Turbulence in astrophysical plasmas at sub-ion scales: electron and Hall MHD



Sébastien Galtier



Advances in Geophysical and Astrophysical Turbulence July 25 - August 5, 2016

Laboratoire de Physique des Plasmas École polytechnique, Palaiseau





Astrophysical turbulence at sub-ion scales

(It is not an exhaustive presentation)

• PART 0: MHD $(\ell > d_i)$ \rightarrow see lecture given by H. Politano

- PART I: Electron MHD $(d_i > \ell > d_e)$
- PART II: Hall MHD $(\ell > d_e)$

$$d_e = c/\omega_{pe} \simeq 5.3 \times 10^6 \, n_e^{-1/2} \, \mathrm{m}$$

Inertial lengths:

$$d_i = c/\omega_{pi} \simeq 2.3 \times 10^8 \, n_i^{-1/2} \, \mathrm{m}$$



If you like MHD...

Gattier | INTRODUCTION TO MODERN MAGNETOHYDRODYNAMICS

- 1. Introduction
- 2. Magnetohydrodynamics
- 3. Conservation laws
- 4. Magnetohydrodynamic waves
- 5. Dynamos
- 6. Discontinuities and shocks
- 7. Magnetic reconnection
- 8. Static equilibrium
- 9. Linear perturbation analysis
- 10. Study of MHD instabilities
- 11. Hydrodynamic turbulence
- 12. MHD turbulence
- 13. Advances in turbulence

+ exercices & solutions

Cover illustration-TBC

INTRODUCTION TO MODERN MAGNETOHYDRODYNAMICS

Sébastien Galtier



Will appear in September

Introduction



Solar wind turbulence

→ see lecture given by O. Alexandrova

Power laws at MHD scales & sub-ion scales



[Kiyani et al., Phil. Trans. R. Soc. A, 2015]

Solar wind turbulence

→ see lecture given by O. Alexandrova

What is the fate of ε ?



[Kiyani et al., Phil. Trans. R. Soc. A, 2015]

Physics of plasmas

To completely describe the state of a plasma one needs to write down all the particle locations and velocities, and describe the E–B fields...

- <u>Kinetic models</u>: particle velocity distribution $f_{i,e}(v,x)$ at each point
 - \rightarrow 6 dimensions !

 $f_e(\mathbf{v}, \mathbf{x_0})$ in the solar wind (Helios, 1 AU) [Pilipp et al., JGR, 1987]



- <u>Fluid models</u>: averaged quantities $\mathbf{u}_{i,e}(\mathbf{x}), \rho_{i,e}(\mathbf{x}), \dots$
 - → multi- and bi-fluids, <u>Hall MHD</u> and MHD
 - → Landau-fluids (see lecture given by T. Passot)

Turbulence

3D incompressible HD turbulence is a difficult subject but

- There are some rigorous results/theories: Kolmogorov laws; weak turbulence
- Direct Numerical Simulations may highlight new properties

Will it be possible one day to understand 6D plasma turbulence ?

✓ There is no strong result in kinetic turbulence

Kinetic simulations are still limited (4D, $f(\mathbf{v}, \mathbf{x}), m_i/m_e$)



[Karimabadi et al., PoP, 2013]

FLUID

PLASMA

 \rightarrow Fluid models are useful to investigate plasma turbulence

Generalized Ohm's law



Hall magnetohydrodynamics

$$\begin{split} & \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \qquad (2.19) \\ \rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla P + \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} + \tilde{\nu} \Delta \mathbf{u} + \frac{\tilde{\nu}}{3} \nabla (\nabla \cdot \mathbf{u}), \qquad (2.20) \\ & \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) - \nabla \times \left(\frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{\mu_0 n e} \right) + \eta \Delta \mathbf{B}, \qquad (2.21) \\ & \nabla \cdot \mathbf{B} = 0. \qquad (2.22) \end{split}$$

$$\tilde{\nu} \text{ is the dynamic viscosity} \qquad \text{Hall effect} \\ d_i = c/\omega_{pi} \\ P_* \equiv P/\rho_0 + b^2/2 \\ \mathbf{b} \equiv \mathbf{B}/\sqrt{\mu_0 \rho_0} \qquad \nabla \cdot \mathbf{u} = 0, \qquad (2.30) \\ & \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P_* + \mathbf{b} \cdot \nabla \mathbf{b} + \nu \Delta \mathbf{u}, \qquad (2.31) \\ & \frac{\partial \mathbf{b}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{b} = \mathbf{b} \cdot \nabla \mathbf{u} - d_0 \nabla \times ((\nabla \times \mathbf{b}) \times \mathbf{b}) + \eta \Delta \mathbf{b}, \qquad (2.32) \\ & \nabla \cdot \mathbf{b} = 0, \qquad (2.33) \end{split}$$

Hall magnetohydrodynamics

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \qquad (2.19)$$

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla P + \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} + \tilde{\nu} \Delta \mathbf{u} + \frac{\tilde{\nu}}{3} \nabla (\nabla \cdot \mathbf{u}), \quad (2.20)$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) - \nabla \times \left(\frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{\mu_0 n e} \right) + \eta \Delta \mathbf{B}, \qquad (2.21)$$

$$\nabla \cdot \mathbf{B} = 0. \qquad (2.22)$$

 ${\bf b} \equiv {\bf B}/\sqrt{\mu_0\rho_0}$

$$\left|\frac{\nabla \times \left(\frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{\mu_0 n e}\right)}{\partial \mathbf{B} / \partial t}\right| \sim \left|\frac{\mathbf{B} / \ell^2}{\mu_0 n e / \tau}\right| \sim \frac{\tau B}{\mu_0 n e \, \ell^2} \sim \frac{d_i}{\ell} \frac{\tau b}{\ell} \sim \mathbf{1} \implies \tau \sim \frac{\ell}{d_i} \frac{\ell}{b} \sim \frac{\ell}{d_i} \tau_b$$

