the logarithmic scale—black/blue: zero density, red/dark brown: maximum density). The enhancement of fluctuations at higher temperatures corresponding to a lower condensate fraction N_0/N (indicated in figure) is evident. (Images provided by Matt Davis—see also [182].)

PHD TUTORIAL

Finite-temperature models of Bose–Einstein condensation

Nick P Proukakis and Brian Jackson[1]

INDICATIVE GUIDELINES FOR POSSIBLE CHOICE OF 'MINIMUM THEORY' ACCORDING TO REGIME UNDER STUDY

Regime under Study			Coherence (Phase Fluctuations)		Temperature			Possible 'Minimum Theory'	Section	Applicability & Related Comments
Equilibrium	Near Equilibrium	Non Equilibrium	BEC	Quasi BEC (Low D)	$T = 0$	$0 < T < T_C$	$T = T_C$	(no unique choice)		(see main text for details & further clarifications)
✓			✓		✓			Static Bogoliubov	2.2	Suitable for very limited regime close to T=0
✓			✓			✓		Hartree-Fock (Static)	3.1.2	Simplest equilibrium theory for describing partially-condensed bosonic gases (Hartree-Fock Energy Spectrum)
								HFB - Popov (Static)	3.3.1	As above - but additionally includes (T=0) dressing to quasiparticles
								Generalized HFB (Static)	3.4	As above - but additionally includes some many-body effects
								Number-conserving Bogoliubov (Static)	5.2.3	Ensures number-conservation by construction. More cumbersome to implement. Includes corrections due to finite size and shape effects
✓				✓		✓		Modified Mean Field (Low dimensions)	5.1	Full treatment of phase fluctuations. Ab initio determination of density profiles and correlation functions at equilibrium for all dimensions d=1, 2 and 3
	✓		✓			✓		Hartree-Fock (Dynamical)	4.2	No particle exchange between condensate & thermal cloud, or many-body effects
								Number-conserving Bogoliubov (Dynamical)	5.3	Essentially as above, but without relying on symmetry-breaking. Additionally includes many-body effects & corrections due to finite size.
		✓	✓			✓		Self-Consistent Gross-Pitaevskii-Boltzmann (' ZNG')	4.4.3	Includes particle exchange between condensate and thermal cloud (not restricted to ergodicity). Describes well both elementary and macroscopic excitations. Not suitable for (low-dimensional) regimes exhibiting strong phase fluctuations.
		✓		✓		✓		Truncated Wigner	6.2.1	Quantum noise included in initial conditions of simulation only, with dynamics governed by the Gross-Pitaevskii equation. Most suitable for study of quantum effects at short times and relatively low temperatures.
							✓*	Classical Field (Projected Gross-Pitaevskii)	6.1	Based on the assumption that all relevant (low-lying) modes of the system are highly occupied and therefore behave predominantly in a classical manner. Arbitrary (non-equilibrium) initial conditions are propagated to equilibrium by the (Projected) Gross-Pitaevskii equation.
							✓	Stochastic Gross-Pitaevskii & Quantum Boltzmann	6.2.2 6.2.3	Accurately describes fluctuations at phase transition. (Quasi)condensate (low-lying modes of the system) equilibrates in contact with thermal cloud (higher-lying modes), including dynamical (thermal / quantum) noise. Existing numerical implementations include noise throughout the simulations but are currently restricted (for purely numerical reasons) to a static thermal cloud (heat bath) and a classical (instead of the usual Bose) distribution function for the low-lying modes.

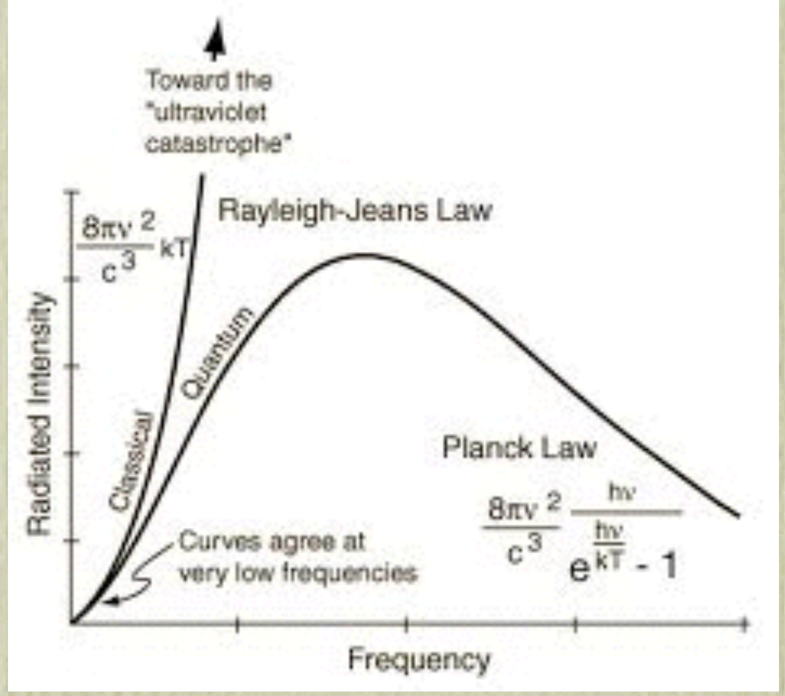
<--TGPE

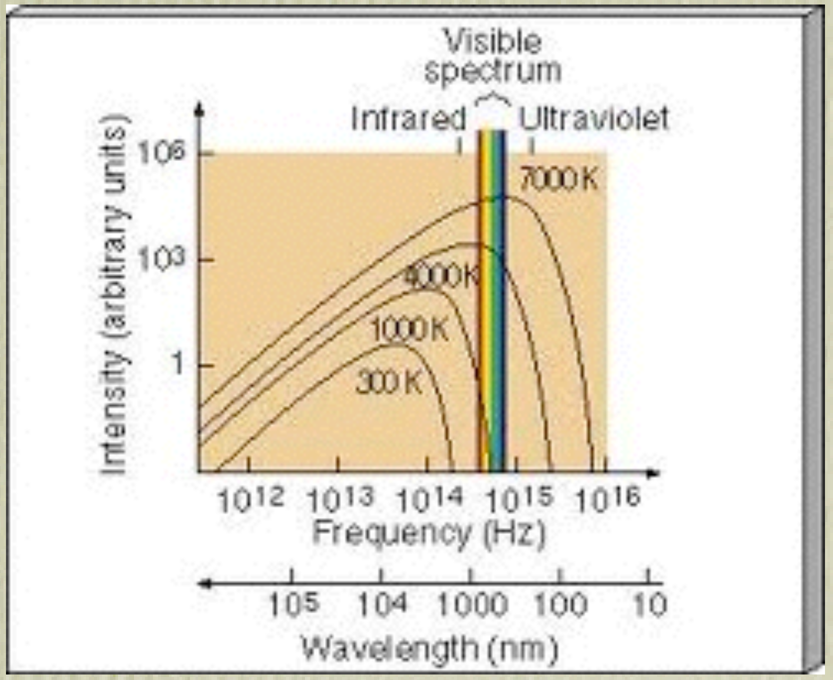
✓* This approach has been used by some authors to describe the route towards condensation and the shift in the critical temperature.

What is TGPE (or PGPE)?

- Clasical GPE contains density waves (sound with dispersive effects at large k)
- These phonon modes should be quantized
- Can they be treated classically?
- Analogy with black body...
- See e.g. www.pnas.org/cgi/doi/10.1073/pnas.1312549111

Black body and truncation...





energy/volume= aT^4

$$a = 7.5657 \times 10^{-16}\ \mathrm{J\ m^{-3}\ K^{-4}}.$$
$$= \frac{8\pi^5 k^4}{15 c^3 h^3},$$

Temperature dependent cutoff

$$\hbar\omega(k_{\rm cut}) = \hbar c k_{\rm cut} \sim k_B T$$

$$k_{\rm cut} \sim k_B T/\hbar c$$

modes below cutoff can be treated classically!

equipartition for wavenumbers $k < k_{\rm cut}$

energy/volume: $k_{\rm cut}^3 k_B T \sim T^4 k_B^4/\hbar^3 c^3$

Classical truncated systems

where first introduced in 1952 by TD Lee in hydrodynamics

T.D. Lee, Quart. Appl. Math., 10(1):69 (1952).

—NOTES—

ON SOME STATISTICAL PROPERTIES OF HYDRODYNAMICAL AND MAGNETO-HYDRODYNAMICAL FIELDS*

BY T. D. LEE (*University of California, Berkeley*)

equilibrium distribution every mode of the Fourier components of magnetic field and velocity field must be in energy equipartition. Let $M(k)$ be the corresponding energy spectrum of magnetic field per unit volume, then we have

$$M(k) = F(k) \propto k^2. \tag{22}$$

3. Turbulence and magneto-turbulence. In the case of a real fluid, due to the energy

General definition of truncated systems

- The basic idea is to perform a truncation (in Fourier space) of the partial differential equation (PDE), as is always done whenever performing an actual numerical computation
- The truncated system is a large number of ordinary differential equations (ODE) with standard statistical mechanical properties
- It contains dissipative processes, thus furnishing a description finite temperature effects

Fourier-Galerkin truncation

Example: Let F be a non-linear function

$$\text{PDE:} \begin{cases} \dfrac{\partial \mathbf{u}}{\partial t}(\mathbf{x},t) = \mathbf{F}[\mathbf{u}, \partial_i \mathbf{u}, \partial_{ij}\mathbf{u}, \ldots] \\ \text{Periodic B.C. on } \quad \Omega = [0, 2\pi]^D \end{cases}$$

with a conserved quantity E

$$\mathbf{u}(\mathbf{x},t) = \sum_{\mathbf{k}} \hat{\mathbf{u}}(\mathbf{k},t) e^{i\mathbf{k}\cdot\mathbf{x}}$$

Non linear terms imply convolutions in Fourier space

$$\frac{\partial \hat{\mathbf{u}}}{\partial t}(\mathbf{k},t) = \hat{\mathbf{F}}[\hat{\mathbf{u}}, \mathbf{k}]$$

$$\mathbf{k} \in \mathbb{Z}^D$$

Galerkin-truncated equation

$$\frac{\partial \hat{\mathbf{u}}}{\partial t}(\mathbf{k}, t) = \hat{\mathbf{F}}[\hat{\mathbf{u}}, \mathbf{k}]$$

$$\hat{\mathbf{u}}(\mathbf{k}, t) = \mathbf{0} \quad \text{if} \quad |\mathbf{k}| \geq k_{\mathrm{max}}$$

- Finite-dimensional system of ODE
- **PDE is approximated by the truncated system only as long as the spectral convergence is ensured**

(dynamics is not influenced by the cut-off)

- Inherits some conservation laws of the original PDE
- Statistical stationary solutions given by the associated Liouville equation $\mathbb{P}[\hat{\mathbf{u}}(\mathbf{k})] = \mathcal{N} e^{-\eta E}$ absolute equilibria

General properties of truncated system

- System relaxes toward the thermodynamical equilibrium
- Partial thermalization at small scales
- Thermalized modes generate an effective dissipation acting at large scales. (Kolmogorov regime for truncated Euler and mutual friction for TGPE)

Classical hydrodynamics

Truncated Euler equation

PRL **95,** 264502 (2005) PHYSICAL REVIEW LETTERS week ending 31 DECEMBER 2005

Effective Dissipation and Turbulence in Spectrally Truncated Euler Flows

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(Received 21 October 2004; published 22 December 2005)

A new transient regime in the relaxation towards absolute equilibrium of the conservative and time-reversible 3D Euler equation with a high-wave-number spectral truncation is characterized. Large-scale dissipative effects, caused by the thermalized modes that spontaneously appear between a transition wave number and the maximum wave number, are calculated using fluctuation dissipation relations. The large-scale dynamics is found to be similar to that of high-Reynolds number Navier-Stokes equations and thus obeys (at least approximately) Kolmogorov scaling.

Truncated Euler equation

TD. LEE (Quart Appl Math 1952), RH. KRAICHNAN 1967-1973, C. Cichowlas et al. (PRL 2005), W. BOS and J. Bertoglio (Phys. Fluids 2005), Frisch et al. (PRL 2008), ...

Euler PDE:
$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p$$
$$\nabla \cdot \mathbf{u} = 0$$

$$\mathbf{u}(\mathbf{x}, t) = \sum_{\mathbf{k}} \hat{\mathbf{u}}(\mathbf{k}, t) e^{i\mathbf{k}\cdot\mathbf{x}}$$

$$\partial_t \hat{u}_\alpha(\mathbf{k}, t) = -\frac{i}{2}\mathcal{P}_{\alpha\beta\gamma}(\mathbf{k}) \sum_{\mathbf{p}} \hat{u}_\beta(\mathbf{p}, t)\hat{u}_\gamma(\mathbf{k}-\mathbf{p}, t)$$

where $\mathcal{P}_{\alpha\beta\gamma} = k_\beta P_{\alpha\gamma} + k_\gamma P_{\alpha\beta}$ with $P_{\alpha\beta} = \delta_{\alpha\beta} - k_\alpha k_\beta / k$

Truncated Euler equation

Conserved quantities

Energy

$$E = \frac{1}{(2\pi)^3} \int \frac{|\mathbf{u}(\mathbf{x})|^2}{2} \, d^3x = \sum_k E(k)$$

Helicity

$$H = \frac{1}{(2\pi)^3} \int \mathbf{u}(\mathbf{x}) \cdot \omega(\mathbf{x}) \, d^3x = \sum_k H(k) \quad , \quad \omega = \nabla \times \mathbf{u}$$

H. Moffatt, J. Moreau in the 60's. Discovered 200 years after Euler work

$$E(k) = \sum_{k-\Delta k/2<|\mathbf{k}'|<k+\Delta k/2} \tfrac{1}{2}|\hat{\mathbf{u}}(\mathbf{k}',t)|^2$$

$$H(k) = \sum_{k-\Delta k/2<|\mathbf{k}'|<k+\Delta k/2} \hat{\mathbf{u}}(\mathbf{k}',t) \cdot \hat{\omega}(-\mathbf{k}',t)$$

Both Energy and Helicity are exactly conserved by the truncated dynamics

Kraichnan's Helical
Absolute Equilibrium
(J. FLuids Mech. 73)

$$\hat{\mathbf{u}}(\mathbf{k}) \sim e^{-\beta E - \alpha H}$$

Gaussian field

$$E(k) = \frac{k^2}{\beta} \frac{4\pi}{1 - \alpha^2 k^2/\beta^2} \sim k^2 \qquad H(k) = \frac{k^4 \alpha}{\beta^2} \frac{8\pi}{1 - \alpha^2 k^2/\beta^2} \sim k^4$$

For the case presented here: $\alpha^2 k_{\text{max}}^2/\beta^2 \ll 1$

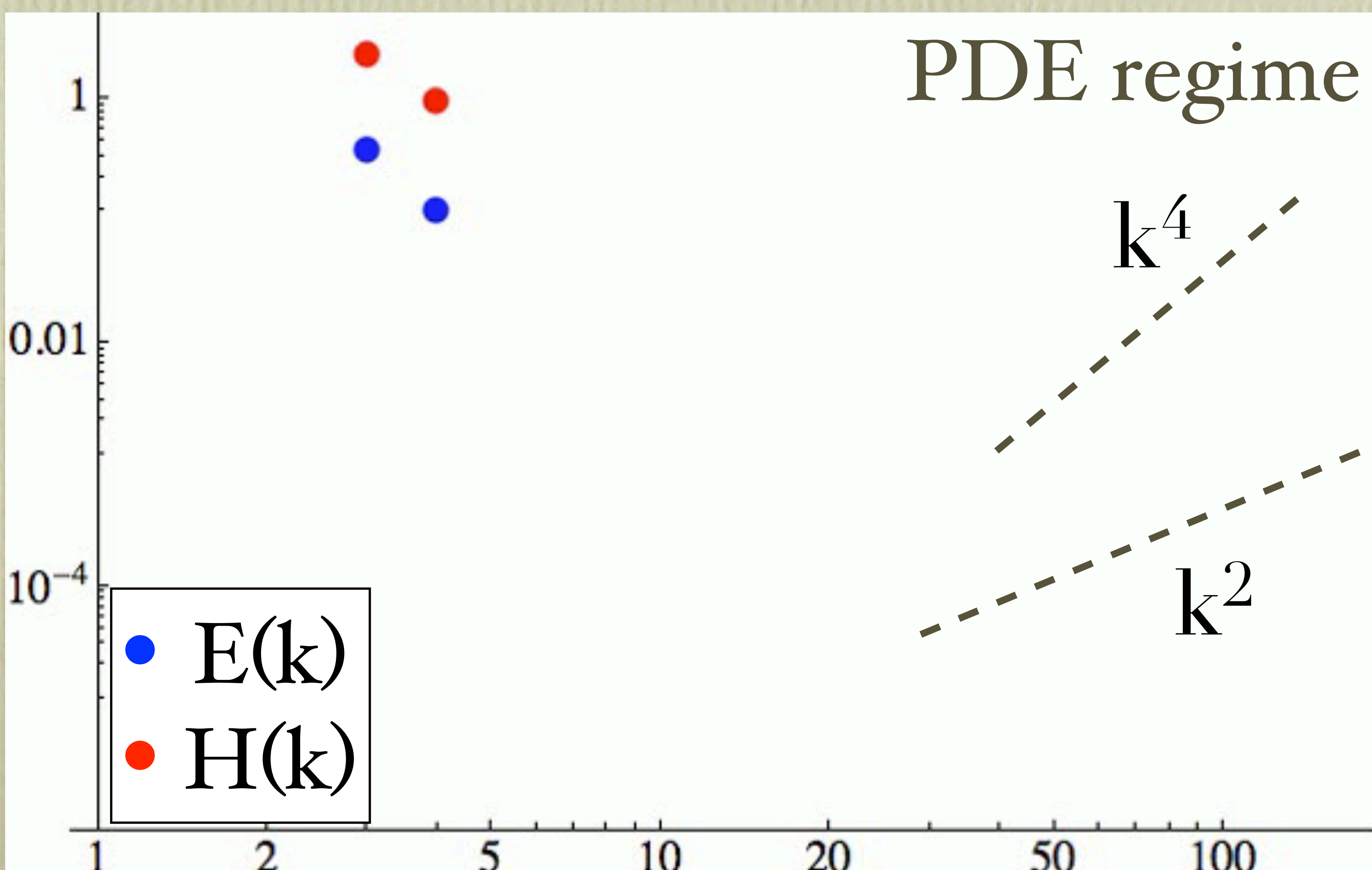
Numerical simulation ABC flow

Resolution of 512^3

$$\nabla \times \mathbf{u}^{(k)}_{\mathrm{ABC}} = \lambda_k \mathbf{u}^{(k)}_{\mathrm{ABC}}$$

G. Krstulovic, P. D. Mininni, M. E. Brachet and A. Pouquet, PRE 79(5) 056304, 2009

$$E(k) = \frac{k^2}{\beta} \frac{4\pi}{1 - \alpha^2 k^2/\beta^2} \sim k^2 \qquad H(k) = \frac{k^4 \alpha}{\beta^2} \frac{8\pi}{1 - \alpha^2 k^2/\beta^2} \sim k^4$$



Truncation effect

Partial Thermalization

near PDE dynamics

Truncated Euler: basic facts

- Relaxation toward Kraichnan helical absolute equilibrium
- Transient mixed energy and helicity cascades
- Thermalized small-scales act as microworld providing an effective dissipation in the system

Superfluid hydrodynamics

Truncated Gross-Pitaevskii equation

PRL **106,** 115303 (2011) PHYSICAL REVIEW LETTERS week ending 18 MARCH 2011

Dispersive Bottleneck Delaying Thermalization of Turbulent Bose-Einstein Condensates

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(Received 26 July 2010; revised manuscript received 10 January 2011; published 16 March 2011)

A new mechanism of thermalization involving a direct energy cascade is obtained in the truncated Gross-Pitaevskii dynamics. A long transient with partial thermalization at small scales is observed before the system reaches equilibrium. Vortices are found to disappear as a prelude to final thermalization. A bottleneck that produces spontaneous effective self-truncation and delays thermalization is characterized when large dispersive effects are present at the truncation wave number. Order of magnitude estimates indicate that self-truncation takes place in turbulent Bose-Einstein condensates. This effect should also be present in classical hydrodynamics and models of turbulence.

