Inferring the internal rotation of the Sun from low degree helioseismology

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ABSTRACT

In this presentation, I investigate the internal rotation of the Sun (below $0.4R_\odot$) from some observations of low degree frequency splittings. I compare the results obtained with two inverse methods applied on the first GOLF eight months dataset combined with the LOWL two years dataset. The emphasis is made on the study of the effects of underestimated errors or bias that may be present in the data or introduced by the inverse method itself.

1. Forward model

For low rotation the observed frequency splittings are given by:

$$\Delta\nu_{nlm} \equiv \frac{\nu_{nlm}-\nu_{nl0}}{m} = \int_0^\pi \int_0^{R_\odot} K_{nlm}(r,\theta)\Omega(r,\theta)\sin\theta dr d\theta + \varepsilon_{nlm}, \tag{1}$$

where $\Omega(r,\theta)$ is the unknown rotation rate versus depth r and colatitude θ and ε_{nlm} are observational errors assumed to be uncorrelated and normally distributed with a standard deviation σ_{nlm} taken equal to the error given by observers on the corresponding splitting. The kernels $K_{nlm}(r,\theta)$ depend on the model and on the eigenfunctions of the modes which are assumed to be known exactly. We note that, most of the observers actually do not give directly the individual splittings but rather a small number of the so-called a-coefficients of their expansions on a set of orthogonal polynomials. Assuming that the relation between individual splittings and these coefficients is linear, an equation similar to Eq. (1) can be established by computing the appropriate kernels $K^a_{nlj}(r,\theta)$ related to each a_j-coefficient.

In order to investigate the rotation below $0.4R_\odot$ where the latitudinal dependence is particularly not well constrained because of the few azimuthal orders provided by the low l degree modes able to sound these zones, one may want to simplify the problem by assuming that the rotation rate has only small (smaller than in the convection zone) variations in latitude. With this assumption it can be shown that Eq. (1) is well approximated by:

$$\Delta\nu_{nlm} = \int_0^\pi \int_0^{R_\odot} \tilde{K}_{nl}(r)[P_l^m(\cos\theta)]^2\Omega(r,\theta)\sin\theta dr d\theta + \varepsilon_{nlm}, \tag{2}$$

where P_l^m are Legendre functions normalized such that $\int_0^\pi [P_l^m(\cos\theta)]^2 \sin\theta d\theta = 1$ and $\tilde{K}_{nl}(r)$ radial kernels (eg. Cuypers (1980)) . One possibility is then to carry out a full 2D inversion of Eq. (2) but constraining the solution to have no latitudinal variation in the deep interior (Corbard et al. 1997a). The solution obtained in the core by this way represents an average of the core rotation between the equator and the pole.

Another possibility is to search for the rotation rate at a given latitude only i.e. to reduce the problem to a 1D inverse problem. Since the most constrained zone is the equatorial one, we try to investigate only the equatorial rotation rate. For this purpose we use the following property of the Legendre functions for $m = l$:

$$[P_l^l(cos(\theta)]^2 = C_l(sin\theta)^{2l} \qquad\qquad C_l = \frac{(2l+1)!}{2^{2l+1}(l!)^2} \tag{3}$$

which shows that, for high degrees l, the major contribution to sectoral splittings $\Delta\nu_{nll}$ comes from the equatorial rotation rate $\Omega_{eq}(r) = \Omega(r,\pi/2)$. This leads to the commonly used 1D integral equation (see e.g. Antia et al. 1996, Corbard et al. 1997b):

$$\Delta\nu_{nll} = \int_0^{R_\odot} \tilde{K}_{nl}(r)\Omega_{eq}(r)dr + \varepsilon_{nll} \tag{4}$$

Nevertheless this approximation is valid only for high degrees l. For lower degrees the sectoral splittings are sensitive not only to the equatorial rotation but also to the rotation rate in a large angular domain around the equator. The extent of this domain will be estimated in the following and I will discuss the influence of this approximation on the estimation of the core rotation using Eq. (4).