Application of Hall MHD



Incompressible Hall MHD

$$\begin{aligned} &\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P_* + \mathbf{b} \cdot \nabla \mathbf{b} + \nu \Delta \mathbf{u} \,, \\ &\frac{\partial \mathbf{b}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{b} = \mathbf{b} \cdot \nabla \mathbf{u} - d_i \nabla \times \left((\nabla \times \mathbf{b}) \times \mathbf{b} \right) + \eta \Delta \mathbf{b} \,, \end{aligned}$$

Three inviscid/ideal invariants:

$$E = \frac{1}{2} \int (\mathbf{u}^2 + \mathbf{b}^2) \, d\mathcal{V} \qquad \text{Total energy}$$

$$H_m = \frac{1}{2} \int \mathbf{a} \cdot \mathbf{b} \, d\mathcal{V} \qquad \text{Magnetic helicity}$$

$$H_G = \frac{1}{2} \int (\mathbf{a} + d_i \mathbf{u}) \cdot (\mathbf{b} + d_i \nabla \times \mathbf{u}) \, d\mathcal{V} \qquad \text{Generalized helicity}$$

$$[Turner, IEEE, 1986]$$

n₂ S₂

Three Reynolds numbers: $R_e = u_\ell \ell / v$; $R_m = u_\ell \ell / \eta$; $R_{Hall} = d_i b_\ell / \eta$ \rightarrow Hyper-resistivity is useful !

Incompressible Hall MHD waves



PART I: Electron MHD turbulence

$$\begin{split} & \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P_* + \mathbf{b} \cdot \nabla \mathbf{b} + \nu \Delta \mathbf{u} \,, \\ & \frac{\partial \mathbf{b}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{b} = \mathbf{b} \cdot \nabla \mathbf{u} - \underline{d_i \nabla \times ((\nabla \times \mathbf{b}) \times \mathbf{b})} + \eta \Delta \mathbf{b} \,, \end{split}$$

Electron magnetohydrodynamics

$$\frac{\partial \mathbf{b}}{\partial t} = -d_i \nabla \times \left((\nabla \times \mathbf{b}) \times \mathbf{b} \right) + \eta \Delta \mathbf{b} \qquad \ell \ll d_i$$

- Ions are too heavy to follow electrons
- Two ideal invariants

$$E^{b} = \frac{1}{2} \int \mathbf{b}^{2} \, d \, \mathcal{V} \qquad H_{m} = \frac{1}{2} \int \mathbf{a} \cdot \mathbf{b} \, d \, \mathcal{V}$$



• Whistler waves



Strong (eddy) turbulence

$$\frac{\partial \mathbf{b}}{\partial t} = -d_i \nabla \times \left((\nabla \times \mathbf{b}) \times \mathbf{b} \right) + \eta \Delta \mathbf{b}$$



Strong wave turbulence

- It is a phenomenology $-A = b_0$ 1 collision
- Scale-by-scale balance: $\tau_{tr} \sim \tau_{nl} \sim \tau_{whistler}$ critical balance –
- Stronger anisotropy at small-scales: $k_{//} \sim k_{\perp}^{-1/3}$

[Cho et al., ApJL, 2004]





Intermittency in strong EMHD turbulence

$$\mathrm{SF}_p(r) = \langle |\mathbf{B}(\mathbf{x}) - \mathbf{B}(\mathbf{x} + \mathbf{r})|^p \rangle$$



[Cho et al., ApJ, 2009]

Whistler wave packet



→ Pure imbalanced electron MHD turbulence is impossible

Whistler wave packet



→ Signature of an inverse cascade of magnetic helicity

Triadic interactions

In Fourier space: helicity basis is used $(\exists \mathbf{b}_0)$ $\mathbf{h}_{\Lambda}(\mathbf{k}) = \mathbf{e}_{\theta} + i\Lambda\mathbf{e}_{\phi}$, $\mathbf{e}_{ heta} = \mathbf{e}_{\phi} imes \mathbf{e}_k \,, \quad \mathbf{e}_{\phi} = rac{\mathbf{e}_{\parallel} imes \mathbf{e}_k}{|\mathbf{e}_{\parallel} imes \mathbf{e}_k|} \,,$ $\partial_t c_{\mathbf{k}}^s = sk \sum_{s_p s_q} \int (s_q q - s_p p) M_{-\mathbf{k}pq}^{ss_p s_q} c_{\mathbf{p}}^{s_p} c_{\mathbf{q}}^{s_q} e^{ig_{k,pq}t} \delta_{k,pq} d_{pq}$ where $M_{kpq}^{ss_ps_q} = \frac{i}{4}(sk + s_pp + s_qq)\frac{\sin\alpha_k}{k}e^{-i(s\Phi_k + s_p\Phi_p + s_q\Phi_q)}$ **k** α_k **p** c_k^{s} is the wave amplitude and $g_{k,pq} = s\omega_k - s_p\omega_p - s_q\omega_q$ $\omega_s^s = sk_{\parallel}kd_ib_0$ $sk_{\parallel} \geq 0$