Kolmorogov regime in the GPE

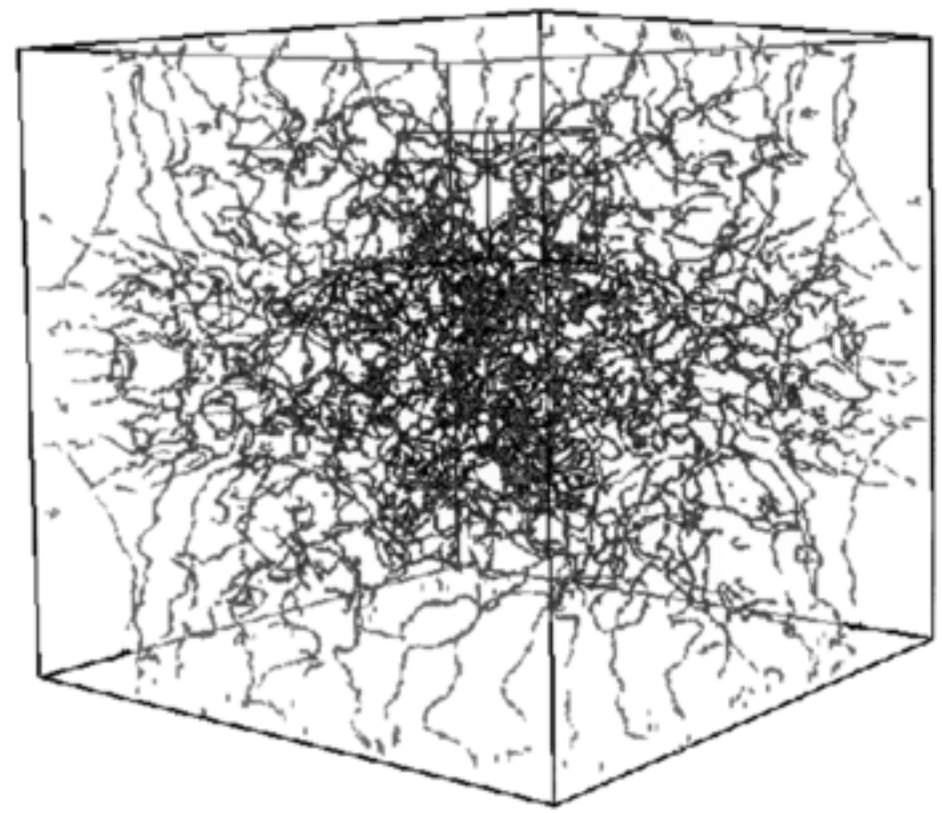


FIG. 5. Same visualization as in Fig. 1, but at time $t = 8$.

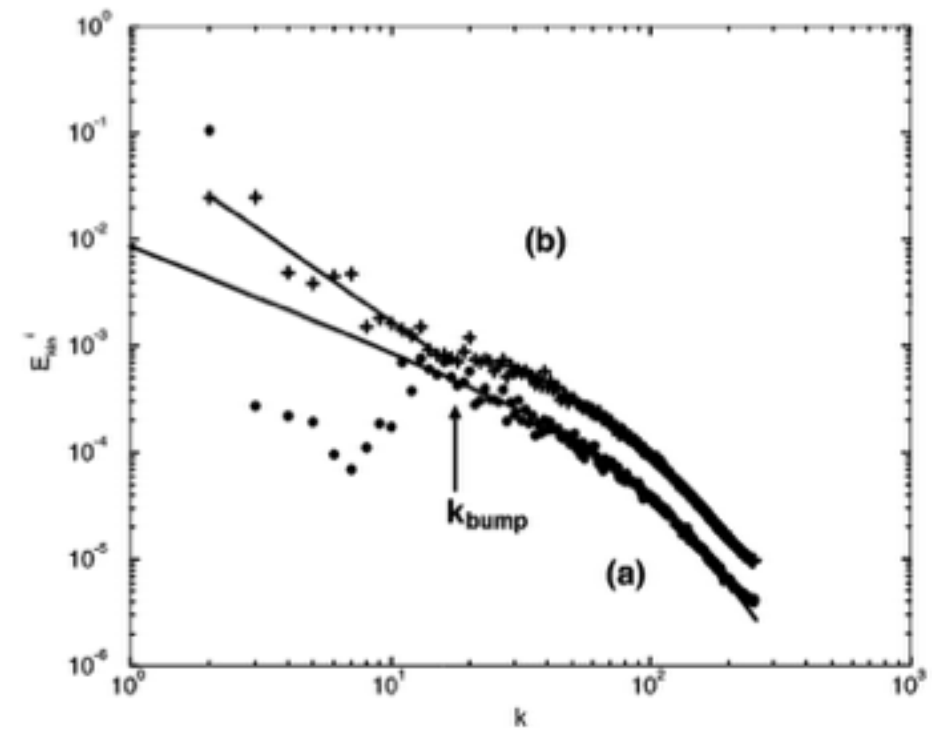


FIG. 2. Plot of the incompressible kinetic energy spectrum, $E^{i}_{\mathrm{kin}}(k)$. The bottom curve (a) (circles) corresponds to time $t = 0$ (same conditions as in Fig. 1). The spectrum of a single axisymmetric 2D vortex multiplied by $(l/2\pi) = 175$ is shown as the bottom solid line. The top curve (b) (plusses) corresponds to time $t = 5.5$. A least-square fit over the interval $2 \leq k \leq 16$ with a power law $E^{i}_{\mathrm{kin}}(k) = Ak^{-n}$ gives $n = 1.70$ (top solid line).

- K41 regime first found in the GPE 19 years ago:
  - C. Nore, M. Abid, and M. E. Brachet, *Phys. Rev. Lett. 78, 3896 (1997)*
  - C. Nore, M. Abid, and M. E. Brachet, *Phys. Fluids 9, 2644 (1997)*
  - M Kobayashi and M Tsubota. *Phy. Rev. Lett. 94(6):065302, Jan 2005.*
  - Yepez et al. *Phys. Rev. Lett. 103(8):084501, Aug 2009*
  - ....

Wave propagation

$$\psi = A_0 e^{-i\frac{\mu}{\hbar}t} + \delta\psi$$

Bogoliubov dispersion relation:

$$\omega(k) = \sqrt{\frac{g|A_0|^2}{m}k^2 + \frac{\hbar^2}{4m^2}k^4}.$$

$$\omega(k) = c\,k\sqrt{1 + \frac{1}{2}\xi^2 k^2}$$

Speed of sound $\quad c = \sqrt{g|A_0|^2/m}$

Coherence length $\quad \xi = \sqrt{\hbar^2/2m|A_\mathbf{0}|^2 g}.$

Important dimensionless parameter for TGPE

$$\xi k_{\mathrm{max}}$$

Amount of dispersion of thermal waves

Hydrodynamic description of GPE

$$\psi(\mathbf{x},t) = \sqrt{\frac{\rho(\mathbf{x},t)}{m}}\exp\left[i\frac{m}{\hbar}\phi(\mathbf{x},t)\right], \qquad \mathbf{v} = \boldsymbol{\nabla}\phi$$

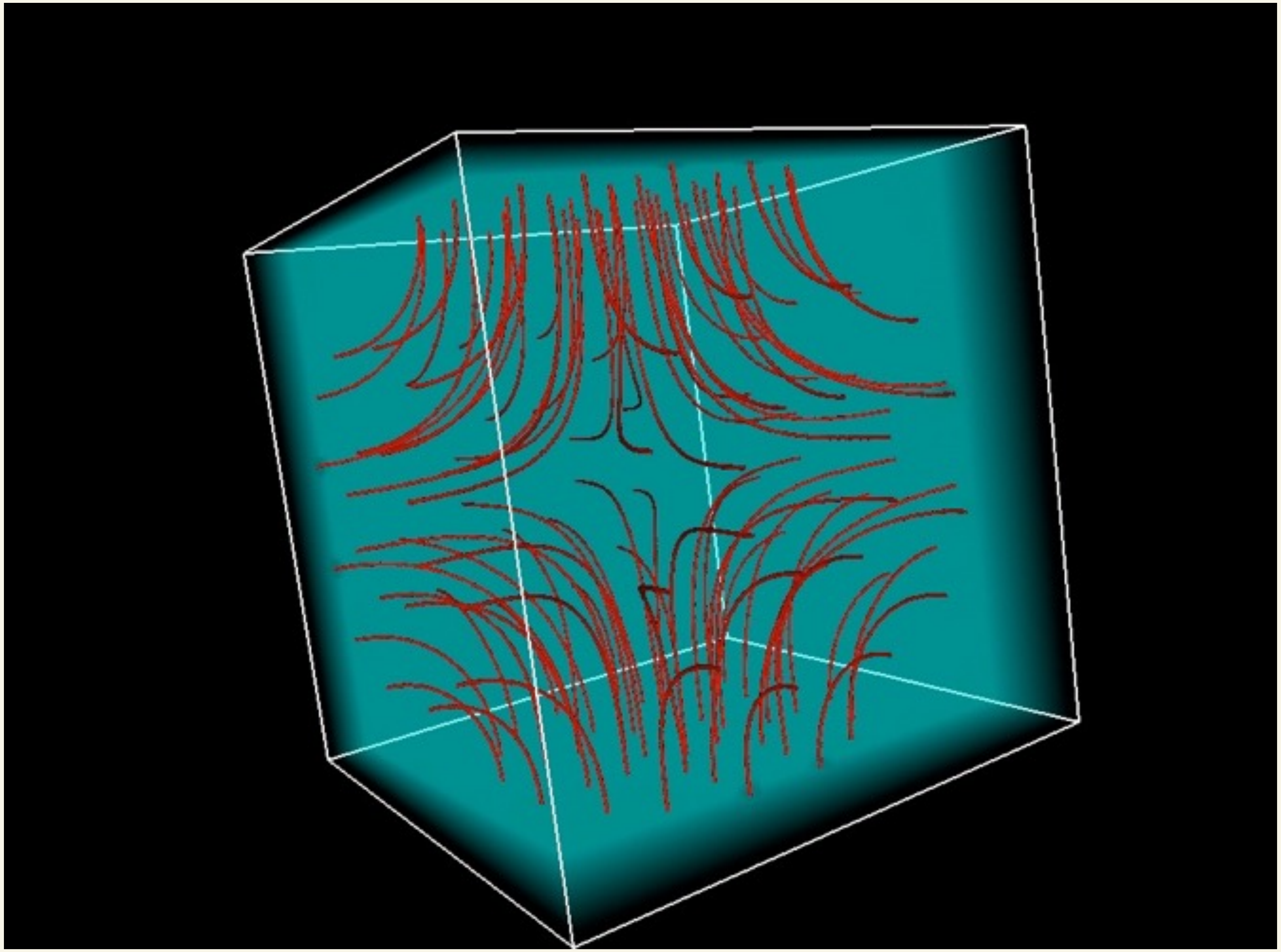
$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\,\boldsymbol{\nabla}\,\phi) = 0, \qquad \frac{\partial\phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2 = c^2(1-\rho) + c^2\xi^2\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}.$$

$$e_{\mathrm{tot}} = \frac{1}{V}\int d^3x\left[\frac{1}{2}(\sqrt{\rho}\mathbf{v})^2 + \frac{c^2}{2}(\rho-1)^2 + \frac{\hbar^2}{2m^2}(\nabla\sqrt{\rho})^2\right]$$

$$E_{\mathrm{tot}} = E_{\mathrm{Kin}} + E_{\mathrm{Int}} + E_{\mathrm{q}}$$

$$E_{\mathrm{Kin}} = E_{\mathrm{Kin}}^{\mathrm{I}} + E_{\mathrm{Kin}}^{\mathrm{C}}$$

Taylor-Green vortex



Energy transfer from incompressible kinetic energy to sound waves.

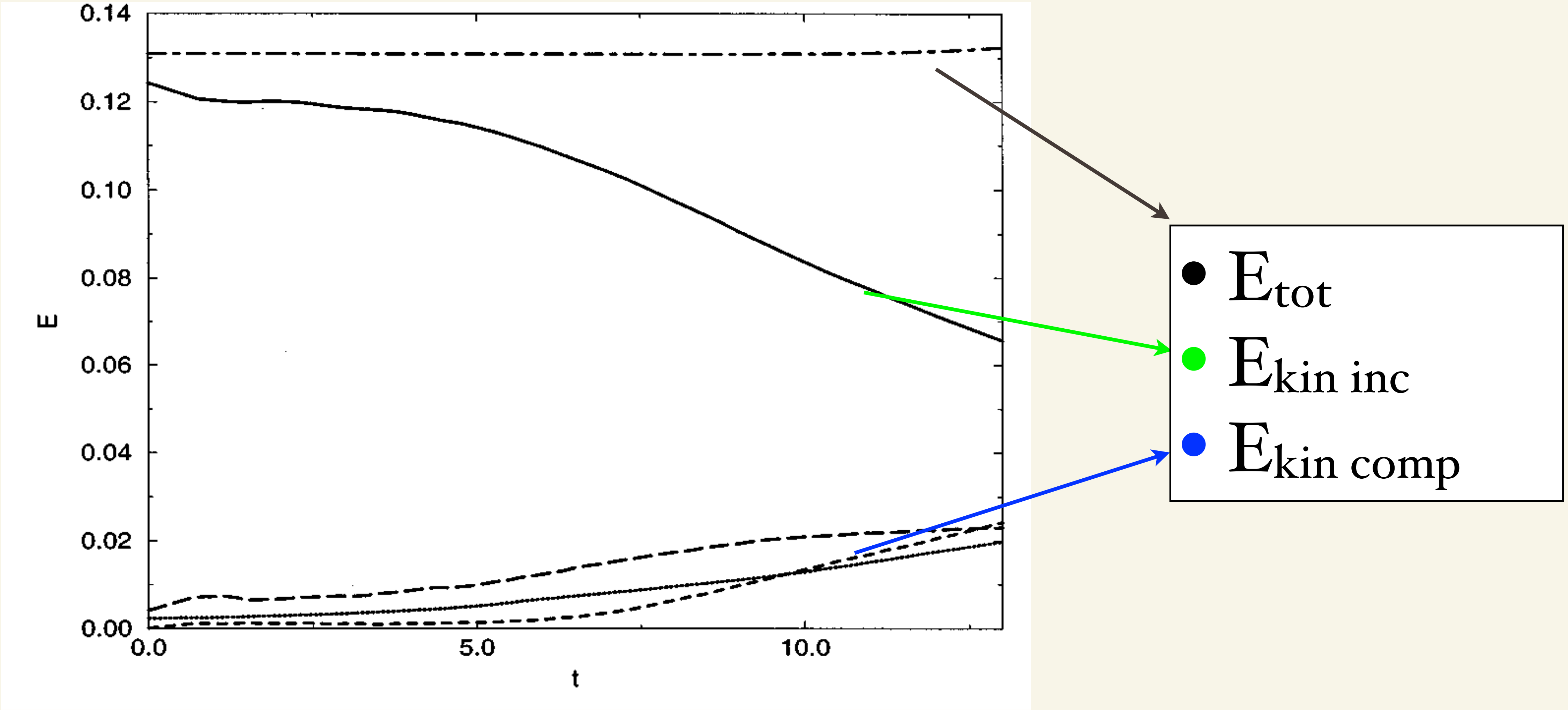


FIG. 13. Time evolution of total energy E_{tot} (dot-dashed), incompressible kinetic energy E^{i}_{kin} (solid), compressible kinetic energy E^{c}_{kin} (dotted), quantum energy E_q (dashed), and internal energy E_{int} (long–dashed) for run d. Note the transfer of energy from the incompressible part to the other contributions.

Nore et al. Phys. Fluids **9** (9), 1997, PRL 78 (30), 1997

Truncation of GPE

$$i\hbar\frac{\partial\psi}{\partial t} = \mathcal{P}_{\mathrm{G}}[-\frac{\hbar^2}{2m}\nabla^2\psi + g\mathcal{P}_{\mathrm{G}}[|\psi|^2]\psi]$$

$$H = \int d^3x \left(\frac{\hbar^2}{2m}|\nabla\psi|^2 + \frac{g}{2}[\mathcal{P}_{\mathrm{G}}|\psi|^2]^2\right).$$

$$\mathcal{P}_{\mathrm{G}}[\hat{\psi}_k] = \theta(k_{\mathrm{max}} - k)\hat{\psi}_k$$

Heaviside function

Description of BEC at finite temperature: Thermal fluctuations overwhelm quantum fluctuations

Conserved quantities

Energy, number of particles and momentum

$$
\begin{aligned}
H &= \int_V d^3x \left(\frac{\hbar^2}{2m} |\nabla\psi|^2 + \frac{g}{2} |\psi|^4 \right) \\
N &= \int_V |\psi|^2 \, d^3x \\
\mathbf{P} &= \int_V \frac{i\hbar}{2} \left(\psi \nabla \overline{\psi} - \overline{\psi} \nabla \psi \right) \, d^3x.
\end{aligned}
$$

Conservation laws are valid in the truncated system, if dealiasing is done carefully enough

Thermalized microcanonical states

Condensation transition in TGPE

It was previously known that

the k=0 mode of ψ vanishes at finite energy

MJ. Davis, SA. Morgan and K. Burnett PRL **87,** (2001)

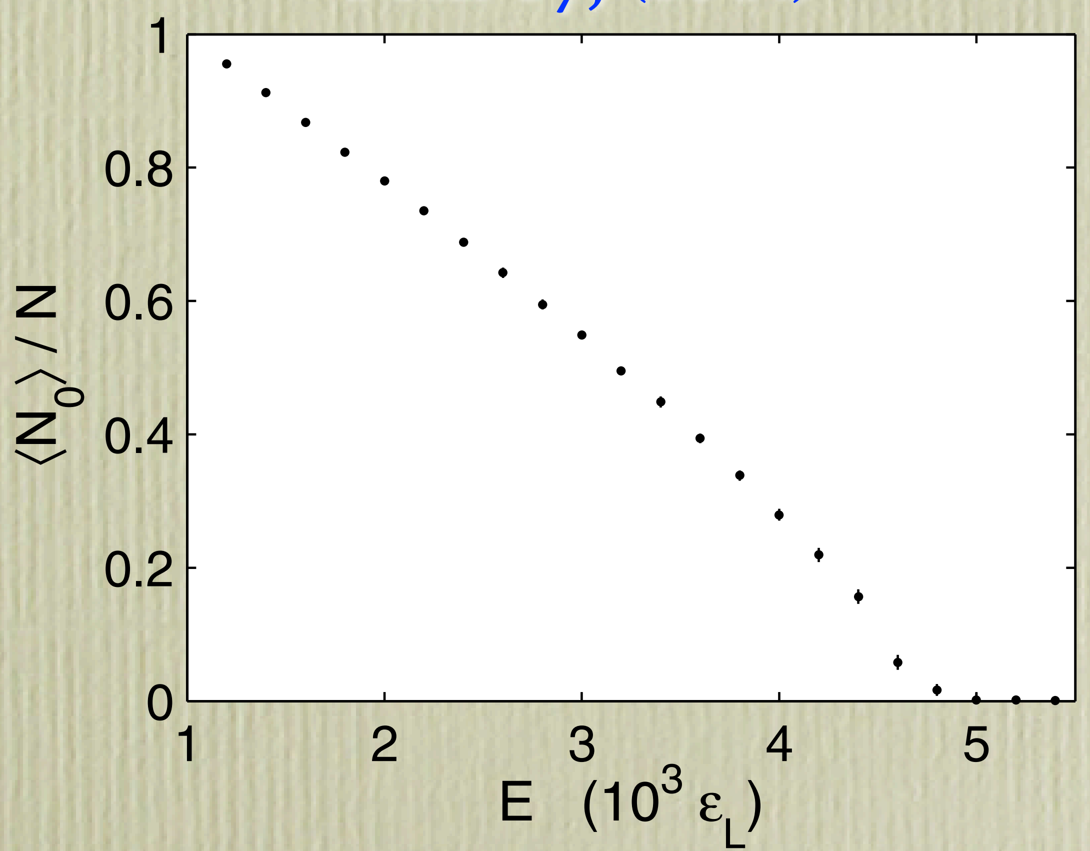


FIG. 1. Condensate fraction plotted against total energy after each individual simulation has reached equilibrium. The barely discernible vertical lines on each point indicate the magnitude of the fluctuations.