2. Inverse methods used

I have used 2 different inverse methods:

- A Regularized Least-Squares method with Tikhonov regularization. For this method we project the equatorial rotation rate on a spline basis $\Omega_{eq}(r) = \sum_{k=1}^{N} \omega_k \varphi_k(r)$ and we search the ω_k coefficients by minimizing the quantity (in the 1D case):

$$\sum_{nl} \left[\frac{\Delta\nu_{nll} - \sum_{k=1}^{N} \omega_k \int_0^{R_\odot} \tilde{K}_{nl}(r)\varphi_k(r)dr}{\sigma_{nll}} \right]^2 + \lambda \sum_{k=1}^{N} \omega_k \int_0^{R_\odot} \left(\frac{d\varphi_k(r)}{dr} \right)^2 dr \tag{5}$$

  where λ is a regularization parameter chosen for establishing a balance between the goodness of the fit of data and the smoothness of the solution. This is a global linear method and then the value of the rotation obtained at any radius r_0 is a linear combination of the data:

$$\Omega_{eq}(r_0) = \sum_{nl} C_{nl}(r_0)\Delta\nu_{nll} \tag{6}$$

  By replacing in Eq. (4) we obtain:

$$\Omega_{eq}(r_0) = \int_0^{R_\odot} \left(\sum_{nl} C_{nl}(r_0)\tilde{K}_{nl}(r) \right) \Omega_{eq}(r)dr + \sum_{nl} C_{nl}(r_0)\varepsilon_{nl} \tag{7}$$

  The function in parentheses is called averaging kernel at r_0. The result obtained at r_0 will be easier to interpret as a local average of the rotation when this kernel is well-peaked and without strong oscillatory behavior. The second term of the sum shows how any bias in the data can lead to bias in the solution which depends on the amplitude of the $C_{nl}(r_0)$ coefficients and may be important.

- A Multiplicative Optimal Localized Average (MOLA) method (or Backus-Gilbert method). With this 'local method' we search directly the coefficients $C_{nl}(r_0)$ which are able to peak the averaging kernel near r_0. This is done by minimizing the quantity:

$$\int_0^{R_\odot} (r-r_0)^2 \left(\sum_{nl} C_{nl}(r_0)\tilde{K}_{nl}(r) \right) dr + \mu \sum_{nl} C_{nl}(r_0)\sigma_{nl}^2 \tag{8}$$

  subject to the constraint: $\int_0^{R_\odot} \sum_{nl} C_{nl}(r_0)\tilde{K}_{nl}(r)dr = 1$
  μ is a regularization parameter which establishes a balance between the thickness of the averaging kernel and the error on the solution.

3. Low degree splitting observations

Some observations of low degree ($l \leq 4$) splittings are summarized in **Fig. 1**. The values plotted are weighted averages over the radial order ranges shown on the bottom of the figure. The inversions using IRIS and BiSON data have been discussed in Corbard et al. (1996) but have not been used for this work. The GONG $l = 1, 2$ splittings have been estimated by Appourchaux and Rabello Soares (1997, private communication). The results discussed in Sec. 4 are for GOLF ($l = 1-3$) + LOWL ($l = 4-99$) data whereas the averaging kernels shown in Sec. 5 have been computed using GONG modeset and errors estimated by Appourchaux and Rabello Soares for $l = 1, 2$ and taken from the GONG months 4-10 dataset for $l = 3-99$.

4. Results with GOLF+LOWL data

On **Fig. 2**, the full 2D RLS inversion of Eq. (2) is shown by continuous lines for ten latitudes between the equator and the pole. Dotted lines represent the 1σ error bars on the solution. The equatorial rotation rate obtained by a MOLA 1D inversion of Eq. (4) is shown by the crosses. The widths of these crosses are related to the full widths at mid height of the averaging kernels shown at the bottom of the figure.

The two solutions are in good agreement down to $0.3R_\odot$ and tend to show a slightly decreasing rotation rate between $0.3R_\odot$ and $0.1R_\odot$. The fact that the MOLA solution obtained at $0.23R_\odot$ is lower than the solution at $0.17R_\odot$ may be related to the fact that the splitting $l=2$ is lower of about 20 nHz than the splitting $l=1$ in GOLF data (cf **Fig. 1**). We note that the averaging kernel computed at $r_0 = 0.1R_\odot$ is peaked at $0.17R_\odot$ and leads to a very important error bar (± 105 nHz). Therefore we estimate that we can not obtain any reliable estimation of the rotation below $0.2R_\odot$.

5. Study of the balance between resolution and error with GONG data

For a given modeset and the corresponding observational errors we can ask two questions concerning the ability of an inverse method to sound the rotation of the core.

- What is the deepest depth below which we can not peak averaging kernels?
- What are the width of this deepest kernel and the corresponding error on the solution?