- \checkmark A monochromatic wave is an exact solution
- \checkmark If p // q => no transfer
- \checkmark Transfer mainly due to collisions of opposite wave-packets

Weak wave turbulence (WWT)

→ see lecture given by S. Nazarenko

Statistical theory of weakly nonlinear dispersive waves





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WWT phenomenology





 \rightarrow Local interactions (k \approx p \approx q) lead to anisotropy ie. a weak cascade along **b**₀

Rigorous mathematical derivation

Weak turbulence = theory for weakly nonlinear dispersive waves

$$\frac{\partial \mathbf{u}(\mathbf{x},t)}{\partial t} = \mathcal{L}(\mathbf{u}) + \varepsilon \mathcal{N}(\mathbf{u},\mathbf{u}) \qquad \varepsilon \ll 1$$

$$\mathbf{A}(\mathbf{k},t) = \frac{1}{(2\pi)^3} \int_{\mathbf{R}^3} \mathbf{u}(\mathbf{x},t) \exp(-i\mathbf{k} \cdot \mathbf{x}) d\mathbf{x}$$

$$\mathbf{A}(\mathbf{k},t) = \mathbf{a}(\mathbf{k},t)e^{-i\omega_k t}$$

$$\frac{\partial a_j(\mathbf{k})}{\partial t} = \varepsilon \int_{\mathbf{R}^6} \mathcal{H}_{jmn}^{\mathbf{k}\mathbf{p}\mathbf{q}} a_m(\mathbf{p}) a_n(\mathbf{q}) e^{i\Omega_{k,pq}t} \delta_{k,pq} d\mathbf{p} d\mathbf{q}$$

$$q_{jj'}(\mathbf{k}') \delta(\mathbf{k} + \mathbf{k}') = \langle a_j(\mathbf{k}) a_{j'}(\mathbf{k}') \rangle$$
Towards a statistical description

Towards a statistical description...

« Sea » of waves

$$\begin{split} \frac{\partial g_{ij'} \cdot \delta(\mathbf{k} + \mathbf{k'})}{\partial t} &= \left\langle a_{j'}(\mathbf{k'}) \frac{\partial a_{i}(\mathbf{k})}{\partial t} \right\rangle + \left\langle a_{j}(\mathbf{k}) \frac{\partial a_{j'}(\mathbf{k'})}{\partial t} \right\rangle = \\ &= \int_{\mathbf{R}^{c}} \mathcal{H}_{jmn}^{\mathbf{kpq}} (a_{m}(\mathbf{p})a_{n}(\mathbf{q})a_{j'}(\mathbf{k'})) e^{i\Omega_{k,pq}} \delta_{k,pq} d\mathbf{p} d\mathbf{q} \\ &+ \\ &= \int_{\mathbf{R}^{c}} \mathcal{H}_{jmn}^{\mathbf{kpq}} (a_{m}(\mathbf{p})a_{n}(\mathbf{q})a_{j}(\mathbf{k})) e^{i\Omega_{k',pq}} \delta_{k',pq} d\mathbf{p} d\mathbf{q} \\ &+ \\ &= \int_{\mathbf{R}^{c}} \mathcal{H}_{jmn}^{\mathbf{kpq}} (a_{m}(\mathbf{p})a_{n}(\mathbf{q})a_{j}(\mathbf{k})) e^{i\Omega_{k',pq}} \delta_{k',pq} d\mathbf{p} d\mathbf{q} \\ &+ \\ &= \int_{\mathbf{R}^{c}} \mathcal{H}_{jmn}^{\mathbf{kpq}} (a_{m}(\mathbf{p})a_{n}(\mathbf{q})a_{j}(\mathbf{k'})) e^{i\Omega_{k',pq}} \delta_{k',pq} d\mathbf{p} d\mathbf{q} \\ &+ \\ &= \int_{\mathbf{R}^{c}} \mathcal{H}_{jmn}^{\mathbf{kpq}} (a_{m}(\mathbf{p})a_{n}(\mathbf{q})a_{j}(\mathbf{k'})) e^{i\Omega_{k',pq}} \delta_{k',pq} d\mathbf{p} d\mathbf{q} \\ &+ \\ &= \int_{\mathbf{R}^{c}} \mathcal{H}_{jmn}^{\mathbf{kpq}} (a_{m}(\mathbf{p})a_{n}(\mathbf{q})a_{j}(\mathbf{k'})) e^{i\Omega_{k',pq}} \delta_{k',pq} d\mathbf{p} d\mathbf{q} \\ &+ \\ &= \int_{\mathbf{R}^{c}} \mathcal{H}_{jmn}^{\mathbf{kpq}} (a_{m}(\mathbf{p})a_{n}(\mathbf{q})) \langle a_{j'}(\mathbf{k'})a_{j''}(\mathbf{k''}) \rangle + \langle a_{m}(\mathbf{p})a_{j'}(\mathbf{k''}) \langle a_{n}(\mathbf{q})a_{j''}(\mathbf{k''}) \rangle \\ &+ \langle a_{m}(\mathbf{p})a_{m}(\mathbf{q})(\mathbf{q})(\mathbf{k'}) \rangle \langle a_{n}(\mathbf{q})a_{j'}(\mathbf{k''}) \rangle \langle a_{n}(\mathbf{q})a_{j''}(\mathbf{k''}) \rangle \\ &+ \langle a_{m}(\mathbf{p})a_{m}(\mathbf{q})(\mathbf{q})(\mathbf{k'},j') \rangle d\mathbf{q} d\mathbf{q} + \\ &= \int_{\mathbf{R}^{c}} \{(\mathbf{k},j) \leftrightarrow (\mathbf{k'},j')\} d\mathbf{q} d\mathbf{q} + \\ &= \int_{\mathbf{R}^{c}} \langle (\mathbf{k}^{*},j^{n}) \leftrightarrow (\mathbf{k'},j') \rangle d\mathbf{q} d\mathbf{q} + \\ &= \int_{\mathbf{R}^{c}} \langle (\mathbf{k}^{*},j^{n}) \leftrightarrow (\mathbf{k'},j') \rangle d\mathbf{q} d\mathbf{q} + \\ &= \int_{\mathbf{R}^{c}} \langle (\mathbf{k}^{*},j^{n}) \leftrightarrow (\mathbf{k'},j') \rangle d\mathbf{q} d\mathbf{q} + \\ &= \int_{\mathbf{R}^{c}} \langle (\mathbf{k}^{*},j^{n}) \leftrightarrow (\mathbf{k'},j') \rangle d\mathbf{q} d\mathbf{q} + \\ &= \int_{\mathbf{R}^{c}} \langle (\mathbf{k}^{*},j^{n}) \leftrightarrow (\mathbf{k'},j') \rangle d\mathbf{q} d\mathbf{q} + \\ &= \int_{\mathbf{R}^{c}} \langle (\mathbf{k}^{*},j^{n}) \leftrightarrow (\mathbf{k'},j') \rangle d\mathbf{q} d\mathbf{q} + \\ &= \int_{\mathbf{R}^{c}} \langle (\mathbf{k}^{*},\mathbf{q}^{*}) \leftrightarrow (\mathbf{k'},j') \rangle d\mathbf{q} d\mathbf{q} + \\ &= \int_{\mathbf{R}^{c}} \langle (\mathbf{k}^{*},\mathbf{q}^{*}) \leftrightarrow (\mathbf{k'},j') \rangle d\mathbf{q} d\mathbf{q} + \\ &= \int_{\mathbf{R}^{c}} \langle (\mathbf{k},\mathbf{q},\mathbf{q}) \leftrightarrow (\mathbf{k'},j') \rangle d\mathbf{q} d\mathbf{q} + \\ &= \int_{\mathbf{R}^{c}} \langle (\mathbf{k},\mathbf{q},\mathbf{q}) \leftrightarrow (\mathbf{k'},j') \rangle d\mathbf{q} d\mathbf{q} + \\ &= \int_{\mathbf{R}^{c}} \langle (\mathbf{k},\mathbf{q},\mathbf{q}) \leftrightarrow (\mathbf{k'},j') \rangle d\mathbf{q} d\mathbf{q} + \\ &= \int_{\mathbf{R}^{c}} \langle (\mathbf{k},\mathbf{q},\mathbf{q}) \leftrightarrow (\mathbf{k'},\mathbf{q}) \leftrightarrow (\mathbf{k'},j') \rangle d\mathbf{q} d\mathbf{q} + \\ &= \int_{\mathbf{R}^{c}} \langle (\mathbf$$