C. Connaughton,C. Josserand, A. Picozzi, Y. Pomeau and S. Rica. PRL **95**, 263901.(2005)
Düring et al. Physica D 2009, vol. 238

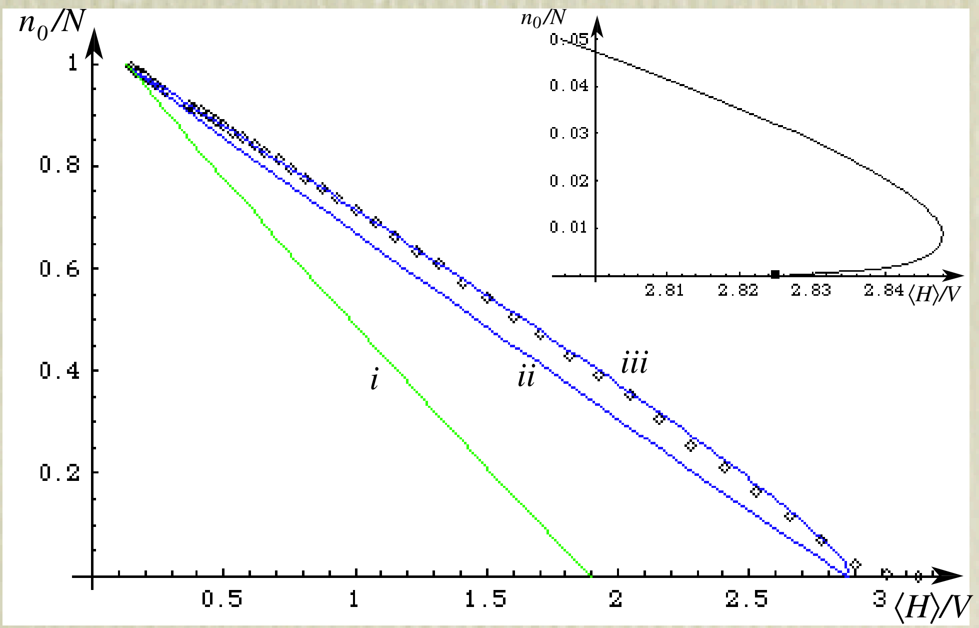


FIG. 2 (color online). Condensate fraction n_0/N vs total energy density $\langle H \rangle / V$, where $\langle H \rangle = E + E_0$, E_0 being the condensate energy [see Eq. (9)]. Points ($\diamond$) refer to numerical simulations of the NLS Eq. (1) with 64^3 modes ($N/V = 1/2$). The straight line (i) [(ii)] corresponds to the continuous Eq. (6) [discretized Eq. (7)] approximation. Curve (iii) refers to condensation in the presence of nonlinear interactions [from Eq. (9)], which makes the transition to condensation subcritical, as illustrated in the inset (with 1024^3 modes). Each point ($\diamond$) corresponds to an average over 10^3 time units.

What is an absolute equilibrium for GPE?

Grand canonical

New algorithm to generate absolute equilibrium

$$P_{\mathrm{stat}} = \frac{1}{\mathcal{Z}} e^{-\beta F} \qquad F = H - \mu N - \cancel{\mathbf{W} \cdot \mathrm{P}}$$

Non Gaussian

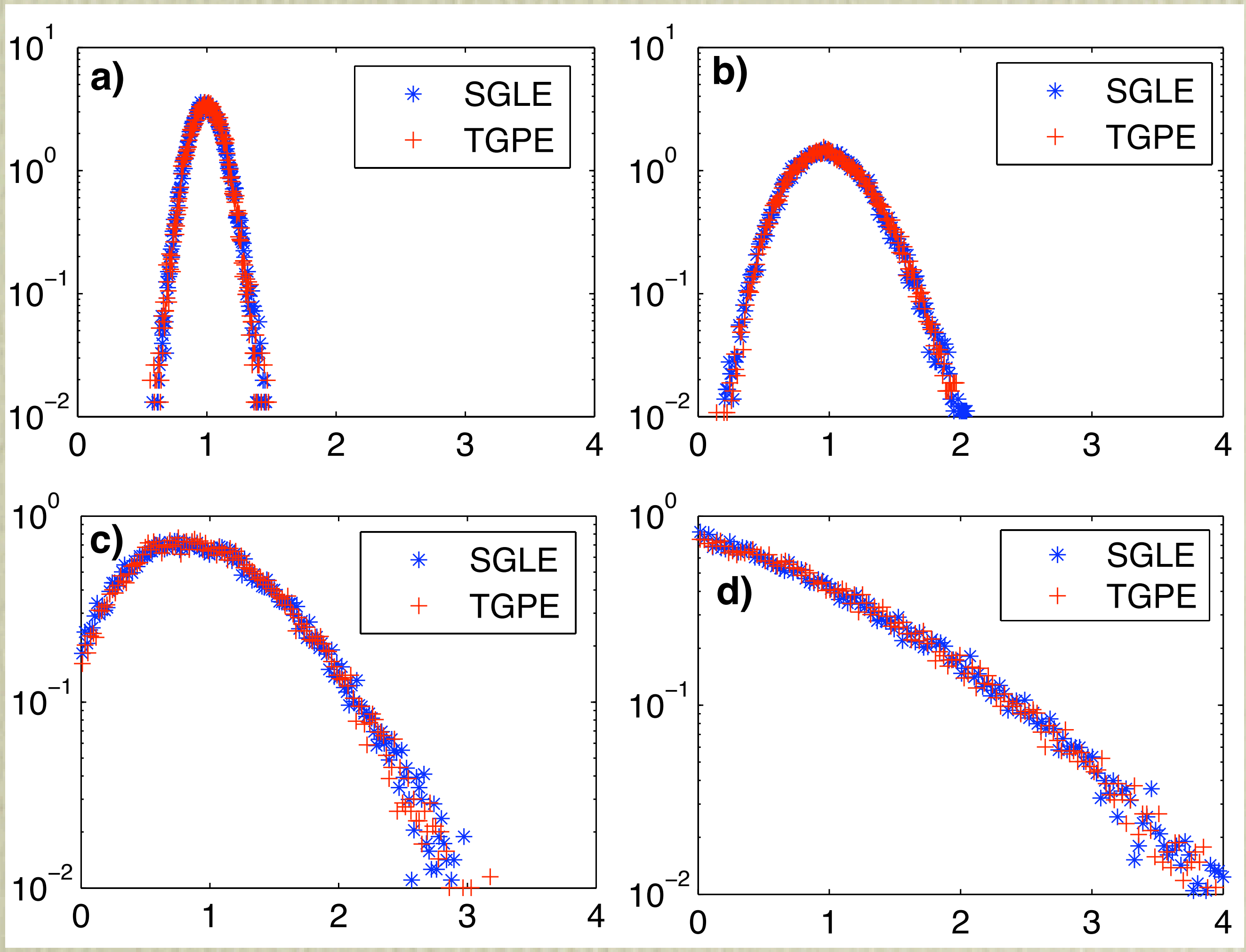
$$\hbar \frac{\partial A_{\mathbf{k}}}{\partial t} = -\frac{1}{V}\frac{\partial F}{\partial A^*_{\mathbf{k}}} + \sqrt{\frac{2\hbar}{V\beta}}\,\hat{\zeta}(\mathbf{k},t)$$

$$\langle \zeta(\mathbf{x},t)\zeta^*(\mathbf{x}',t')\rangle = \delta(t-t')\delta(\mathbf{x}-\mathbf{x}'),$$

$$\hbar\frac{\partial \psi}{\partial t} = \mathcal{P}_{\mathrm{G}}[\frac{\hbar^2}{2m}\nabla^2\psi + \mu\psi - g\mathcal{P}_{\mathrm{G}}[|\psi|^2]\psi - \cancel{i\hbar\mathbf{W}\cdot\nabla\psi}] + \sqrt{\frac{2\hbar}{V\beta}}\mathcal{P}_{\mathrm{G}}[\zeta(\mathbf{x},t)]$$

Partition function can be analytically obtained at low temperatures

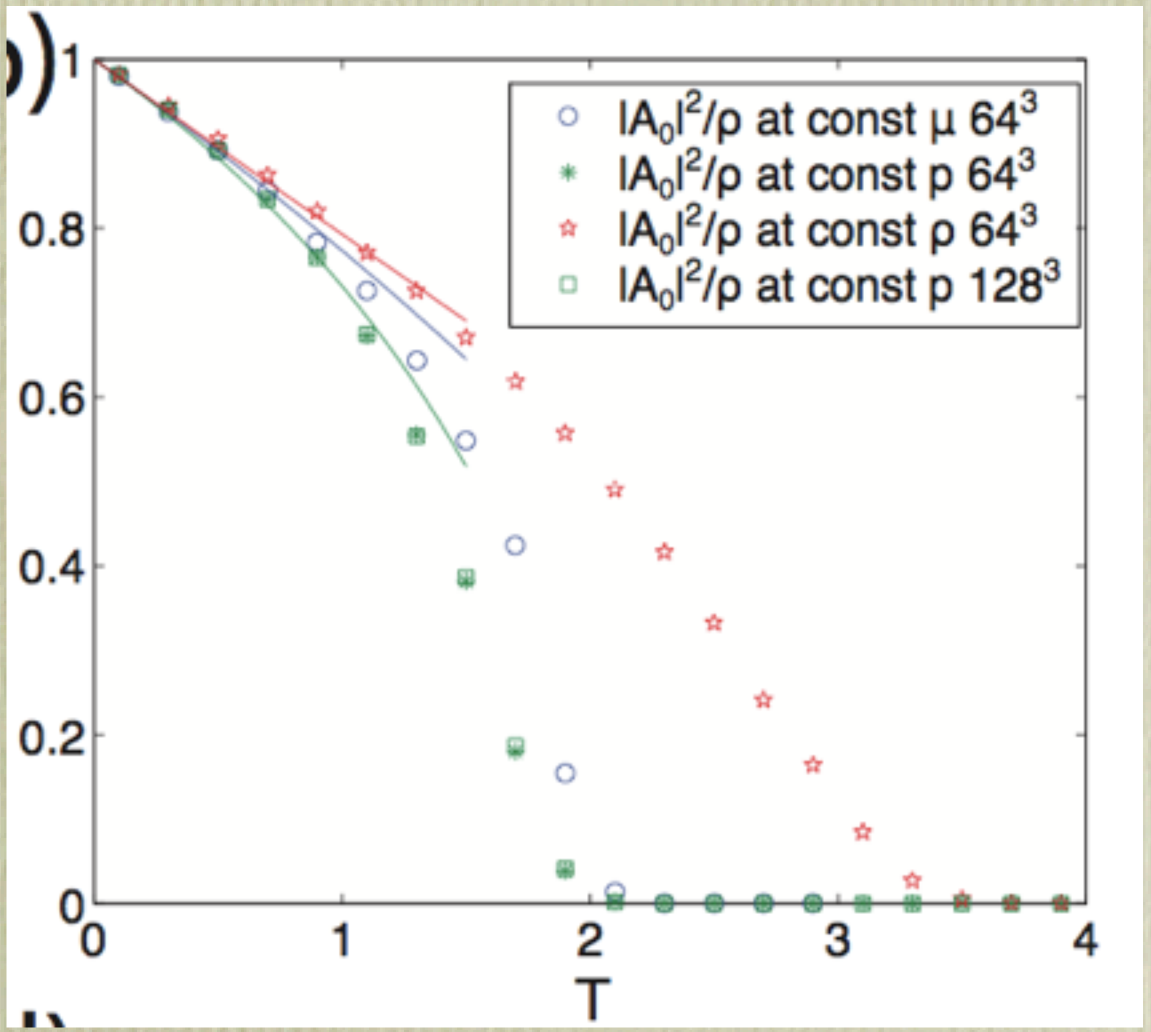
Micro canonical versus grand canonical

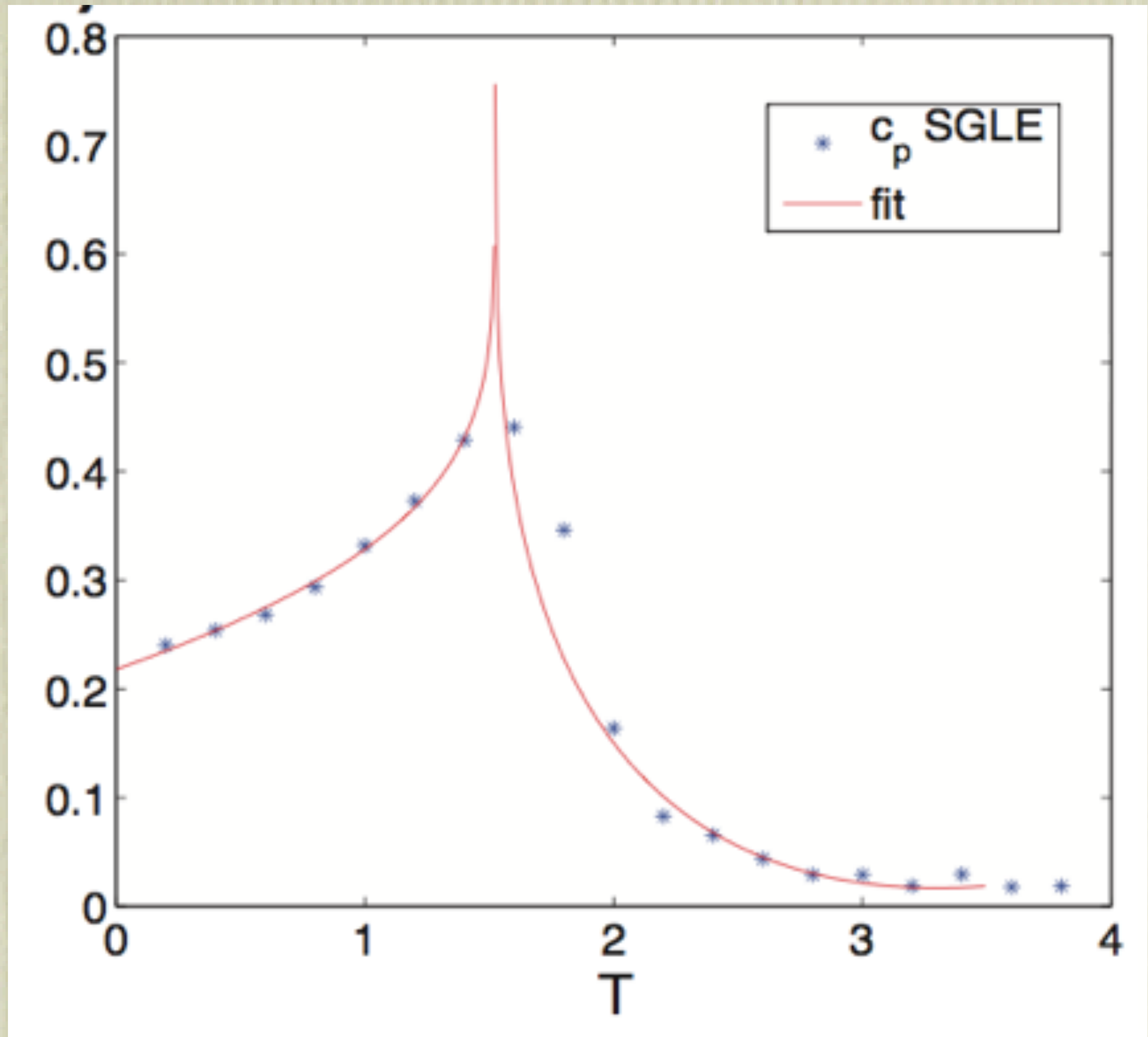


Density histograms

H	T	TGPE time steps	SGLE time steps
0.09	0.09	40000	9600
0.5	0.5	20000	9600
1.96	1.8	20000	9600
4.68	4	20000	5000