Figure 3 shows the individual GONG $l = 1, 2$ splittings plotted against the turning points of the modes. **Figure 4** shows the averaging kernels obtained from these data (adding GONG 4-10 data for $3 \leq l \leq 99$) with MOLA method and two different regularization parameters μ (cf Eq. (8)). These two choices lead to 'thin kernel' (upper figure) or 'large kernels' (lower figure) but we can not peak kernels below $0.15R_\odot$. This gives an answer to the first question. The corresponding error on the solution (question 2) is around ± 100nHz. Therefore, in principle, we are able to give an average of the equatorial rotation rate between 0.1 and $0.2R_\odot$ within ± 100 nHz. In fact, as expected, tests show that the solutions computed using the two averaging kernels peaked below $0.2R_\odot$ are very sensitive to any bias in the data (a bias of 10 nHz in the $l = 1, 2$ splittings leads to a change of more than 100 nHz in the solutions). The dispersion of the individual splittings (**Fig. 3**) and of the weighted averages obtained from the different instruments (**Fig. 1**) show that a bias of more than 10 nHz can not be excluded. Therefore the solutions computed using these 'thin kernels' are not thought to be reliable and it seems more reasonable to use the 'large kernels' which are less sensitive to bias and lead to an error bar of about only ± 20 nHz.

The final answers to the two questions for the GONG dataset are then that we can obtain an average of the equatorial rotation rate between 0.15 and $0.35R_\odot$ within ± 20 nHz. Nevertheless this is valid only under the assumption that:

1. errors given on individual splittings are not underestimated,
2. there is no bias introduced by the inverse method.

The unique way to become confident concerning the point 1 will be to obtain less dispersion on the estimation of individual splittings (**Fig. 3**) and to obtain fully consistent estimations from all the space and ground-based instruments (**Fig. 1**).

For the point 2, as explained in Sec. 1, the reduction to a 1D inverse problem is not valid for low degrees. This may lead to systematic errors in the determination of the core rotation and we have to study this point more in details.

6. Study of the bias introduced by 1D inverse method

It is known that any bias in the data leads to very important changes on the solution obtained below $0.2R_\odot$. The 1D inverse method can also introduce a bias just because the sectoral splittings of low degrees are not sensitive only to the equatorial rotation. The latitudinal extent of the domain around the equator that influences the sectoral splittings can be estimated for each degree l by approximating the latitudinal kernel $[P_l^l(\cos\theta)]^2 \sin\theta$ that arises in Eq. (2) by a rectangular window of the same area and with a height given by the maximum value of the latitudinal kernel (i.e. C_l) which, according to Eq. (3), is its value at $\theta = \pi/2$. The width of the window $\theta_l = 1/(2C_l)$ is then such that:

$$\Omega_{eq}^l(r) \equiv \int_0^\pi \Omega(r,\theta)[P_l^l(\cos\theta)]^2 \sin\theta d\theta \simeq C_l \int_{\frac{\pi}{2}-\theta_l}^{\frac{\pi}{2}+\theta_l} \Omega(r,\theta) d\theta \quad (9)$$

Simple calculus gives for example $\theta_1 \simeq 38°$, $\theta_2 \simeq 31°$, $\theta_3 \simeq 26°$and $\theta_4 \simeq 23°$ for the low degrees. For higher degrees, the width decrease with increasing l ($\theta_{10} \simeq 15°$ down to $\theta_{99} \simeq 5°$ which corresponds to the highest degree used in this work). If we consider the surface rotation law deduced from Doppler velocity measurements (Snodgrass & Ulrich, 1990) (and which is consistent with LOWL data (Corbard et al., 1997)), the surface rotation at 38° of latitude is slower of about 63 nHz than at the equator. In first approximation, this difference remains in the whole convective zone and this may lead to an important under-estimation of the core rotation. In order to study this effect, we can calculate the difference between the sectoral splittings computed from Eq. (2) and from Eq. (4) by assuming that the rotation rate is given by the surface law in the whole convection zone and is constant below. This difference is plotted against the turning points of the modes in **Fig. 5** showing that this difference becomes rapidly important below $0.2R_\odot$ up to more than 10 nHz for the $l = 1$ modes. Adding artificially 10 nHz to the $l = 1$ splittings and 7 nHz to the $l = 2$ splittings produces no difference in the solution above $0.3R_\odot$, a slightly more rapid ($\simeq +10$ nHz) rotation between 0.2 and $0.3R_\odot$, and very important changes ($\geq +90$nHz) if we use the 'thin kernels' to infer the rotation below $0.2R_\odot$. Therefore we conclude again that the results obtained below $0.2R_\odot$ are not reliable. Furthermore this shows that a 2D MOLA method should be better to infer the rotation between 0.2 and $0.3R_\odot$. Another possibility is to calibrate the effect of neglecting the latitudinal dependence of the rotation and to take this effect into account by assuming a known rotation law in the convection zone.