Weak wave turbulence: theory

$$egin{aligned} &k_{\perp}\gg k_{\parallel}\ \partial_tigg\{ egin{subarray}{l} E_k\ H_kigg\} &=rac{\epsilon^2}{16}\sum_{ss_ps_q}\intrac{s_pp_{\perp}k_{\parallel}p_{\parallel}}{q_{\perp}}\left(rac{s_qq_{\perp}-s_pp_{\perp}}{k_{\parallel}}
ight)^2(sk_{\perp}+s_pp_{\perp}+s_qq_{\perp})^2\sin heta_q\ &igg\{ sk_{\perp}\left[E_q(p_{\perp}E_k-k_{\perp}E_p)/(k_{\perp}p_{\perp}q_{\perp})+s_qH_q\left(sH_k-s_pH_p
ight)
igh] \ &E_q(sH_k-s_pH_p)/q_{\perp}+s_qH_q(p_{\perp}E_k-k_{\perp}E_p)/(k_{\perp}p_{\perp}) \ &\delta(k_{\parallel}+p_{\parallel}+q_{\parallel})\,\delta(sk_{\perp}k_{\parallel}+s_pp_{\perp}p_{\parallel}+s_qq_{\perp}q_{\parallel})\,dp_{\perp}dq_{\perp}dp_{\parallel}dq_{\parallel}\,. \end{aligned}$$

Exact solution at constant magnetic energy flux: [SG et al., PoP, 2003]

$E_k \sim k_\perp^n k_\parallel ^m$	Zakharov	n = -5/2, m = -1/2
$H_k \sim k_\perp^{ ilde{n}} k_\parallel ^{ ilde{m}}$	transformation	$\tilde{n} = -7/2$ and $\tilde{m} = -1/2$