Condensation transition





$$\lambda - \phi^4 \quad (D = 3, n = 2)$$

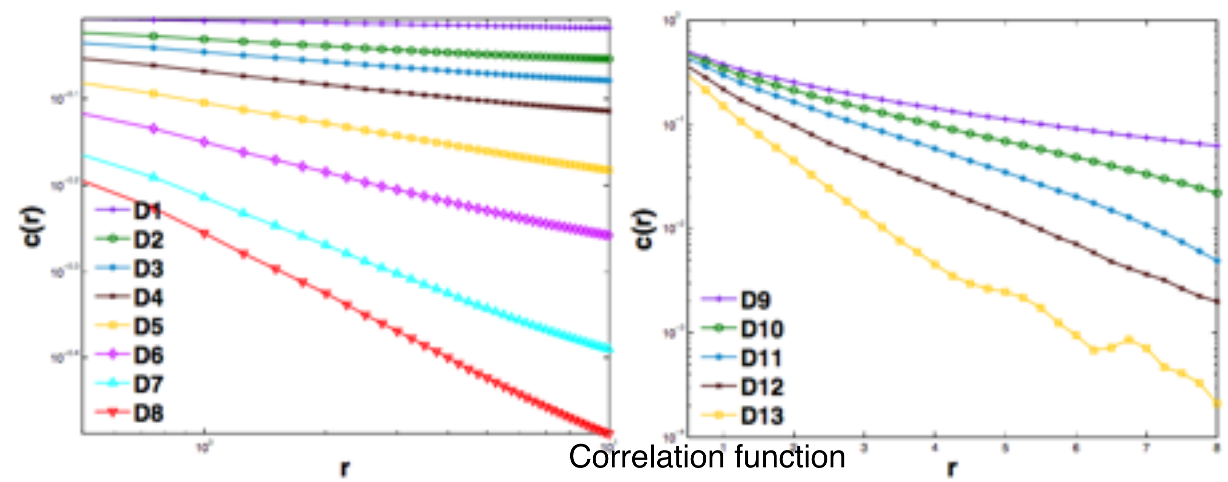
2D BKT transition

Vishwanath Shukla, Marc Brachet and Rahul Pandit

Turbulence in the two-dimensional Fourier-truncated Gross–Pitaevskii equation

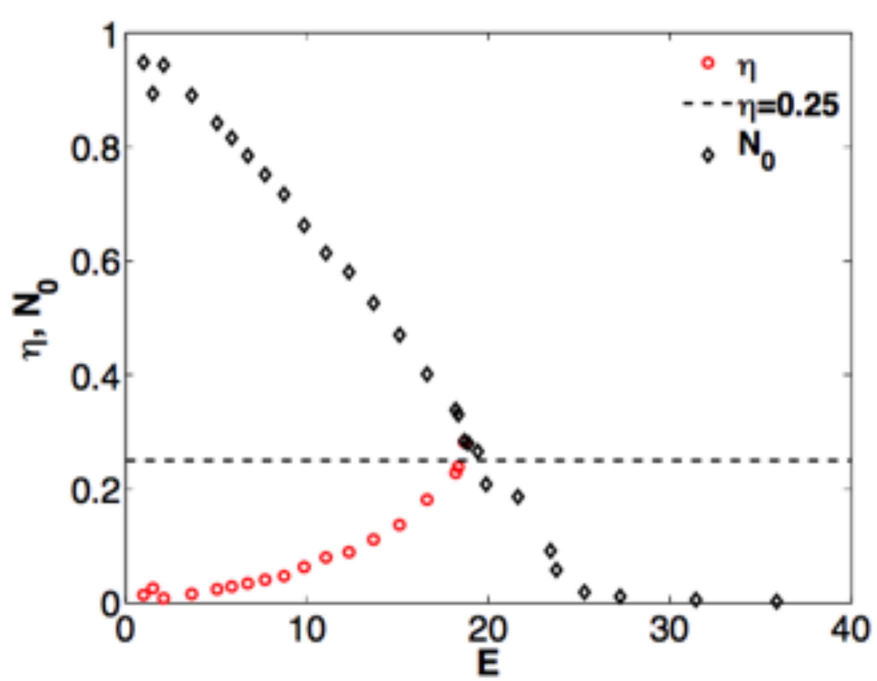
New J. Phys. 15 113025 (2013)

Below transition



Above transition

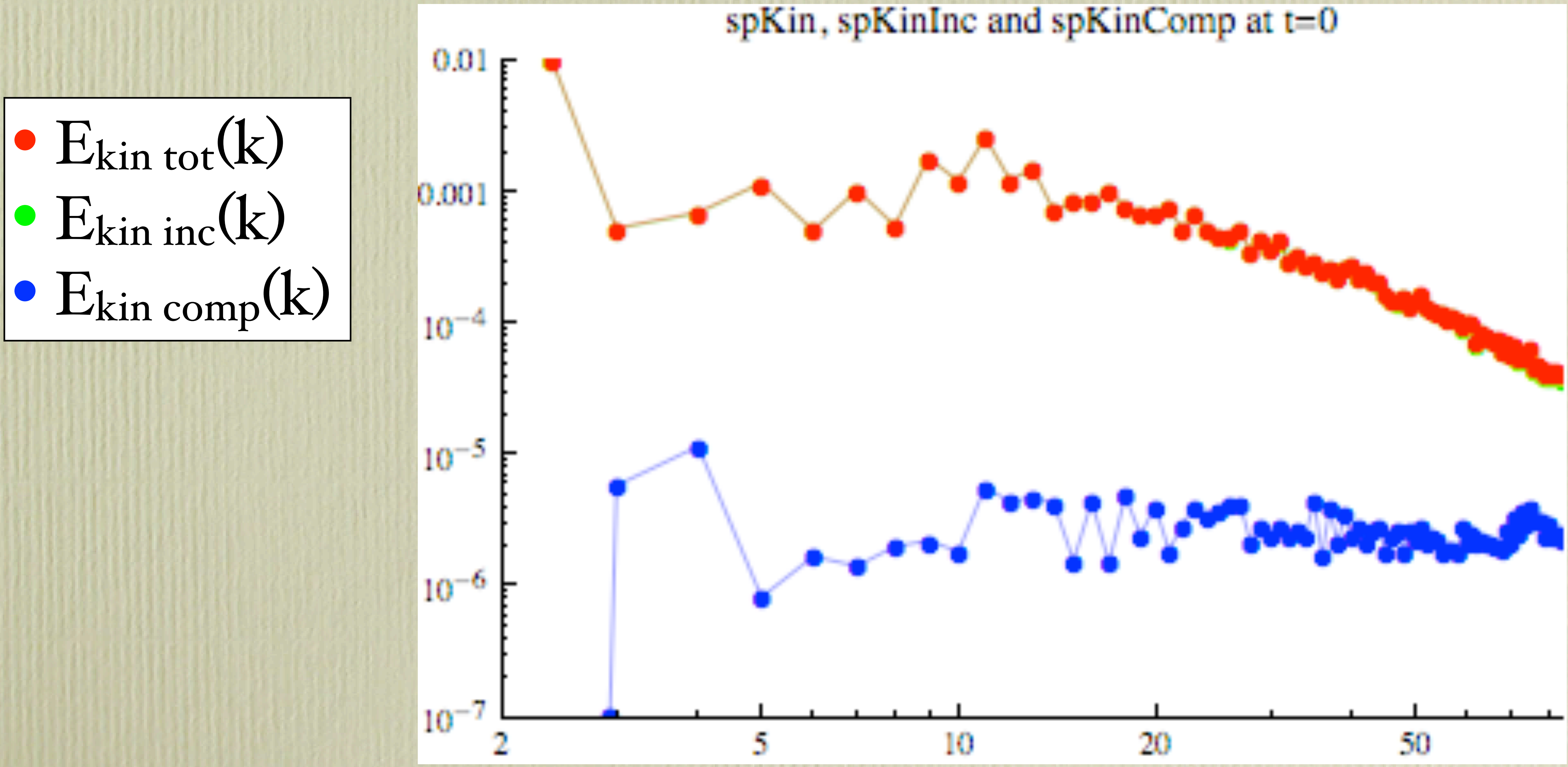
[Left] Log-log plot of $c(r)$ vs. r ($E < E_{\mathrm{BKT}}$, $N_c = 128$); [Right] Semilog-y plot $c(r)$ vs. r ($E > E_{\mathrm{BKT}}$, $N_c = 128$).



