Overview of work in collaboration with Balk, Bustamante, Connaughton, Dyachenko, Harper, Manin, Medvedev, Nadiga, Quinn, Zakharov

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Outline

- Importance of resonant wave interactions in QG turbulence
- Generation of zonal jets by local anisotropic cascades and nonlocal mechanisms.
- Quadratic invariants.
- Self-regulating turbulence – zonal jet system
- Continuous spectrum vs discrete-wave clusters
A chapter on Rossby wave turbulence in:

Wave Turbulence

Sergey Nazarenko

LECTURE NOTES IN PHYSICS 825

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Rossby and drift wave turbulence and zonal flows: The Charney–Hasegawa–Mima model and its extensions

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Zonal Jets
The ISSI Team

\textbf{Editors:} Boris Galperin and Peter Read
Rossby waves and jets

Earth’s atmosphere and ocean

Atmospheres of giant planets
CHARNEY-HASEGAWA-MIMA EQUATION

\[
\frac{\partial}{\partial t} \left( \rho^2 \nabla^2 \psi - \psi \right) - \beta \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial x} \frac{\partial \nabla^2 \psi}{\partial y} - \frac{\partial \psi}{\partial y} \frac{\partial \nabla^2 \psi}{\partial x} = 0
\]

- \( \psi \) – streamfunction (electrostatic potential).
- \( \rho \) – Deformation radius (ion Larmor radius).
- \( \beta \) – PV gradient (diamagnetic drift).
- \( x \) – east-west (poloidal arc-length).
- \( y \) – south-north (radial length).
Turbulence/Zonal-Flow feedback loop

- Small-scale turbulence generates zonal flows.
- Negative feedback loop: turbulence is suppressed by ZFs
- Suppressed turbulence → reduced anomalous transport

Balk, SN and Zakharov 1990
Barotropic governor in GFG

James and Gray’ 1986

FIG. 1. Schematic illustration of the “barotropic governor,” summarizing the effect of horizontal shears on baroclinic instability as postulated in JG. Energy conversion from available potential energy (AZ) to eddy kinetic energy (KE) results in momentum fluxes which increase the barotropic contribution to the zonal kinetic energy (KZ). As barotropic shears build up in the zonal flow, the baroclinic conversions are inhibited.
Nonlocal mechanism of ZF generation:
Modulational Instability
Loretz 1972, Gill 1973, Manin, Nazarenko, 1994
Numerics: Connaughton, Nadiga, SN, Quinn, 2009.

- Cf. Benjamin-Fair Instability of water waves
Modulational Instability

\[ \psi_0(x, t) = \Psi_0 e^{i k \cdot x - i \omega t} + \overline{\Psi_0} e^{-i k \cdot x + i \omega t} \]

\[ \omega(k) = - \frac{\beta k_x}{k^2 + F} \quad \text{frequency of linear waves.} \]

- These waves are solutions of CHM equation for any amplitude. Are they stable? (Lorentz 1972, Gill 1973).

\[ \psi(x, 0) = \psi_0(x) + \dot{\psi}_1(x), \]
\[ \psi_1(x) = \psi_z(x) + \psi^+(x) + \psi^-(x) \quad \text{perturbation.} \]

\[ \psi_z(x) = ae^{iq \cdot x} + \overline{a} e^{-iq \cdot x} \quad \text{zonal part} \quad q = (0, q), \]
\[ \psi^+(x) = b^+ e^{ip_+ \cdot x} + \overline{b^+} e^{-ip_+ \cdot x} \quad \text{satellite} \quad p_+ = k + q, \]
\[ \psi^-(x) = b^- e^{ip_- \cdot x} + \overline{b^-} e^{-ip_- \cdot x} \quad \text{satellite} \quad p_- = k - q. \]
Instability dispersion relation

\[(q^2 + F)\Omega + \beta q_x + |\Psi_0|^2 |k \times q|^2 (k^2 - q^2) \left[ \frac{p_+^2 - k^2}{(p_+^2 + F)(\Omega + \omega) + \beta p_+} - \frac{p_-^2 - k^2}{(p_-^2 + F)(\Omega - \omega) + \beta p_-} \right] = 0 \]

\[M = \frac{\Psi_0 k^3}{\beta} \quad \text{-- nonlinearity parameter.}\]

\[M \to \infty \quad \text{-- Euler limit (Rayleigh instability)};\]
\[M \to 0 \quad \text{-- weak nonlinearity: resonant wave interaction.}\]
Structure of instability as a function of $M$

For small $M$ the unstable region collapses onto the resonant curve and the most unstable disturbance is not zonal.

$$\mathbf{k} + \mathbf{k}_1 = \mathbf{k}_2,$$

$$\omega(\mathbf{k}) + \omega(\mathbf{k}_1) = \omega(\mathbf{k}_2)$$
Continuous spectrum theory: Kinetic equation for weakly nonlinear Rossby waves
(Longuet-Higgens & Gill, 1967)

\[ \dot{n}_k = \int |V_{12k}|^2 \delta(k_1 + k_2 - k) \delta(\omega(k_1) + \omega(k_2) - \omega(k)) \times 
\left[ n(k_1)n(k_2) - 2n(k)n(k_1) \text{sign}(k_x k_{1x}) \right] dk_1 dk_2, \]

\[ \omega(k) = -\beta k_x / k^2, \text{ - frequency} \]

\[ V_{12k} = |k_x k_{1x} k_{2x}|^{1/2} \left( \frac{k_{1y}}{k_1^2} + \frac{k_{2y}}{k_2^2} - \frac{k_y}{k^2} \right) \text{ - interaction coefficient} \]

\[ n(k) = k^4 |\hat{\psi}_k|^2 / (\beta k_x), \text{ - waveaction spectrum} \]