Energy spectrum compatible with a simple phenomenology

Weak wave turbulence: theory

$$\begin{split} k_{\perp} \gg k_{\parallel} \\ \partial_t \Big\{ \begin{matrix} E_k \\ H_k \end{matrix} \Big\} &= \frac{\epsilon^2}{16} \sum_{ss_p s_q} \int \frac{s_p p_{\perp} k_{\parallel} p_{\parallel}}{q_{\perp}} \left(\frac{s_q q_{\perp} - s_p p_{\perp}}{k_{\parallel}} \right)^2 (sk_{\perp} + s_p p_{\perp} + s_q q_{\perp})^2 \sin \theta_q \\ & \left\{ \begin{matrix} sk_{\perp} \left[E_q (p_{\perp} E_k - k_{\perp} E_p) / (k_{\perp} p_{\perp} q_{\perp}) + s_q H_q (sH_k - s_p H_p) \right] \\ E_q (sH_k - s_p H_p) / q_{\perp} + s_q H_q (p_{\perp} E_k - k_{\perp} E_p) / (k_{\perp} p_{\perp}) \end{matrix} \right\} \qquad k_{//} \ge 0 \\ & \delta(k_{\parallel} + p_{\parallel} + q_{\parallel}) \, \delta(sk_{\perp} k_{\parallel} + s_p p_{\perp} p_{\parallel} + s_q q_{\perp} q_{\parallel}) \, dp_{\perp} dq_{\perp} dp_{\parallel} dq_{\parallel} \, . \end{split}$$

Exact solution at constant magnetic helicity flux: [SG et al., JPP, 2015]



Weak inertial wave turbulence

 $\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\nabla P + \mu \Delta \boldsymbol{v} + 2 \,\Omega \,\boldsymbol{v} \times \hat{\boldsymbol{e}}_z$ Coriolis force

→ see lecture given by P. Mininni





It is also the exact constant helicity flux solutions of the weak turbulence equations ! [SG, PRE, 2003, 2014]

New phenomenology



Weak wave turbulence: simulation



END OF PART I

Turbulence in astrophysical plasmas at sub-ion scales: electron and Hall MHD



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Astrophysical turbulence at sub-ion scales

(It is not an exhaustive presentation)

• PART 0: MHD $(\ell > d_i)$ \rightarrow see lectures given by H. Politano

- **PART I: Electron MHD** $(d_i > \ell > d_e)$
- PART II: Hall MHD $(\ell > d_e)$

$$d_e = c/\omega_{pe} \simeq 5.3 \times 10^6 \, n_e^{-1/2} \mbox{ m}$$

$$d_i = c/\omega_{pi} \simeq 2.3 \times 10^8 \, n_i^{-1/2} \mbox{ m}$$



PART II: Hall MHD turbulence

Solar wind turbulence

→ see lecture given by O. Alexandrova

Power laws at MHD scales & sub-ion scales



[Kiyani et al., Phil. Trans. R. Soc. A, 2015]

Is Hall MHD just the addition of EMHD + MHD ?

Or, is it more subtle ??

Incompressible Hall MHD

$$\begin{split} & \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P_* + \mathbf{b} \cdot \nabla \mathbf{b} + \nu \Delta \mathbf{u} \,, \\ & \frac{\partial \mathbf{b}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{b} = \mathbf{b} \cdot \nabla \mathbf{u} - d_i \nabla \times \left((\nabla \times \mathbf{b}) \times \mathbf{b} \right) + \eta \Delta \mathbf{b} \,, \end{split}$$

$$\mathbf{b} \equiv \mathbf{B}/\sqrt{\mu_0
ho_0}$$

 $d_i = c/\omega_{pi}$
 $\omega_{pi} = \sqrt{rac{n_i e^2}{m_i \epsilon_0}}$

Incompressible Hall MHD turbulence

There is an exact solution:

[Mahajan et al., MNRAS, 2005]

$$\mathbf{v} = B(\hat{\mathbf{e}}_{x} + i\hat{\mathbf{e}}_{y})exp\left[ikz + i\alpha_{k\pm}k\left(\hat{\mathbf{e}}_{z}\cdot\mathbf{B}_{0}\right)t\right] \\ + C(\hat{\mathbf{e}}_{y} + i\hat{\mathbf{e}}_{z})exp\left[ikx + i\alpha_{k\pm}k\left(\hat{\mathbf{e}}_{x}\cdot\mathbf{B}_{0}\right)t\right] \\ + A(\hat{\mathbf{e}}_{z} + i\hat{\mathbf{e}}_{x})exp\left[iky + i\alpha_{k\pm}k\left(\hat{\mathbf{e}}_{y}\cdot\mathbf{B}_{0}\right)t\right]$$

$$lpha_{k\pm} = \left[-rac{d_i k}{2} \pm \left(rac{(d_i k)^2}{4} + 1
ight)^{1/2}
ight]$$
 $oldsymbol{b}_k = oldsymbol{lpha} oldsymbol{v}_k$

\rightarrow monochromatic wave



Structures in Hall MHD turbulence

→ Transition of vorticies from sheet-like to tubular structures [Miura et al., PoP, 2014]

Hall MHD $(d_i=0.05)$ $\mathbf{MHD} \ (\mathbf{d_i}=0)$ 1024^3 ; decay

Isosurfaces of current density (grey) and vorticity density (colored)

Exact laws in turbulence

• Incompressible HD:

$$\left| \begin{array}{c} -rac{4}{5}arepsilon\ell = \langle (\delta u_\ell)^3
angle \, , \end{array}
ight.$$

[Kolmogorov, DAN, 1941]

 $-rac{4}{3}arepsilon\ell=\langle (\delta u_i)^2\delta u_\ell
angle\,,$

[Antonia et al., JFM, 1997]