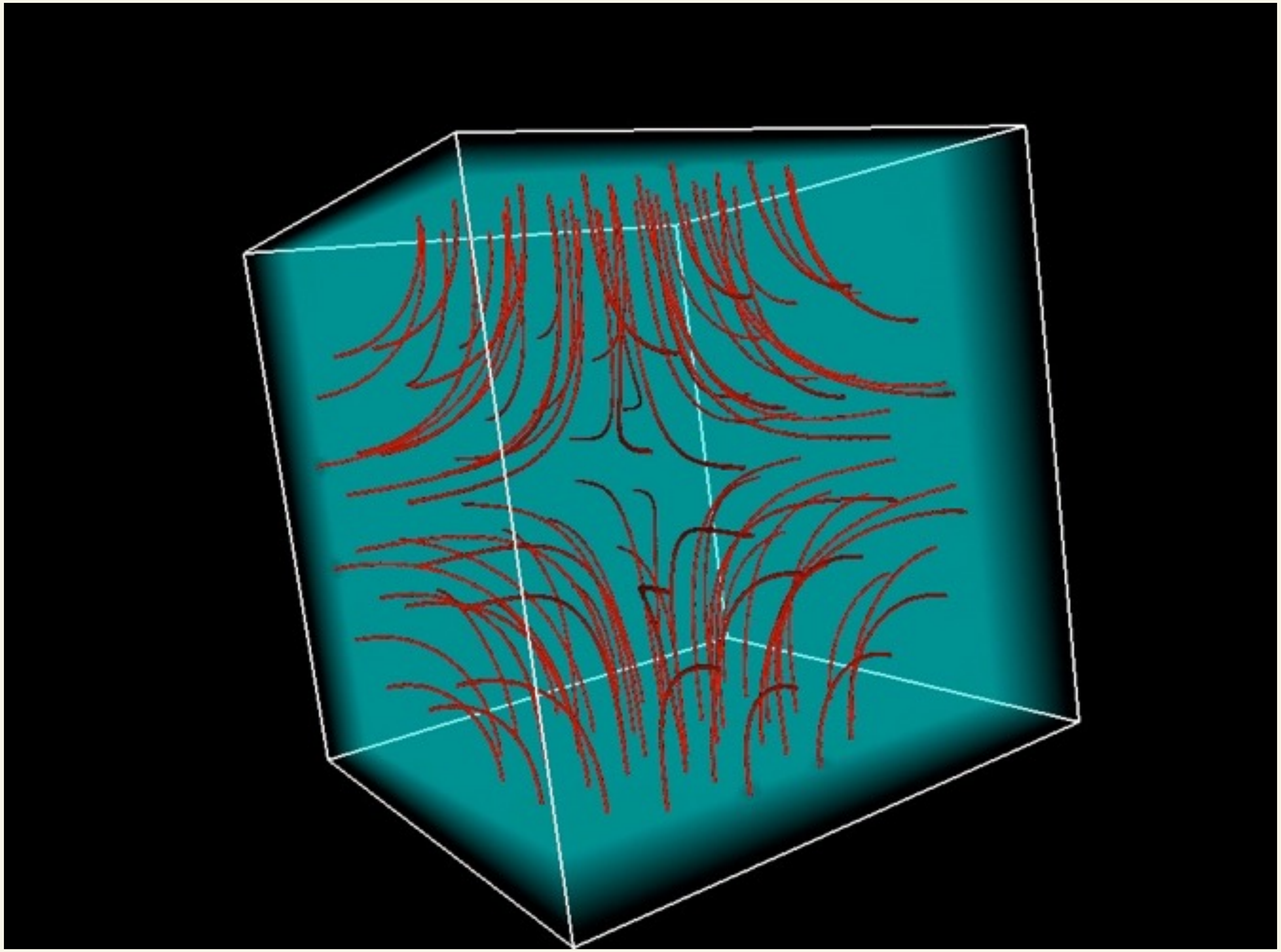
Condensate fraction
and
Power exponent

Dynamics of thermalization in the GPE

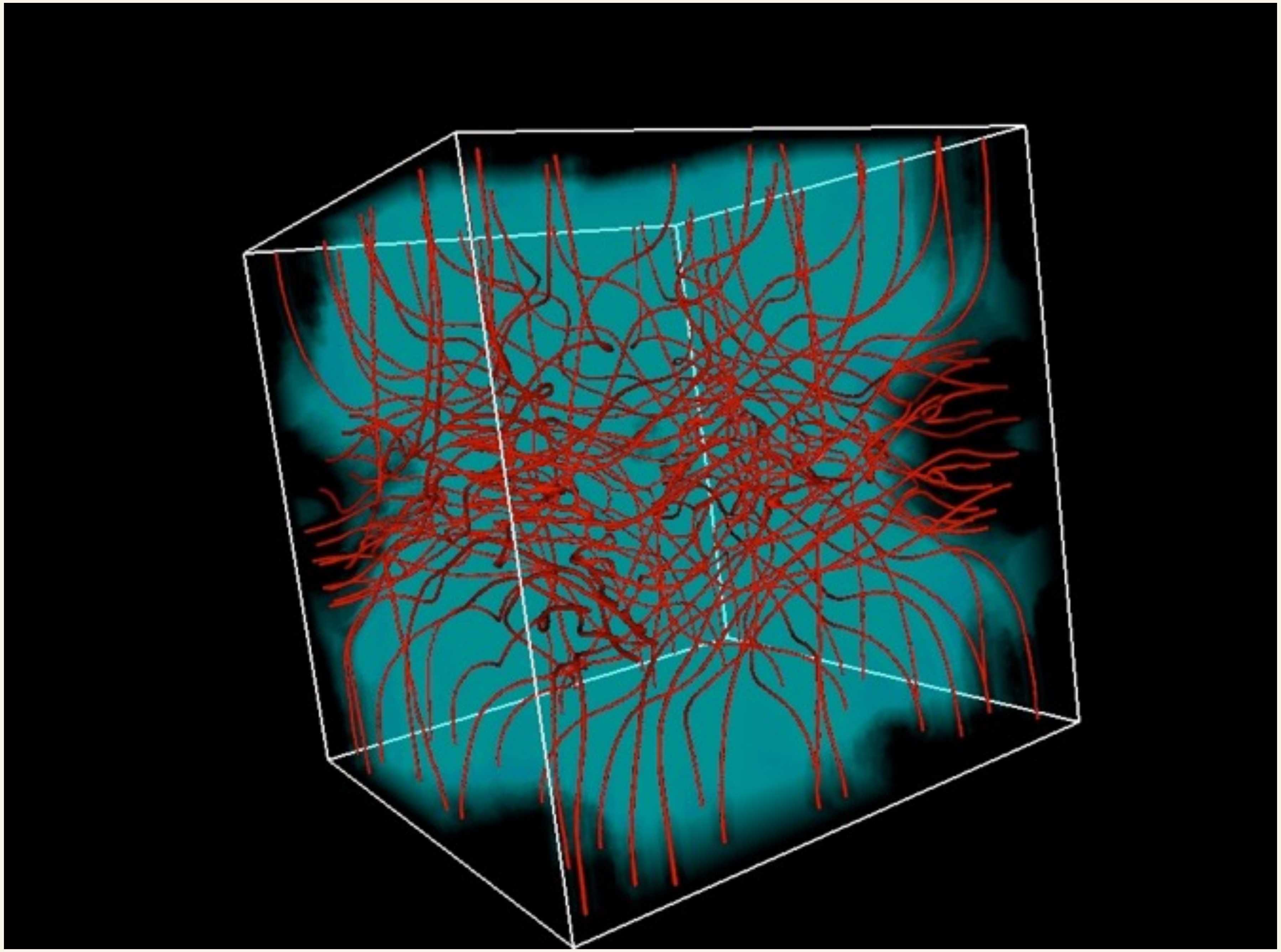
Kinetic energy spectrum



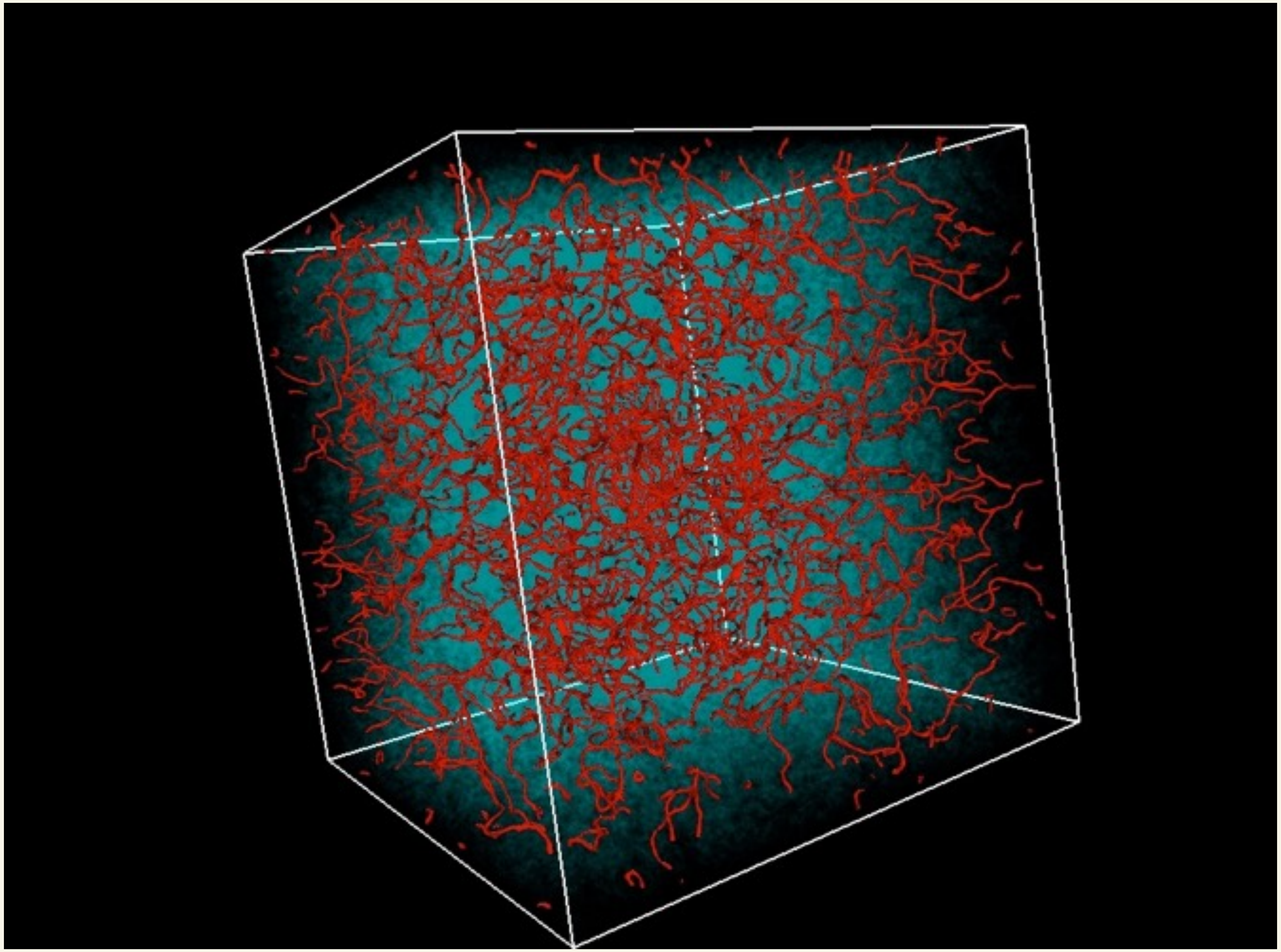
Taylor-Green vortex



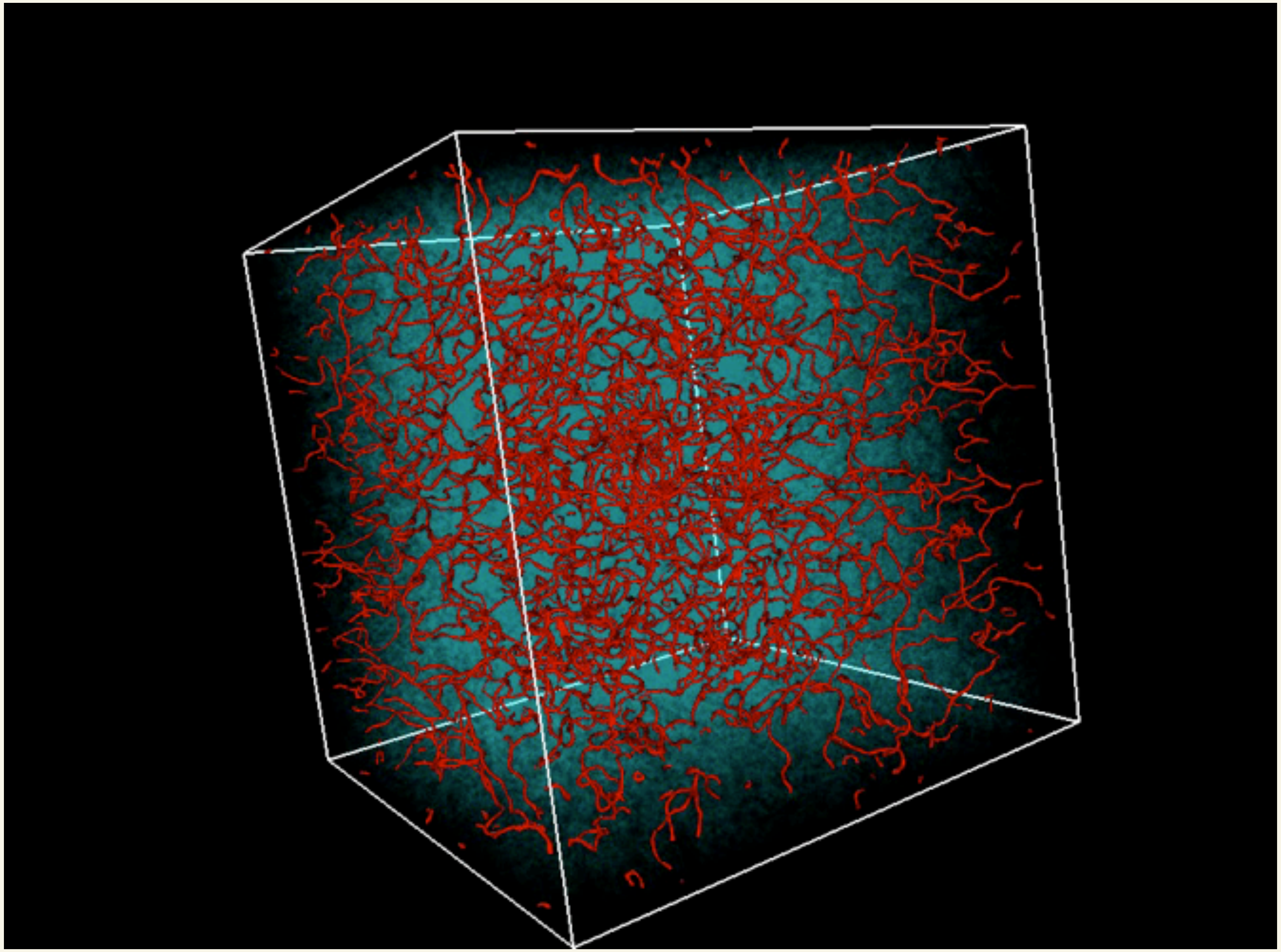
Taylor-Green vortex



Taylor-Green vortex



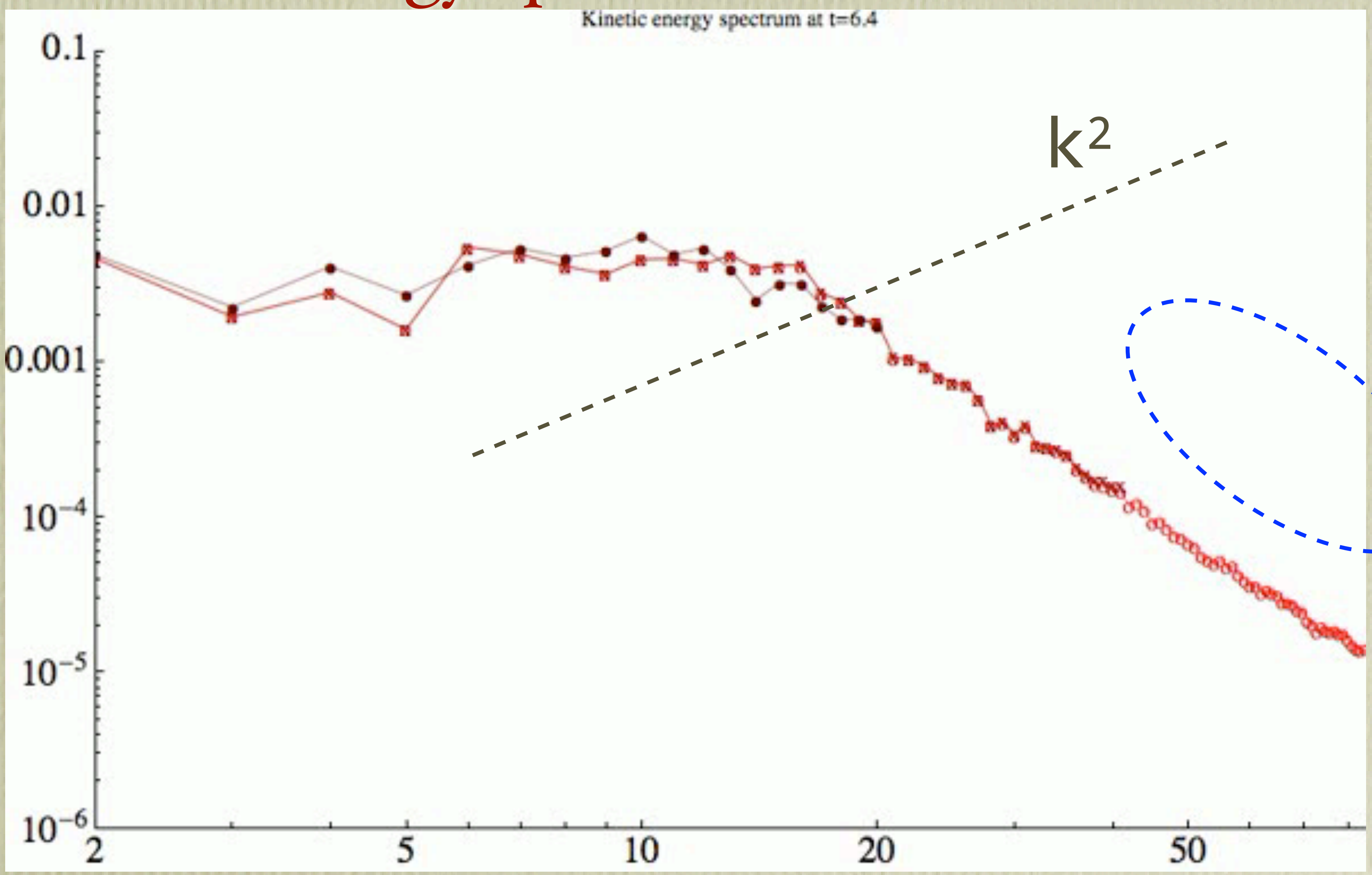
Taylor-Green vortex



Dispersive "bottleneck" for thermalization of waves

Variable ξk_{max} (ξ fixed, different resolutions)

Kinetic energy spectrum



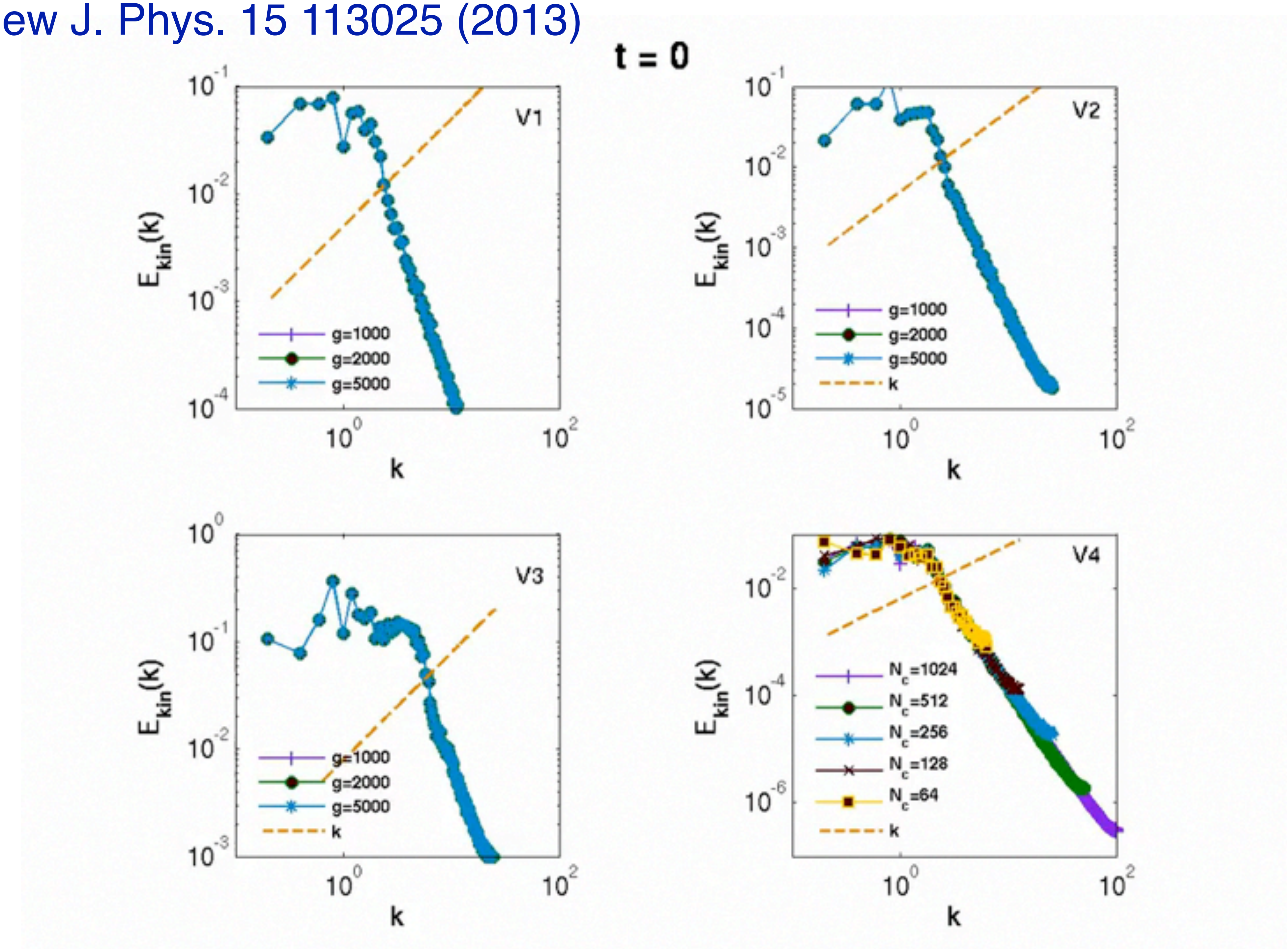
Partial thermalization starts

Spectral convergence

k_{max}-independent bottleneck

Self truncation in 2D

Vishwanath Shukla, Marc Brachet and Rahul Pandit
Turbulence in the two-dimensional Fourier-truncated Gross–Pitaevskii equation
New J. Phys. 15 113025 (2013)



[Top Left] k0 = 5Δk and σ = 2Δk, [Top Right] k0 = 15Δk and σ = 2Δk, [Bottom Left] k0 = 35Δk and σ = 5Δk, and [Bottom Right] different Nc.

Mutual friction and counterflow effects in Truncated Gross-Pitaevskii equation

PHYSICAL REVIEW B **83**, 132506 (2011)

Anomalous vortex-ring velocities induced by thermally excited Kelvin waves and counterflow effects in superfluids

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(Received 15 February 2011; published 21 April 2011)

Dynamical counterflow effects on vortex evolution under the truncated Gross-Pitaevskii equation are investigated. Standard longitudinal mutual-friction effects are produced and a dilatation of vortex rings is obtained at large counterflows. A strong temperature-dependent anomalous slowdown of vortex rings is observed and attributed to the presence of thermally excited Kelvin waves. This generic effect of finite-temperature superfluids is estimated using energy equipartition and orders of magnitude are given for weakly interacting Bose-Einstein condensates and superfluid ^{4}He. The relevance of thermally excited Kelvin waves is discussed in the context of quantum turbulence.

Grand-canonical Algorithm

$$\mathbb{P}_{\mathrm{st}}[\psi] = \frac{1}{\mathcal{Z}} e^{-\beta F} \qquad F = H - \mu N - \mathbf{W} \cdot \mathbf{P}$$

$$\hbar \frac{\partial A_{\mathbf{k}}}{\partial t} = -\frac{1}{V} \frac{\partial F}{\partial A^*_{\mathbf{k}}} + \sqrt{\frac{2\hbar}{V\beta}} \hat{\zeta}(\mathbf{k}, t)$$

$$\langle \zeta(\mathbf{x}, t) \zeta^*(\mathbf{x}', t') \rangle = \delta(t - t')\delta(\mathbf{x} - \mathbf{x}'),$$

$$\hbar \frac{\partial \psi}{\partial t} = \mathcal{P}_{\mathrm{G}}[\frac{\hbar^2}{2m} \nabla^2 \psi + \mu \psi - g \mathcal{P}_{\mathrm{G}}[|\psi|^2]\psi - i\hbar \mathbf{W} \cdot \nabla \psi] + \sqrt{\frac{2\hbar}{V\beta}} \mathcal{P}_{\mathrm{G}}[\zeta(\mathbf{x}, t)]$$

Stochastic Ginzburg-Landau equation
generates counterflow w

Finite-temperature vortex dynamics
Phenomenology

$\mathbf{V}_\mathrm{L}$: vortex line velocity

$$\mathbf{v}_\mathrm{L} = \mathbf{v}_\mathrm{sl} + \alpha \mathbf{s}' \times (\mathbf{v}_\mathrm{n} - \mathbf{v}_\mathrm{sl}) - \alpha' \mathbf{s}' \times [\mathbf{s}' \times (\mathbf{v}_\mathrm{n} - \mathbf{v}_\mathrm{sl})],$$

$$\boxed{\alpha = B\frac{\rho_n}{2\rho} \qquad \alpha' = B'\frac{\rho_n}{2\rho}}$$

$v_\mathrm{sl} = v_\mathrm{s} + u_i$: Local superfluid velocity

v_s : Superfluid velocity

u_i : Self-induced velocity

v_n : Normal velocity

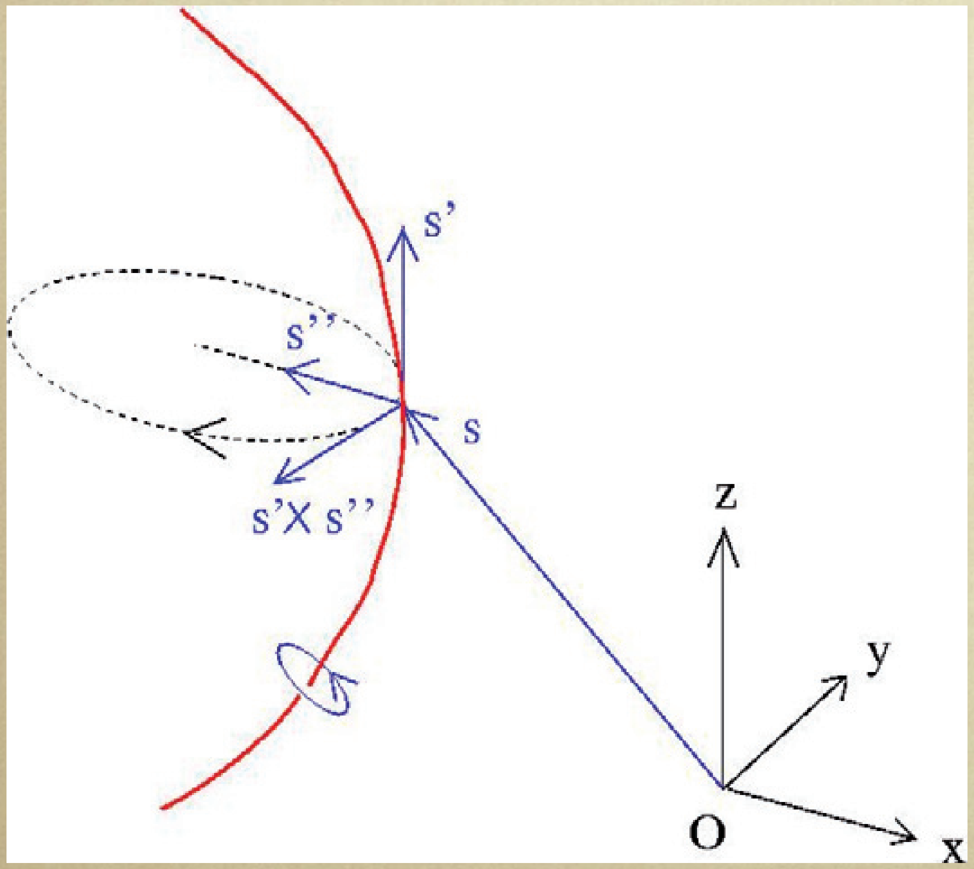


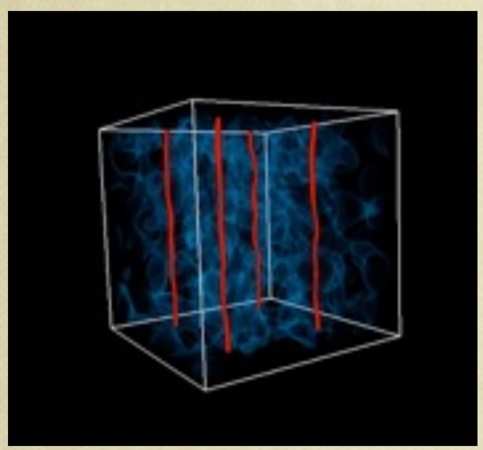
Fig. from C. F. Barenghi and R. J. Donnelly.
Fluid Dyn. Res. **41** (2009) 051401

Finite-temperature TGPE vortex dynamics

Counterflow effect on TGPE

Simplest configuration in a periodical system:

$$\psi_{\rm ini} \quad = \quad \psi_{\rm crystal} \quad \times \quad \psi_{\rm eq}$$



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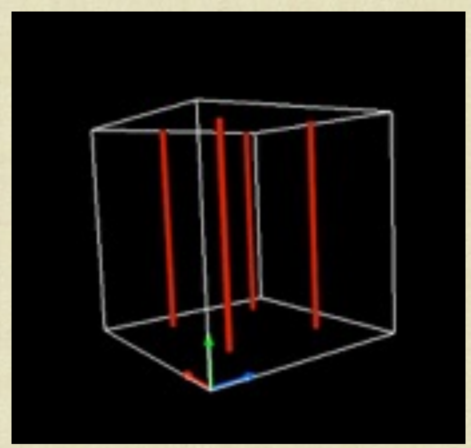