For case \( k\rho >> 1 \). Resonant three-wave interactions.
Conservation laws in 2D

\[ E(k) = \int \langle u(x + r) \times u(x) \rangle e^{-i k \cdot r} \, dr \]  - energy spectrum

\[ \langle u^2 \rangle = \int E(k) \, dk \]  - energy

\[ \langle (\nabla \times u)^2 \rangle = \int k^2 E(k) \, dk \]  - enstrophy
Extra quadratic invariant on $\beta$-plane

- Balk, Nazarenko & Zakharov (1990)
- Adiabatic for the original $\beta$-plane equation: requires small nonlinearity.
- For case $k\rho >> 1$:

\[
\Phi = \int \frac{k_x^2}{k^6} (k_x^2 + 5k_y^2) |\hat{\psi}_k|^2 dk, \quad \text{- Zonostrophy invariant.}
\]
Energy flows into the zonal flow sector
Generalised Fjortoft’s Theorem

• Consider a statistically steady state in a forced-dissipated system which has (in absence of forcing and dissipation) positive quadratic invariants $I_1, I_2, \ldots, I_n$. Let forcing be in vicinity of $k_0=(k_{0x}, ky_0)$. The dissipation rate of $I_m$ in the regions where its relative spectral density (w.r.t. to the one at $k_0$) is vanishingly small compared to the relative spectral density of at least one other invariant is vanishingly small w.r.t. to its production rate.

• No assumption about locality of interactions, nor about continuity or discreteness of the k-spectrum.
TRIPLE CASCADE IN QG TURBULENCE: NUMERICS OF UNFORCED CHM

SN and B. Quinn, 2009.
Trajectories of the 3 centroids.
Fjortoft works well even for strong turbulence.
Self-regulation and Feedback loop in QG turbulence

- Instability generates small-scale turbulence.
- Inverse cascade leads to energy condensation into zonal jets.
- Jets kill small-scale turbulence and saturate.

Rossby wave turbulence.
More important for large betas
Evolution in the k-space

Energy of Rossby wave packets is partially transferred to ZF and partially dissipated at large \( k \)'s. (Balk et al, 1990).
Kinetic equation for weakly nonlinear Rossby/Drift waves (Longuet-Higgsens & Gill, 1967)

\[ \dot{n}_k = \int |V_{12k}|^2 \delta(k_1 + k_2 - k) \delta(\omega(k_1) + \omega(k_2) - \omega(k)) \times \]
\[ [n(k_1)n(k_2) - 2n(k)n(k_1) \text{sign}(k_x k_{1x})] \quad dk_1 dk_2, \]

\[ \omega(k) = -\beta k_x / k^2, \text{ - frequency} \]

\[ V_{12k} = |k_x k_{1x} k_{2x}|^{1/2} \left( \frac{k_{1y}}{k_1^2} + \frac{k_{2y}}{k_2^2} - \frac{k_y}{k^2} \right) \text{ - interaction coefficient} \]

\[ n(k) = k^4 |\hat{\psi}_k|^2 / (\beta k_x), \text{ - waveaction spectrum} \]

Resonant three-wave interactions.
Baroclinic instability forcing

Accessing the stored free energy via instability
Maximum on the $k_x$-axis at $k \rho \sim 1$. 
Evolution of nonlocal drift turbulence:
retain only interaction with small $k$’s and Taylor-expand the integrand of the wave-collision integral; integrate.

- Diffusion along curves
  \[ \Omega_k = \omega_k - \beta k_x = \text{conts.} \]
- $S \sim \text{ZF intensity}$
Initial evolution

- Solve the eigenvalue problem at each curve.
- Max eigenvalue $< 0 \rightarrow$ spectrum on this curve decay.
- Max eigenvalue $> 0 \rightarrow$ spectrum on this curve grow.
- Growing curves pass through the instability scales
ZF growth

- Waves pass energy from the growing curves to ZF.
- ZF accelerates wave energy transfer to the dissipation scales via the increased diffusion coefficient.
ZF growth

- Hence the growing region shrink.
- Wave Turbulence - ZF loop closed!
Steady state

- Saturated ZF.
- Jet spectrum on a k-curve passing through the maximum of instability.
- Suppressed intermediate scales
- Balanced/correlated turbulence and ZF
Zonal scales form.
Small-scale turbulence is suppressed.
NUMERICS OF INSTABILITY-FORCED CHM

\[ \text{Zonal scales form.} \]
\[ \text{Small-scale turbulence is suppressed.} \]

Evolution in time of energies:
\text{Read} – zonal sector,
\text{Green} – off-zonal sector;
\text{Blue} – instability scales.

Summary

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- Quadratic invariants.
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Thank you 😊