- Incompressible MHD: $-\frac{4}{3}\varepsilon^{T}\ell = \langle (\delta \mathbf{u} \cdot \delta \mathbf{u} + \delta \mathbf{b} \cdot \delta \mathbf{b}) \delta u_{\ell} \rangle 2\langle (\delta \mathbf{u} \cdot \delta \mathbf{b}) \delta b_{\ell} \rangle$ $\mathbf{z}^{\pm} \equiv \mathbf{u} \pm \mathbf{b} \qquad \boxed{-\frac{4}{3}\varepsilon^{\pm}\ell = \langle (\delta \mathbf{z}^{\pm} \cdot \delta \mathbf{z}^{\pm}) \delta z_{\ell}^{\mp} \rangle}, \qquad [Politano \& Pouquet, PRE, 1998]$
- Compressible isothermal HD: $-2\varepsilon = \langle (\nabla' \cdot \mathbf{u}')(R-E) \rangle + \langle (\nabla \cdot \mathbf{u})(\tilde{R}-E') \rangle + \nabla_{\mathbf{r}} \cdot \langle \left[\frac{\delta(\rho \mathbf{u}) \cdot \delta \mathbf{u}}{2} + \delta\rho \delta e C_s^2 \bar{\delta}\rho \right] \delta \mathbf{u} + \bar{\delta}e \delta(\rho \mathbf{u}) \rangle,$ [SG et al., PRL, 2011]
- Compressible isothermal MHD:

$$-2\varepsilon = \frac{1}{2}\nabla_{\mathbf{r}} \cdot \left\langle \left[\frac{1}{2}\delta(\rho\mathbf{z}^{-}) \cdot \delta\mathbf{z}^{-} + \delta\rho\delta e \right] \delta\mathbf{z}^{+} + \left[\frac{1}{2}\delta(\rho\mathbf{z}^{+}) \cdot \delta\mathbf{z}^{+} + \delta\rho\delta e \right] \delta\mathbf{z}^{-} + \overline{\delta}\left(e + \frac{v_{A}^{2}}{2}\right) \delta(\rho\mathbf{z}^{-} + \rho\mathbf{z}^{+}) \right\rangle \\ - \frac{1}{8} \left\langle \frac{1}{\beta'}\nabla' \cdot (\rho\mathbf{z}^{+}e') + \frac{1}{\beta}\nabla \cdot (\rho'\mathbf{z}'+e) + \frac{1}{\beta'}\nabla' \cdot (\rho\mathbf{z}^{-}e') + \frac{1}{\beta}\nabla \cdot (\rho'\mathbf{z}'-e) \right\rangle \\ + \left\langle (\nabla \cdot \mathbf{v}) \left[R'_{E} - E' - \frac{\overline{\delta}\rho}{2} (\mathbf{v}_{A'} \cdot \mathbf{v}_{A}) + \frac{P'_{M} - P'}{2} \right] \right\rangle + \left\langle (\nabla' \cdot \mathbf{v}') \left[R_{E} - E - \frac{\overline{\delta}\rho}{2} (\mathbf{v}_{A} \cdot \mathbf{v}_{A'}) + \frac{P_{M} - P}{2} \right] \right\rangle \\ + \left\langle (\nabla \cdot \mathbf{v}_{A}) [R_{H} - R'_{H} + H' - \overline{\delta}\rho(\mathbf{v}' \cdot \mathbf{v}_{A})] \right\rangle + \left\langle (\nabla' \cdot \mathbf{v}_{A'}) [R'_{H} - R_{H} + H - \overline{\delta}\rho(\mathbf{v} \cdot \mathbf{v}_{A'})] \right\rangle,$$
[Banerjee et al., PRE, 2013]

Application to the (slow) solar wind



[[]Hadid et al., 2016]

Exact law in Hall MHD turbulence

Tensor analysis à la Kolmogorov:

$$S_{ijk}^{1}(\mathbf{r}) = \langle v_{i}(\mathbf{x})v_{j}(\mathbf{x})v_{k}(\mathbf{x}') \rangle$$

$$= A_{1}r_{i}r_{j}r_{k} + B_{1}(r_{i}\delta_{jk} + r_{j}\delta_{ik}) + D_{1}r_{k}\delta_{ij},$$

$$S_{ijk}^{2}(\mathbf{r}) = \langle b_{i}(\mathbf{x})b_{j}(\mathbf{x})v_{k}(\mathbf{x}') \rangle$$

$$= A_{2}r_{i}r_{j}r_{k} + B_{2}(r_{i}\delta_{jk} + r_{j}\delta_{ik}) + D_{2}r_{k}\delta_{ij},$$

$$S_{ijk}^{4}(\mathbf{r}) = \langle J_{i}(\mathbf{x})b_{j}(\mathbf{x})b_{k}(\mathbf{x}') \rangle$$

$$= A_{4}r_{i}r_{j}r_{k} + B_{4}r_{i}\delta_{jk} + C_{4}r_{j}\delta_{ik} + D_{4}r_{k}\delta_{ij},$$

$$\Box$$

$$\Box$$

Incompressible isotropic Hall MHD: [SG, PRE, 2008]

$$-\frac{4}{3}\varepsilon^{T}r = \langle [(\delta \mathbf{v})^{2} + (\delta \mathbf{b})^{2}]\delta v_{r} \rangle - 2\langle [\delta \mathbf{v} \cdot \delta \mathbf{b}]\delta b_{r} \rangle \Big\} \quad \mathbf{E}^{\text{tot}} \sim \mathbf{k}^{-5/3} \\ + 4d_{I} \langle [(\mathbf{J} \times \mathbf{b}) \times \delta \mathbf{b}]_{r} \rangle \Big\} \mathbf{E}^{\mathbf{b}} \sim \mathbf{k}^{-7/3}$$

Two inertial ranges in a single law !