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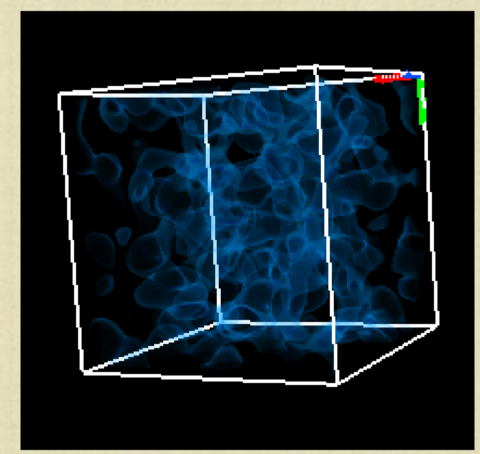
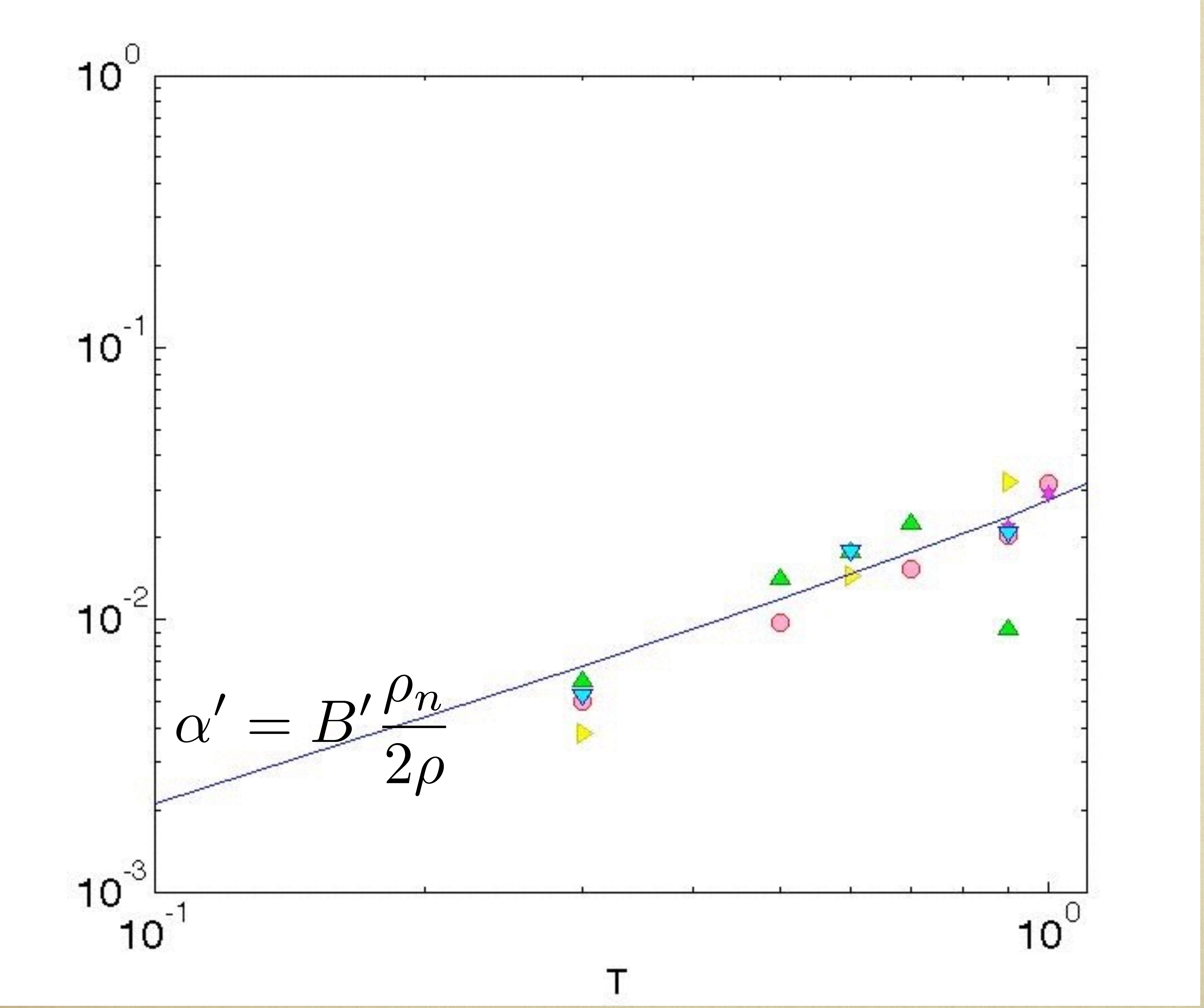


ξ :Vortex core size

d :inter-vortex distance

Limit $\xi/d \to 0$:isolated vortex.

Mutual friction



Different resolutions (ξ/d) and counterflow values

$$\alpha' = \frac{v_{\parallel}}{w}$$

ρ_n obtained by using Grand canonical algorithm

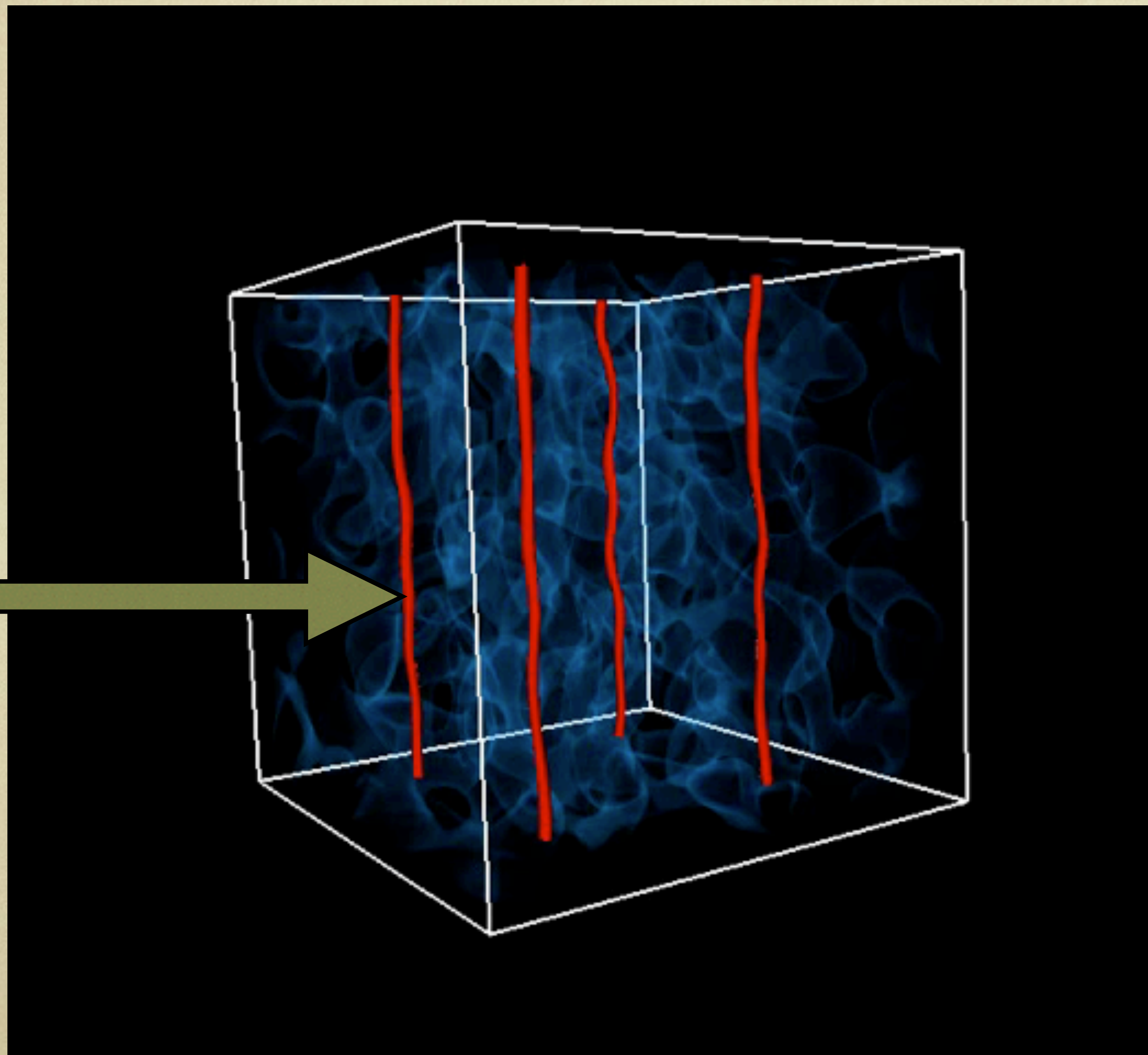
$$B' = 0.85$$

Simplest configuration

$$\mathbf{v}_{\mathrm{L}} = \mathbf{v}_{\mathrm{sl}} + \alpha \mathbf{s}' \times (\mathbf{v}_{\mathrm{n}} - \mathbf{v}_{\mathrm{sl}}) - \alpha' \mathbf{s}' \times [\mathbf{s}' \times (\mathbf{v}_{\mathrm{n}} - \mathbf{v}_{\mathrm{sl}})],$$

$$\alpha' = B' \frac{\rho_n}{2\rho} \qquad \alpha = B \frac{\rho_n}{2\rho}$$

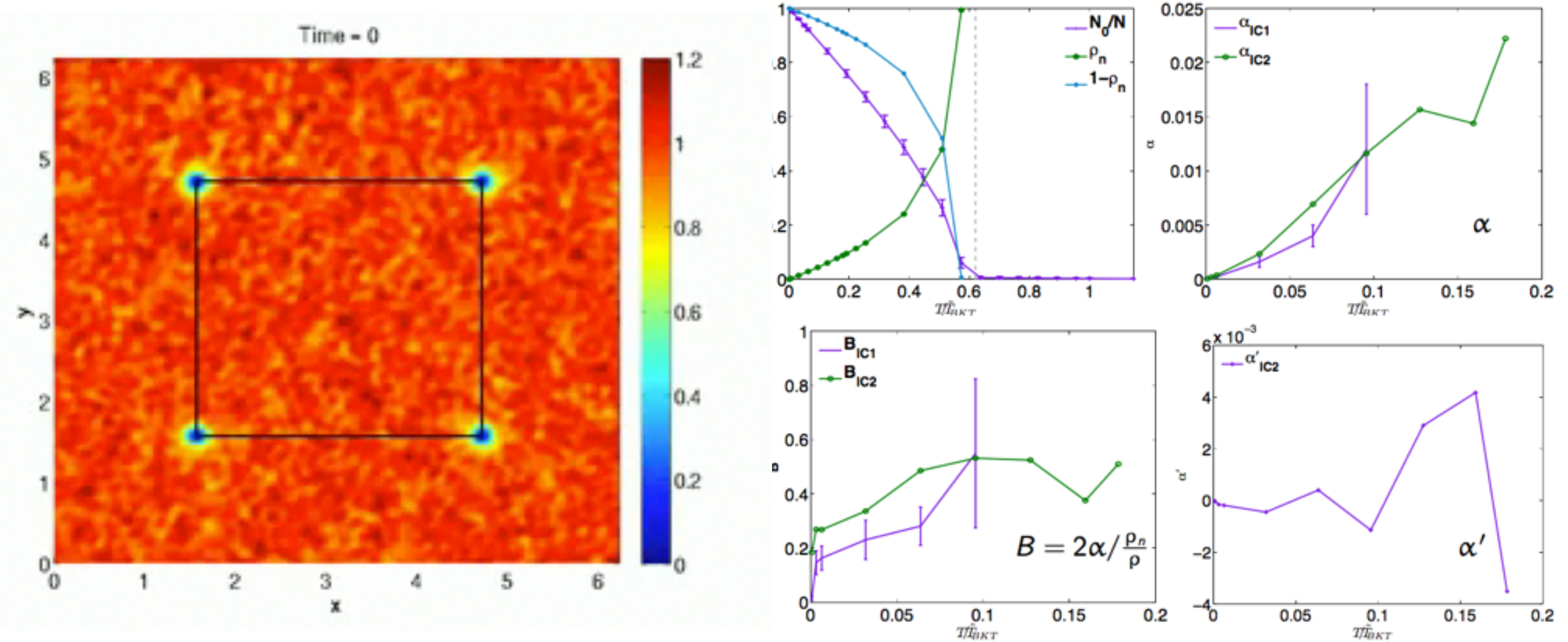
w



$$\alpha' = \frac{v_{\|}}{w}$$

2D equivalent is more complicated!

Temperature scan



$\tilde{T}_{\mathrm{BKT}}$: energy-entropy-argument based estimate of the BKT transition temperature.

Superfluid Mutual-friction Coefficients from Vortex Dynamics in the Two-dimensional Galerkin-truncated Gross-Pitaevskii Equation, Vishwanath Shukla, Marc Brachet, Rahul Pandit, http://arxiv.org/abs/1412.0706

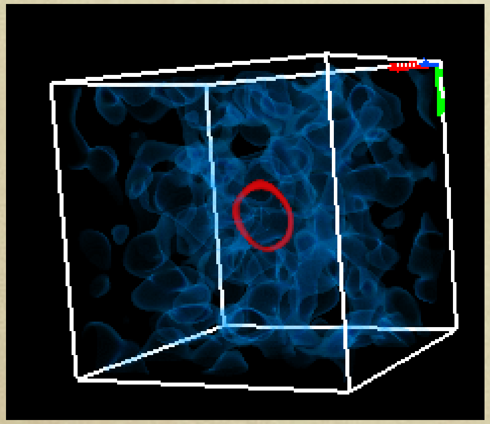
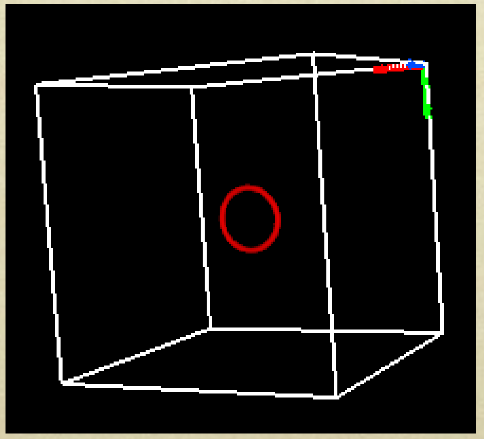
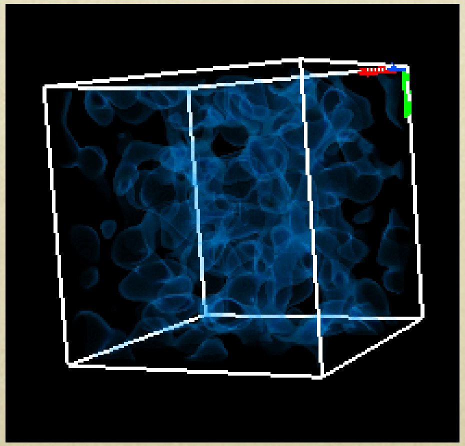
MUTUAL FRICTION AND COUNTER-FLOW EFFECTS ON RINGS

$$\mathbf{v}_\mathrm{L} = \mathbf{v}_\mathrm{sl} + \alpha \mathbf{s}' \times (\mathbf{v}_\mathrm{n} - \mathbf{v}_\mathrm{sl}) - \alpha' \mathbf{s}' \times [\mathbf{s}' \times (\mathbf{v}_\mathrm{n} - \mathbf{v}_\mathrm{sl})],$$

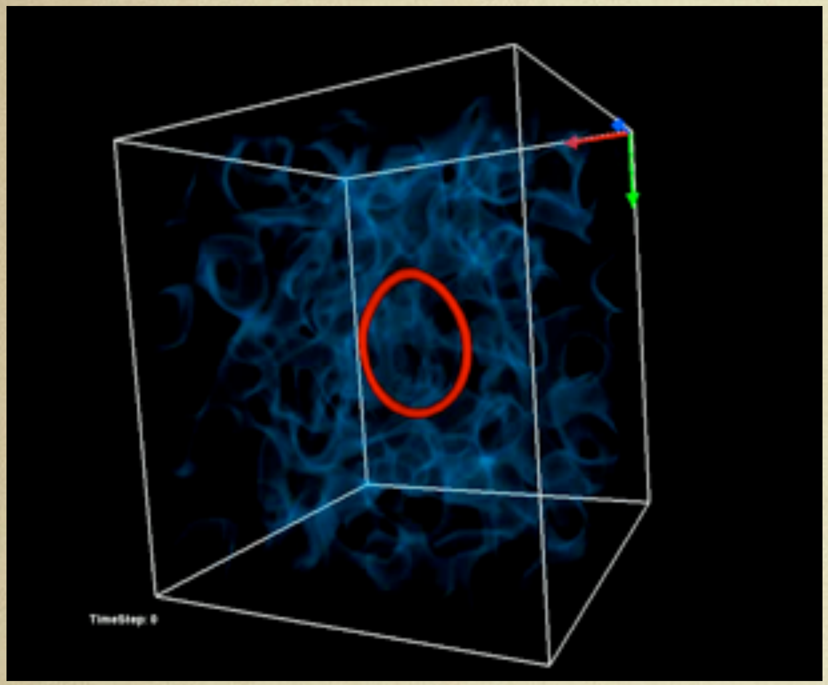
$$\dot{R} = -\alpha(u_i - w_z) \qquad w = v_n - v_s$$

$$v_\mathrm{L} = v_\mathrm{s} + (1 - \alpha')u_\mathrm{i} + \alpha' w_z,$$

$$\psi_\mathrm{ini} = \psi_\mathrm{ring} \times \psi_\mathrm{eq}$$

 $=$  $\times$ 

MUTUAL FRICTION AND COUNTER-FLOW EFFECTS



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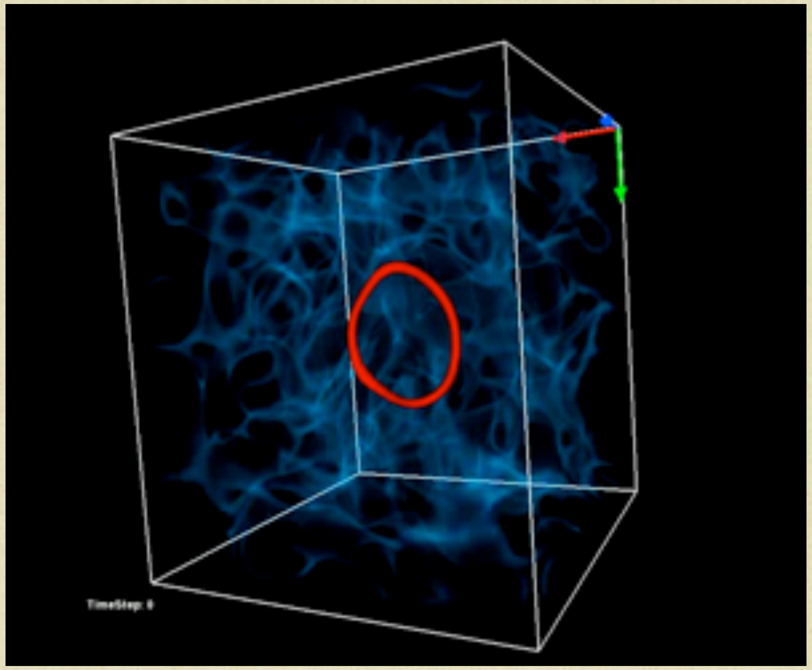

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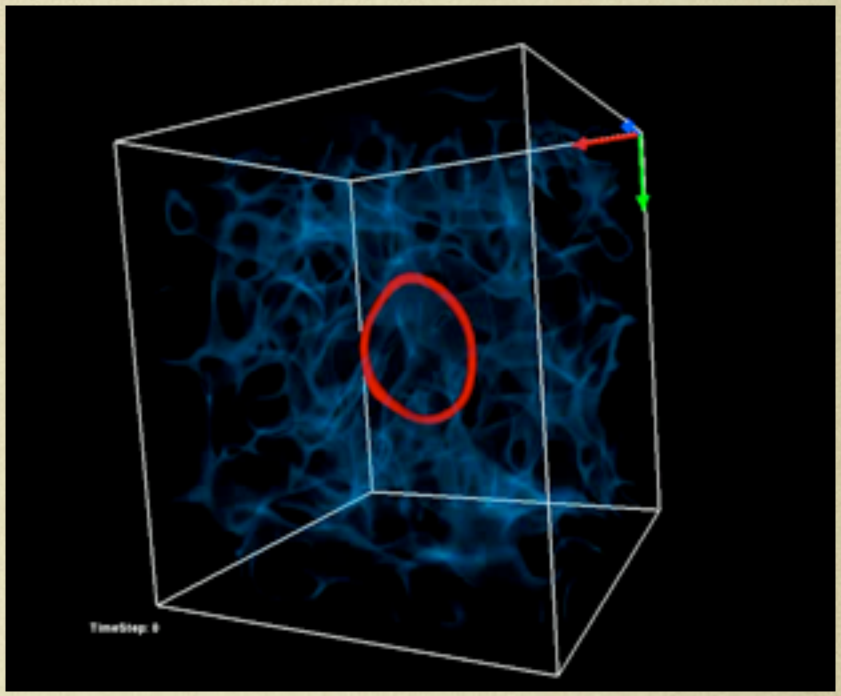

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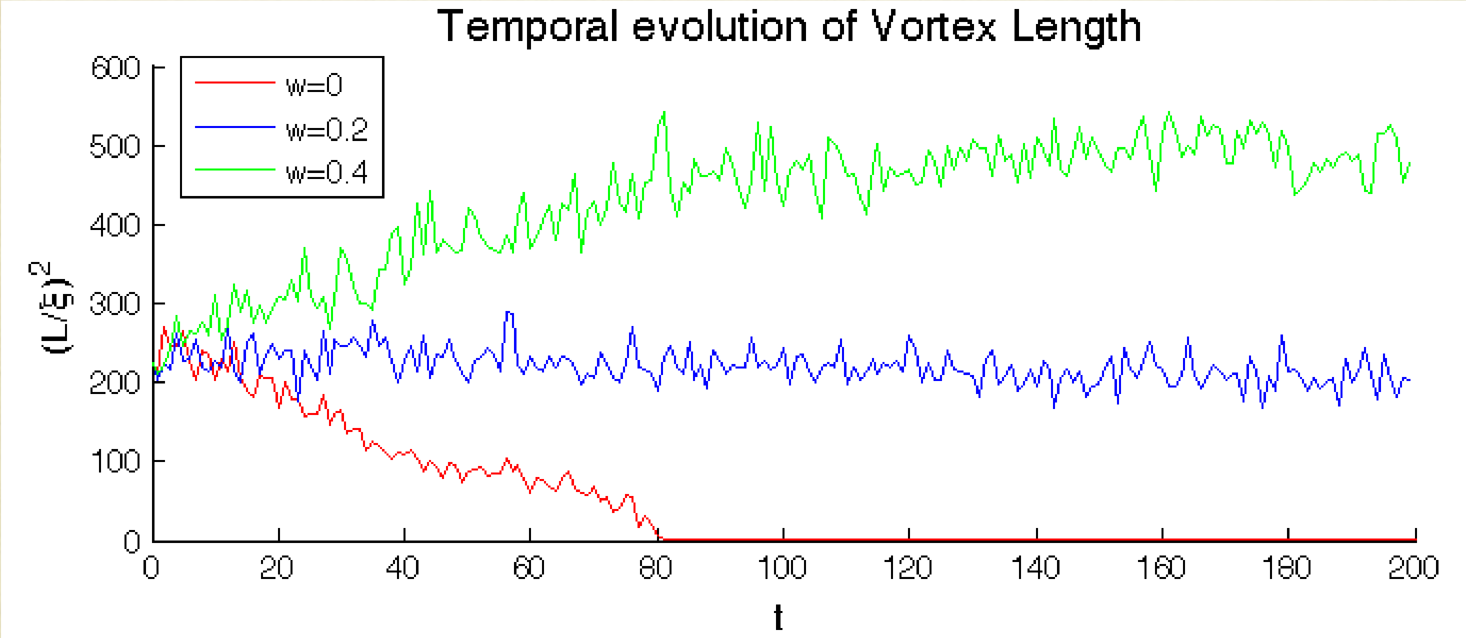
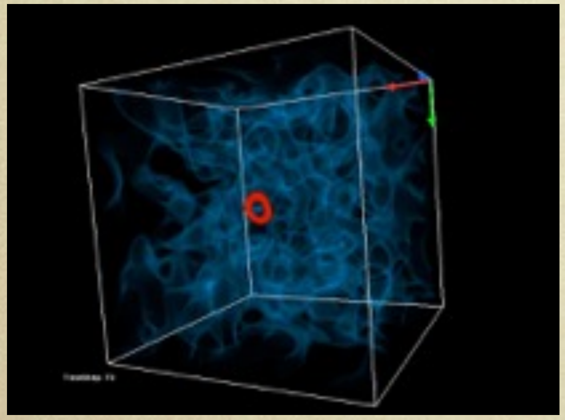
MUTUAL FRICTION AND COUNTER-FLOW EFFECTS



N.G. BERLOFF AND J. YOUD
PRL VOL. 99, 145301 (2007)



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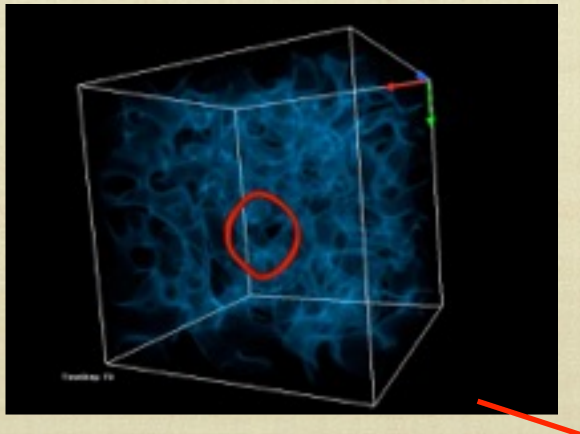

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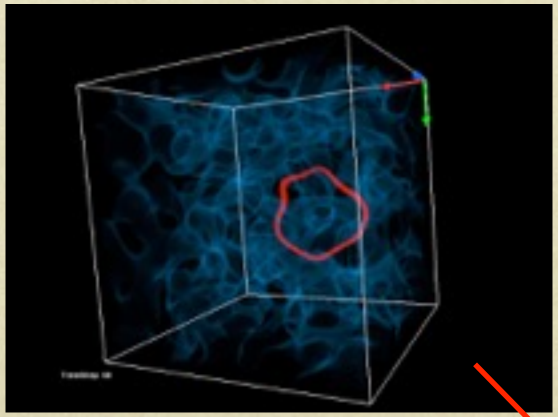

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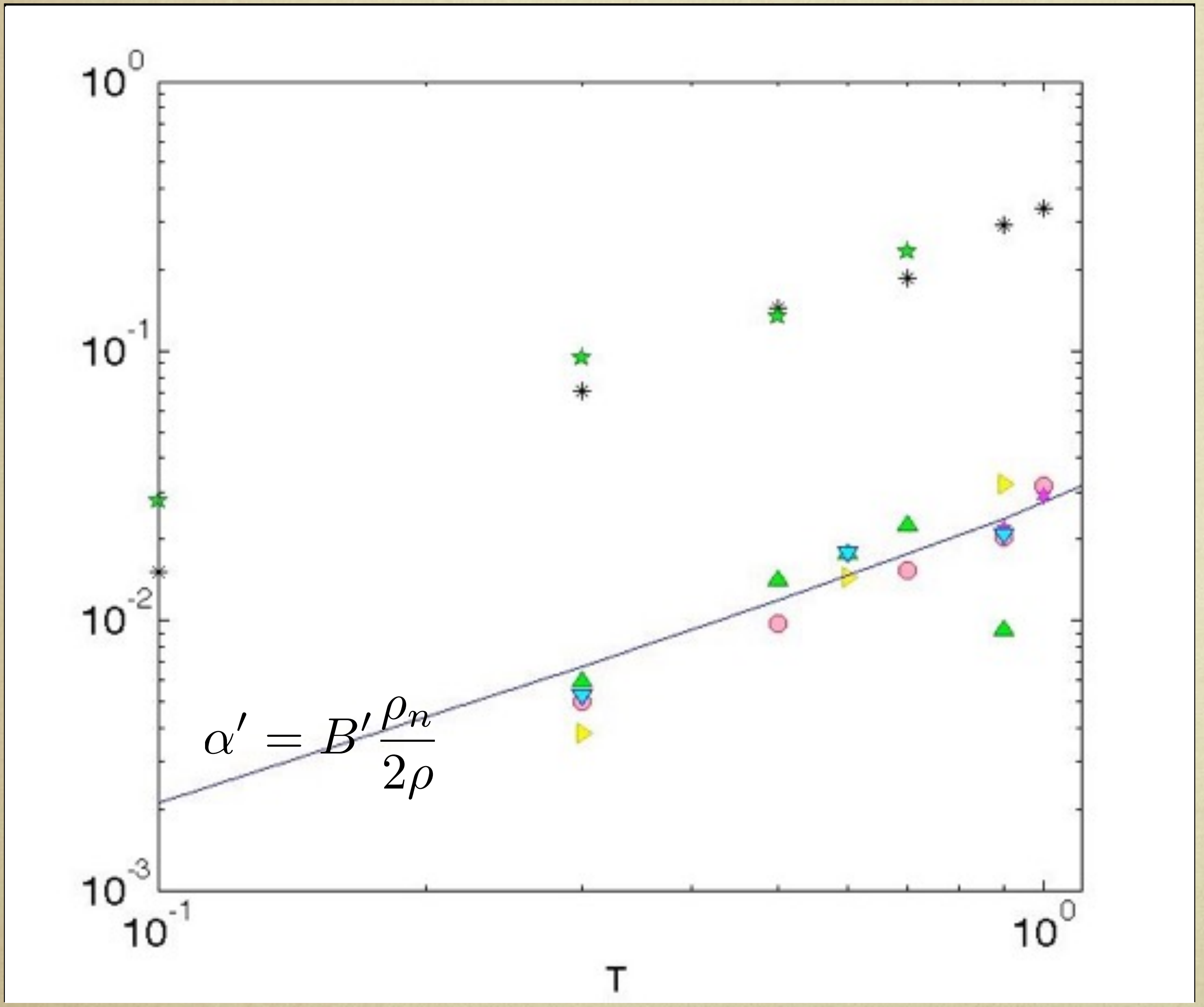
NEW EFFECT

Translational velocity slowdown

$$\mathbf{v}_{\mathrm{L}} = \mathbf{v}_{\mathrm{sl}} + \alpha \mathbf{s}' \times (\mathbf{v}_{\mathrm{n}} - \mathbf{v}_{\mathrm{sl}}) - \alpha' \mathbf{s}' \times [\mathbf{s}' \times (\mathbf{v}_{\mathrm{n}} - \mathbf{v}_{\mathrm{sl}})],$$



w=0

$$\frac{\Delta v_L}{u_i} = \frac{u_i - v_L}{u_i} = \alpha'$$

Slowdown is one order of magnitude above standard phenomenology!!

Kelvin waves & vortex rings

C.F. Barenghi, R. Hänninen and M. Tsubota

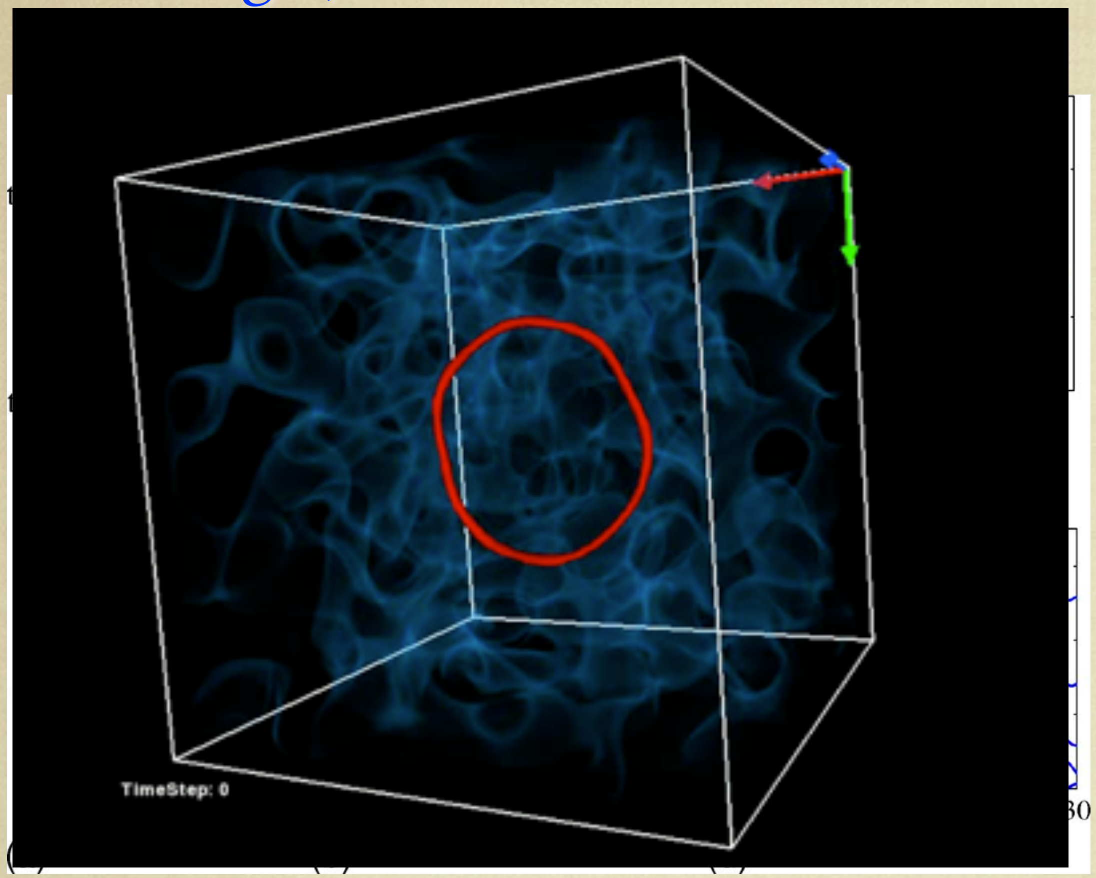


FIG. 1. (Color online) Snapshots of the vortex ring of radius $R=0.1$ cm perturbed by $N=10$ Kelvin waves of various amplitude A taken during the motion of the vortex. In the left panel (a) the amplitude of the Kelvin waves is small, $A/R=0.05$, but the perturbed vortex ring (red color) already moves slower than the unperturbed vortex (blue color). In the center panel (b) the Kelvin waves have large amplitude, $A/R=0.35$, and the perturbed vortex ring moves backwards (negative z direction) on average. The top right panel (c) shows the top (xy) view of the large amplitude vortex at $t=0$ s (blue) and $t=26$ s (red, outermost). For comparison, a non-disturbed vortex is shown with dashed line (green). The lower right panel (d) gives the averaged location of the ring as a function of time. From top to bottom the curves correspond to $A/R=0.0, 0.05, 0.10, \ldots, 0.35$.

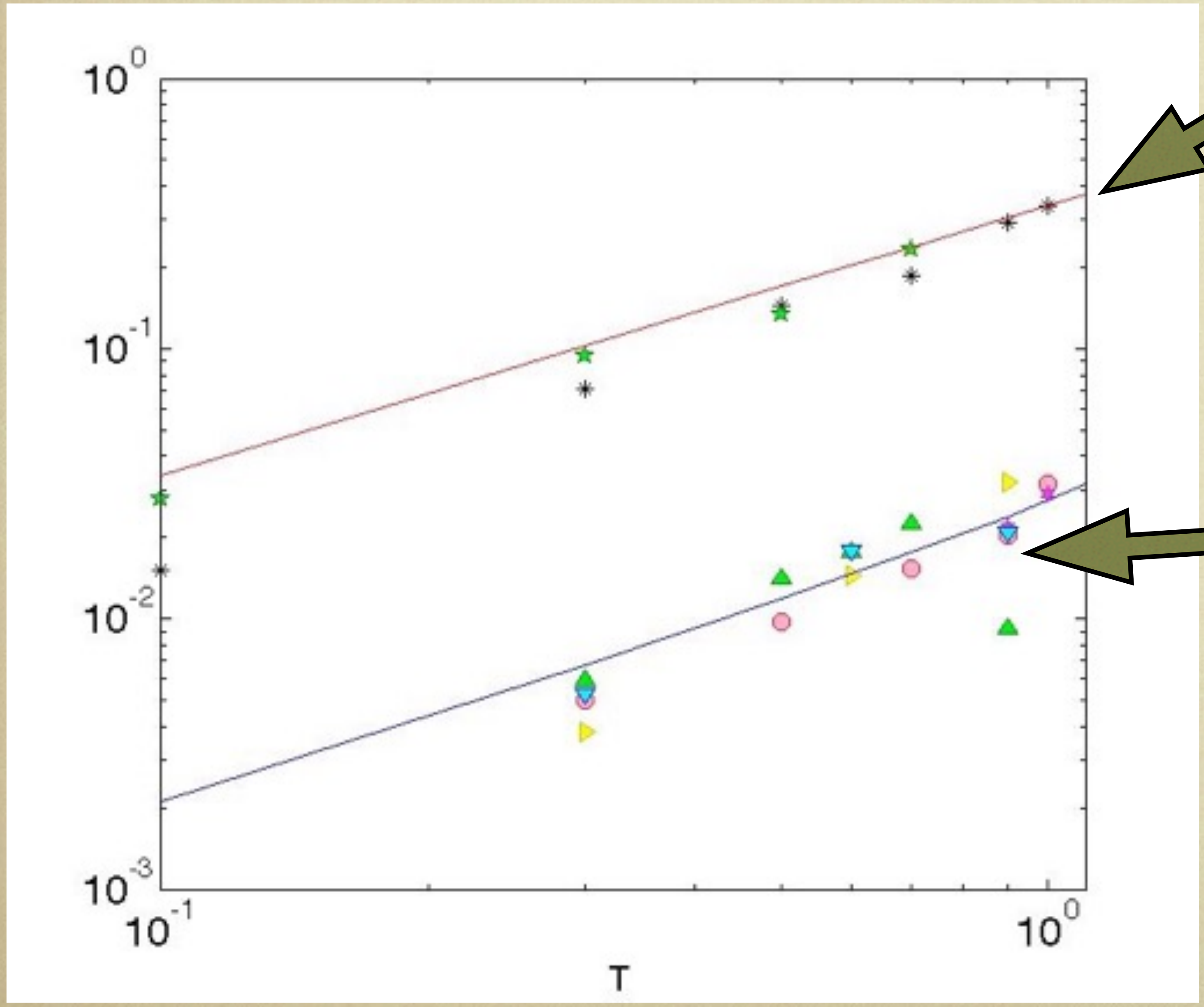
- Kelvin waves induce anomalous translational velocity.
  - L Kiknadze and Y Mamaladze, JLTP,126(1-2): 321–326, 2002.
  - C. F Barenghi, R. Hanninen and M.Tsubota. PRE, 74(4):046303, 2006

$$\frac{\Delta v_L}{u_i} \sim \frac{A^2}{R^2} n^2$$

- Equipartition of energy between waves and thermal bath

Thermally induced Kelvin waves effect stronger than mutual friction!

$$\frac{\Delta v_L}{u_i} \equiv \frac{u_i - v_{\mathrm{a}}}{v_{\mathrm{a}}} \approx \frac{2k_B T m^2}{\pi\rho\xi\hbar^2}\frac{1}{\log\frac{8R}{\xi} - a}$$



Kelvin waves effect

Mutual friction effect

Mutual friction and counterflow Summary

- Mutual Friction and counter-flow effects are present in TGPE dynamics
- TGPE description naturally includes thermal fluctuations
- Thermally excited Kelvin waves induce slowdown of vortex ring velocity

Particles in the GPE

Galerkin-truncated GP equation

$$i\frac{\partial\psi(\mathbf{x},t)}{\partial t}=\mathcal{P}_G\left[\left(-\alpha_0\nabla^2+g\mathcal{P}_G[|\psi|^2]-\mu+\sum_{\mathrm{i=1}}^{\mathcal{N}_\mathrm{o}}V_{\mathcal{P}}(\mathbf{r}-\mathbf{q}_i)\right)\psi(\mathbf{x},t)\right];$$

Newtonian dynamics for the particle

$$m_\mathrm{o}\ddot{\mathbf{q}}_i=\mathbf{f}_{\mathrm{o},i}+\mathbf{F}_{\mathrm{ext,i}};$$

Force exerted by the fluid on the particle

$$\mathbf{f}_{\mathrm{o},i}=2\alpha_0\int_{\mathcal{A}}|\psi|^2\nabla V_{\mathcal{P}}(\mathbf{r}-\mathbf{q}_i)d^2x.$$

Vishwanath Shukla, Particles and Fields in Superfluid Turbulence: Numerical and Theoretical Studies, PhD Thesis, Indian Institute of Science, Bangalore, India, 2015.