3D DNS of Hall MHD turbulence



Numerical simulation

- It is still very challenging to do DNS in 3D Hall MHD turbulence
 - \rightarrow impossible to get clearly two inertial ranges

Solution: shell models

[see review by Plunian et al., Phys. Rep., 2013]

Incompressible hydrodynamics:

$$(d/dt + \nu k_n^2)u_n = ik_n(\alpha u_{n+2}u_{n+1} + \beta u_{n-1}u_{n+1} + \gamma u_{n-2}u_{n-1})^*$$
$$k_n = k_0\lambda^n, \text{ with } \lambda \equiv 2$$



The cascade can be non local in Hall MHD [Mininni et al, JPP, 2005]

[Hori et al., JPP, 2006]



Hall MHD





 $S_p^u(k_n) = \langle |u_n|^p \rangle \sim k_n^{\zeta_p^u}$

 $S_p^b(k_n) = \langle |b_n|^p \rangle \quad \sim k_n^{\zeta_p^{b,1}} \quad (k < k_I)$ $\sim k_n^{\zeta_p^{b,2}} \quad (k_d > k > k_I)$



3D DNS of isotropic Hall MHD turbulence

Analysis in terms of polarization: $P_m = \sigma_m \sigma_c$

[Ghosh et al., JGR, 1996]

$$\sigma_m = \frac{\hat{\mathbf{a}} \cdot \hat{\mathbf{b}}^* + \hat{\mathbf{a}}^* \cdot \hat{\mathbf{b}}}{2|\hat{\mathbf{a}}||\hat{\mathbf{b}}|}, \qquad \sigma_c = \frac{\hat{\mathbf{u}} \cdot \hat{\mathbf{b}}^* + \hat{\mathbf{u}}^* \cdot \hat{\mathbf{b}}}{2|\hat{\mathbf{u}}||\hat{\mathbf{b}}|},$$

$$E^{R}(\mathbf{k}) = \frac{1}{2} (|\mathbf{\hat{u}}|^{2} + |\mathbf{\hat{b}}|^{2}), P_{m} > 0$$
$$E^{L}(\mathbf{k}) = \frac{1}{2} (|\mathbf{\hat{u}}|^{2} + |\mathbf{\hat{b}}|^{2}), P_{m} < 0$$

 $P_m = +1$ for whistler waves $P_m = -1$ for ion-cyclotron waves

3D DNS of isotropic Hall MHD turbulence



Weak wave turbulence theory

- ✓ Assumption: $\mathbf{B}(\mathbf{x},t) = \mathbf{b}_0 \mathbf{e}_{//} + \mathbf{\varepsilon} \mathbf{b}(\mathbf{x},t)$ with $0 < \varepsilon << 1$
- ✓ Complex helicity decomposition (helical waves)
- ✓ Wave amplitude equation (three wave interactions):

✓ Perturbative theory + statistical description give:

s= ± 1 , $\Lambda = \pm 1$

$$\partial_{t}q_{\Lambda}^{s}(\mathbf{k}) = \frac{\pi\epsilon^{2}}{4d_{i}^{2}B_{0}^{2}} \int \sum_{\Lambda_{p},\Lambda_{q} \atop s_{p},s_{q}} \left(\frac{\sin\psi_{k}}{k}\right)^{2} (\Lambda k + \Lambda_{p}p + \Lambda_{q}q)^{2} \qquad [SG, JPP, 2006]$$

$$\times (1 - \xi_{\Lambda}^{-s^{2}}\xi_{\Lambda_{p}}^{-s_{p}^{2}}\xi_{\Lambda_{q}}^{-s_{q}^{2}})^{2} \left(\frac{\xi_{\Lambda_{q}}^{s_{q}} - \xi_{\Lambda_{p}}^{s_{p}}}{k_{\parallel}}\right)^{2} \left(\frac{\omega_{\Lambda}^{s}}{1 + \xi_{\Lambda}^{-s^{2}}}\right) \\ \times \left[\left(\frac{\omega_{\Lambda}^{s}}{1 + \xi_{\Lambda}^{-s^{2}}}\right)\frac{1}{q_{\Lambda}^{s}(\mathbf{k})} - \left(\frac{\omega_{\Lambda_{p}}^{s_{p}}}{1 + \xi_{\Lambda_{p}}^{-s_{p}^{2}}}\right)\frac{1}{q_{\Lambda_{p}}^{s_{p}}(\mathbf{p})} - \left(\frac{\omega_{\Lambda_{q}}^{s_{q}}}{1 + \xi_{\Lambda_{q}}^{-s_{q}^{2}}}\right)\frac{1}{q_{\Lambda_{q}}^{s_{q}}(\mathbf{q})}\right] \\ \times q_{\Lambda}^{s}(\mathbf{k})q_{\Lambda_{p}}^{s_{p}}(\mathbf{p})q_{\Lambda_{q}}^{s_{q}}(\mathbf{q})\delta(\Omega_{k,pq})\delta_{k,pq}d\mathbf{p}d\mathbf{q}. \qquad (3.23)$$

Weak wave turbulence theory



→ Anisotropic system

3D DNS of anisotropic Hall MHD turbulence



Weak and strong turbulence may coexist at the same scale

3D DNS of anisotropic Hall MHD turbulence



3D DNS of anisotropic Hall MHD turbulence



3D DNS of anisotropic Hall MHD turbulence

 $E_L^u(k,\omega)$ energy spectra



[Meyrand et al., 2016]

For comparison with MHD



[Meyrand et al., PRL, 2016]

Summary

Isotropic Hall MHD turbulence



Anisotropic Hall MHD turbulence



Beyond Hall MHD

Inertial electron MHD turbulence

$\ell < d_e$

The inviscid three-dimensional electron MHD equations can be written in SI units as (Biskamp et al. 1996)

$$\partial_t \left(1 - d_e^2 \Delta \right) \mathbf{B} = -d_i \nabla \times \left[\mathbf{J} \times \left(1 - d_e^2 \Delta \right) \mathbf{B} \right], \tag{1}$$





Exact relations

 $\ell < d_e$

$$4d_i \langle [(\overline{\mathbf{J} \times \mathbf{B}}) \times \delta \mathbf{B}]_L \rangle - d_i d_e^2 \langle (\delta \mathbf{J})^2 \delta J_L \rangle = -\frac{4}{3} \varepsilon^T r.$$

Exact relation
$$d_i d_e^2 J^3 \sim \varepsilon^J r,$$

[Meyrand et al., ApJ, 2010]

$$B^2(k) \sim \left(\frac{\varepsilon^J}{d_i d_e^2}\right)^{2/3} k^{-11/3}.$$

Bi-fluid approach: \rightarrow see talk given by N. Andrés

$$-\frac{4}{3}r\varepsilon_{T} = \begin{bmatrix} B_{\parallel ii}^{\upsilon\upsilon\upsilon}(\mathbf{r}) + B_{\parallel ii}^{\upsilon BB}(\mathbf{r}) - B_{\parallel ii}^{B\upsilon B}(\mathbf{r}) - B_{\parallel ii}^{BuB}(\mathbf{r}) \end{bmatrix} \\ + \mu \begin{bmatrix} B_{\parallel ii}^{uuu}(\mathbf{r}) + B_{\parallel ii}^{uBB}(\mathbf{r}) - B_{\parallel ii}^{\upsilon\upsilon\upsilon}(\mathbf{r}) - B_{\parallel ii}^{\upsilon BB}(\mathbf{r}) \end{bmatrix}$$

[Andrés et al., PRE, 2016]

Proposition for a new space mission

THOR (Turbulence Heating ObserveR)

Mission M4/ESA selected (27 \rightarrow 3) for 2026 (only 1 in 2017)







ABSTRACT

[Franci et al., ApJ, 2015]

We present results from a high-resolution and large-scale hybrid (fluid electrons and particle-in-cell protons) two-dimensional numerical simulation of decaying turbulence. Two distinct spectral regions (separated by a smooth break at proton scales) develop with clear power-law scaling, each one occupying about a decade in wave numbers. The simulation results exhibit simultaneously several properties of the observed solar wind fluctuations: spectral indices of the magnetic, kinetic, and residual energy spectra in the magneto-hydrodynamic (MHD) inertial range along with a flattening of the electric field spectrum, an increase in magnetic compressibility, and a strong coupling of the cascade with the density and the parallel component of the magnetic fluctuations at sub-proton scales. Our findings support the interpretation that in the solar wind large-scale MHD fluctuations naturally evolve beyond proton scales into a turbulent regime that is governed by the generalized Ohm's law.



Thank you for your attention !



New formulation of exact laws

$$\begin{split} \mathbf{\Omega}_{L} &= \mathbf{b} + d_{i} \nabla \times \mathbf{v}, & \eta_{M} &= \langle \delta(\mathbf{v}_{R} \times \mathbf{\Omega}_{R}) \cdot \delta \mathbf{\Omega}_{R} \rangle, \\ \mathbf{\Omega}_{R} &= \mathbf{b}, & \pm \eta_{G} &= \langle \delta(\mathbf{v}_{L} \times \mathbf{\Omega}_{L}) \cdot \delta \mathbf{\Omega}_{L} \rangle. \\ \mathbf{v}_{L} &= \mathbf{v}, & \text{Isotropy is not assumed !} \end{split}$$

$$\mathbf{v}_R = \mathbf{v} - d_i \boldsymbol{\nabla} \times \mathbf{b} \,,$$



$$H_G = \frac{1}{2} \int (\mathbf{a} + d_i \mathbf{u}) \cdot (\mathbf{b} + d_i \nabla \times \mathbf{u}) \, d\mathcal{V} \qquad \text{MHD}$$



Relation with the dynamo problem

Balance between Coriolis and Lorentz-Laplace forces

 $\mathbf{b}_{\mathbf{0}} = b_0 \mathbf{\hat{e}}_{\parallel} \qquad \mathbf{\Omega}_{\mathbf{0}} = \Omega_0 \mathbf{\hat{e}}_{\parallel}$

MHD under rotation: $\begin{aligned}
\mathbf{b_0} &= d\mathbf{\Omega_0} \\
\mathbf{b_0} &= \mathbf{0} \\
\mathbf{b_0} &= \mathbf{$

$$2\mathbf{u} = -d\mathbf{j}$$