Particle potential:

Two particle Collision

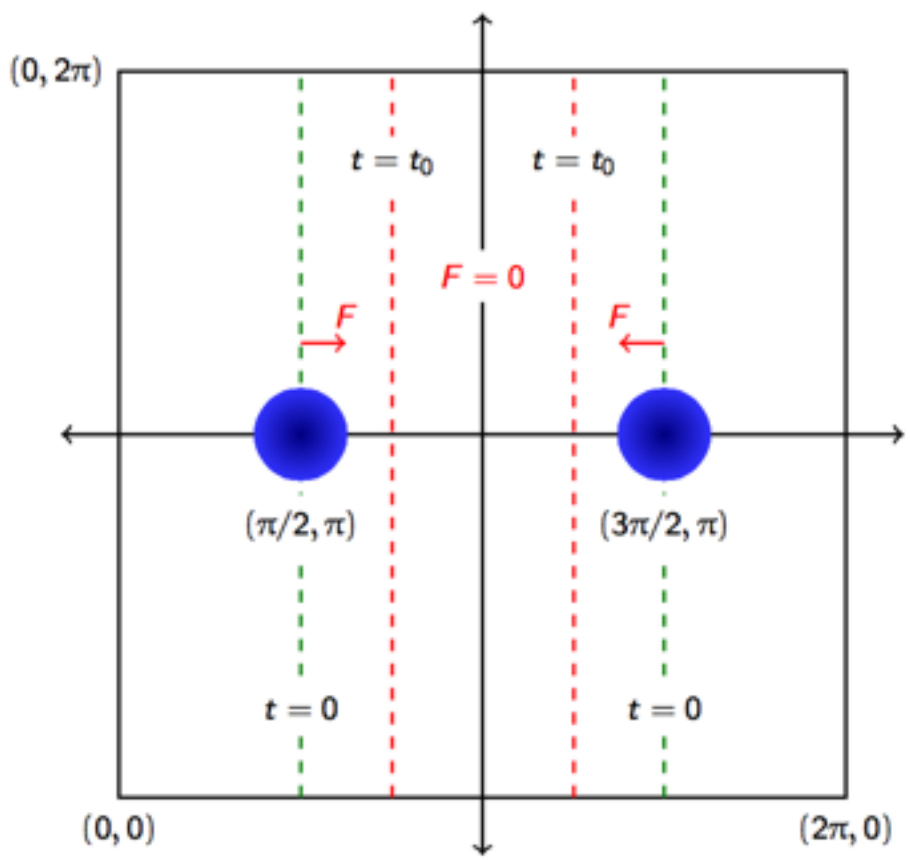
$$V_{\mathcal{P}}(r) = V_{\rm o} \exp(-r^2/(2d_{\rm p}^2);$$

Short-range repulsion energy for the many-particle case:

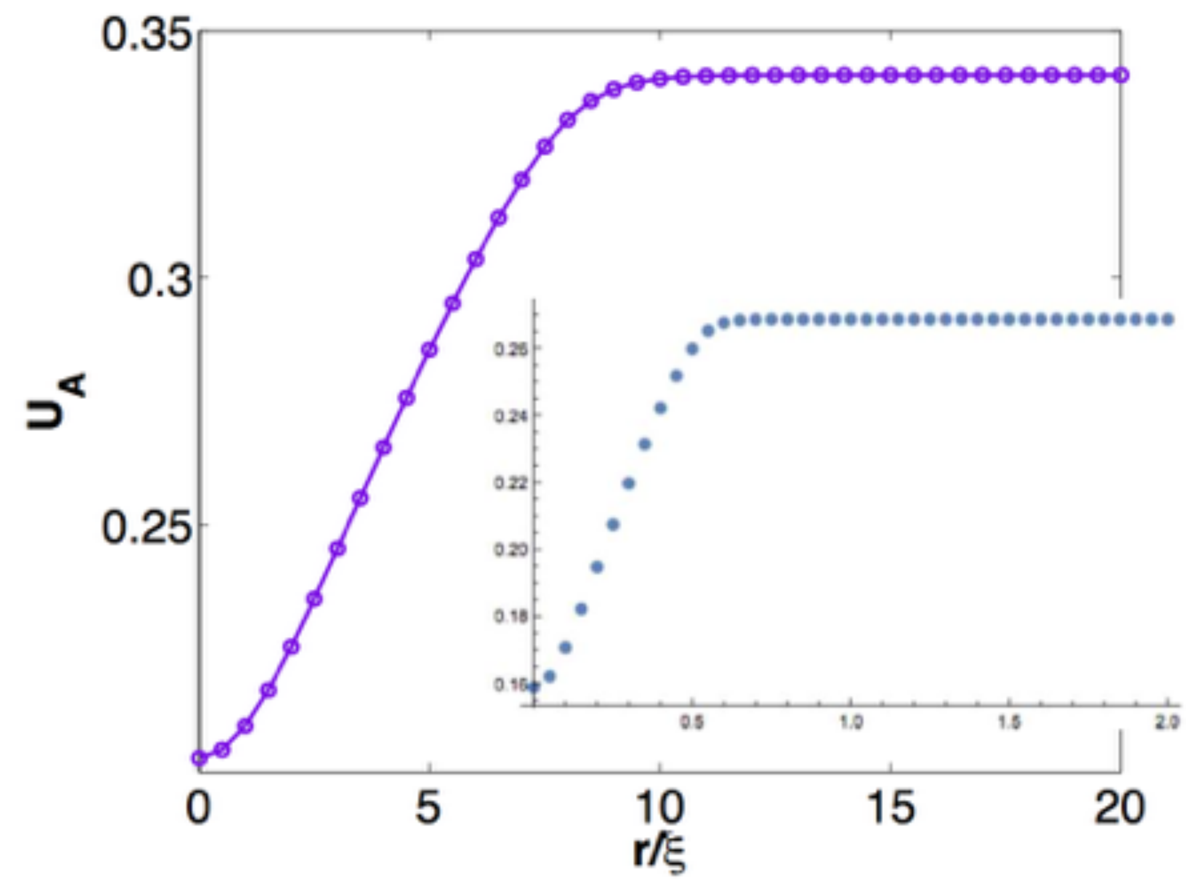
$$E_{\rm SR} = \frac{1}{2}\sum_{\rm i,j,i\neq j}^{\mathcal{N}_{\rm o},\mathcal{N}_{\rm o}} \mathcal{U}_{{\rm SR},i,j};$$

$$\mathcal{U}_{\rm SR} = \frac{\Delta E r_{\rm SR}^{12}}{r^{12}};$$

Head-on collision

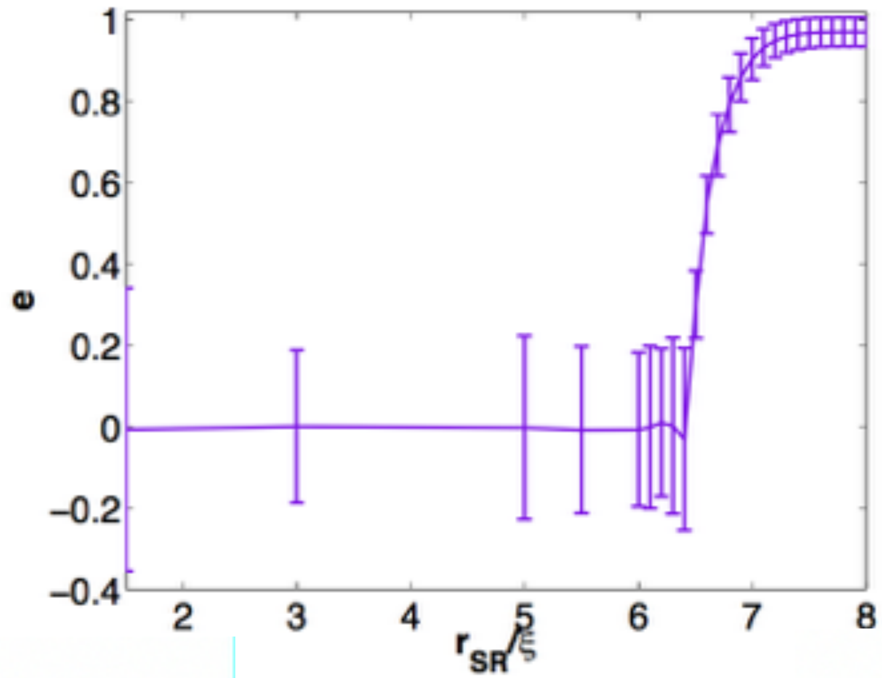


Attractive Potential



Sticking transition

Coefficient of restitution: $e = \frac{v_2 - v_1}{u_1 - u_2}$



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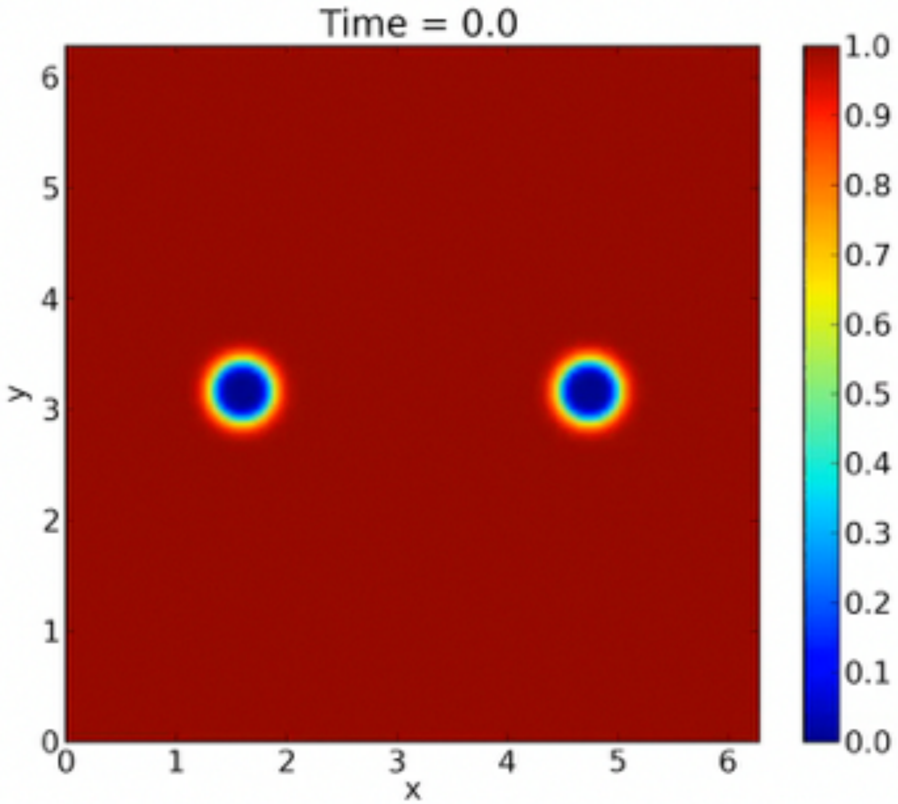

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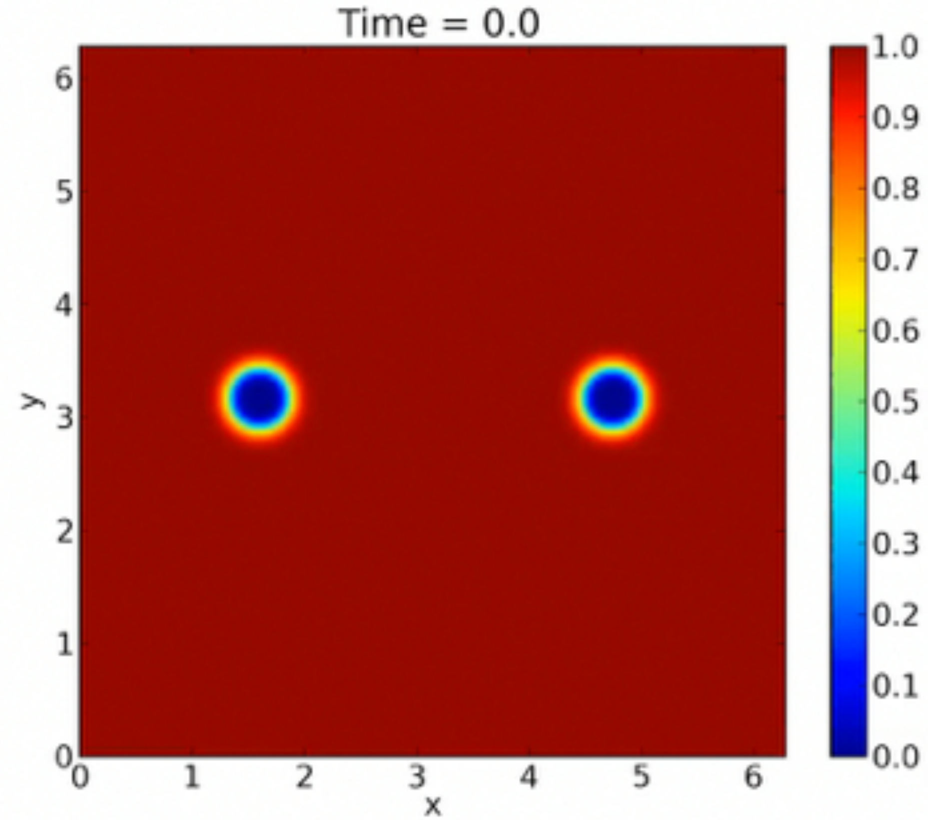


Vishwanath Shukla, Marc Brachet and Rahul Pandit, manuscript in preparation, 2015.

Conclusion

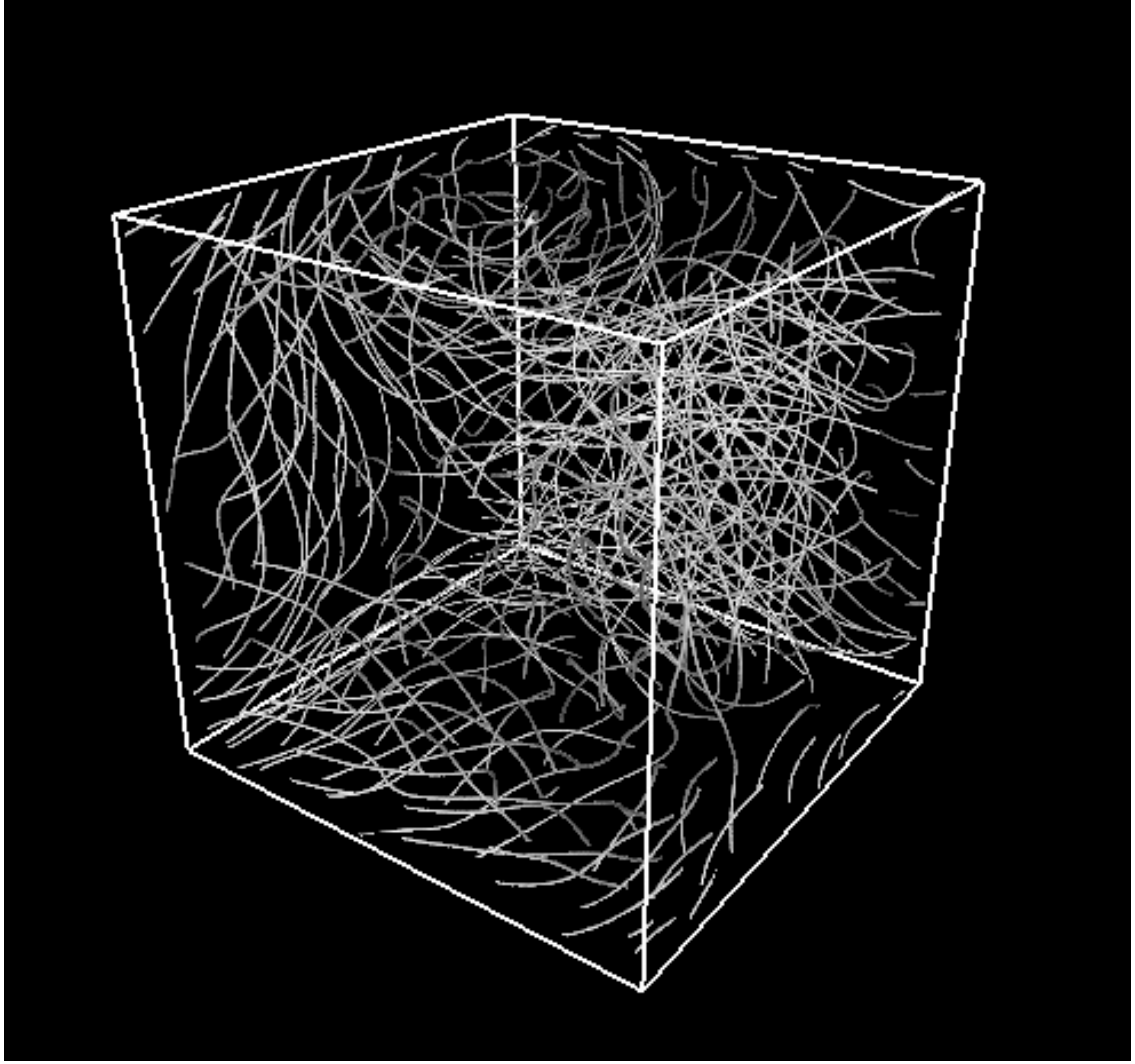
- Turbulence is still an open problem [physically and mathematically]
- It is, perhaps, the most important unsolved problem of nonlinear science
- Analogy between classical viscous and coflow superfluid turbulence is challenging
- Navier-Stokes versus Gross-Pitaeveskii

Conclusion

- New experiments are under construction
- SHREK will study quantum versus classical regimes
- Computer power is still increasing exponentially
- New ideas are needed...

Conclusion

- Perhaps turbulence is simpler to resolve starting from GPE rather than Navier-Stokes?
- Statistical mechanics of interacting and reconnecting vortex lines?
- Anyway, large-scale TGPE computations are needed to study finite-T effects



Thank you!