7. Conclusion

This is a very preliminary work on the determination of the core rotation from low degree helioseismology. I have tried to show which are the difficulties and the limits of such investigations through inverse methods. The most important results obtained are:

- From actual data (i.e. modeset and errors) we can not obtain a reliable estimation of the core rotation below $0.2R_\odot$. If we try to use the 'thin' kernels that we can peak below this point, this leads to an error of over ± 100 nHz on the solution and the result is very sensitive to any bias in the data which can not be excluded from the comparison of the different low degree splitting observations. In turn, we expect to be able to produce an estimation of an average of the rotation rate between 0.15 and $0.3R_\odot$ which may be reliable if there is no important variation of the rotation in this domain.
- The 1D approximation (Eq. (4)) concerning only the equatorial rotation may lead to an underestimation of the core rotation. Thus, in future work, we have to use a full 2D inversion or, at least, to calibrate this effect when using 1D inversions.
- The results obtained below $0.3R_\odot$ with the global RLS method are difficult to interpret because of the oscillatory behavior of the averaging kernels. Nevertheless, it can be used in combination with MOLA method by showing modes which are suspected to have underestimated errors.

From this work, there is no real conclusion about the value of the rotation rate below $0.3R_\odot$. More work and calibrations of the methods, taking into account the above points, are needed for this. We note however, that if the results obtained from the 2D inversion performed on LOWL+GOLF data are to be believed, the solution is consistent with a constant or slightly decreasing rotation rate between 0.4 and $0.2R_\odot$.

Acknowledgments. The numerical computations have been performed on the Cray C98 (IDRIS, CNRS, Orsay). We gratefully acknowledge S. Tomczyk and J. Schou for providing LOWL data, T. Appourchaux, C. Rabello Soares and the GONG team for providing GONG data and M. Lazrek and the GOLF team for providing GOLF data.

REFERENCES

1. Antia H.M., Chitre S.M., Thompson M.J., 1996 The Sun's rotation in its equatorial plane, A&A 308, 656
2. Chaplin W.J., Elsworth Y., Howe R., 1996, Solar core rotation: low-degree solar p-mode rotational splitting results from BISON, MNRAS 280, 849
3. Corbard T., Berthomieu G., Provost J., Fossat E., 1996, The solar core rotation from LOWL and IRIS or BiSON data, To appear in Proceedings of IAU181 Symposium
4. Corbard T., Berthomieu G., Provost J., et al., 1997a, The solar rotation rate from LOWL data: A 2D regularized least-squares inversion using B-splines, 1996, A&A 324, 298
5. Corbard T., Berthomieu G., Provost J., Morel P., 1997b, Inferring the equatorial solar tachocline from frequency splittings, submitted to A&A
6. Cuypers J., 1980, On the calculation of the frequency splitting of adiabatic nonradial stellar oscillations by slow differential rotation, A&A 89, 207
7. Gelly B., Fierry-Fraillon D. et al., 1997, Solar P-mode frequencies from the IRIS network, A&A 323, 235
8. Lazrek M., Pantel A., Fossat E., et al., 1996, Is the solar core rotating faster or slower the the envelope?, Sol. Phys., 166, 1
9. Snodgrass H.B., Ulrich R.K., 1990, Rotation of Doppler features in the solar photosphere, ApJ 351, 309

Figures

Fig.1: Weighted averaged low degree splittings from various observations. The corresponding radial order ranges are shown at the bottom.

- IRIS measurements cover a total of 19 months of observations with 4 time series obtained in 1989, 1990, 1991 and 1992 (Gelly et al. 1996). The given splittings are the weighted averages of the results obtained by different methods applied on the same averaged spectra (Lazrek et al. 1996).
- BiSON data have been collected between 05/01/93 and 08/23/94 by Chaplin et al. (1996)
- LOWL values have been calculated from the data covering 2 years of observation between 2/26/94 and 2/25/96 (communicated by S. Tomczyk).
- GONG data ($l = 1, 2$) have been analyzed by T. Appourchaux & C. Rabello Soares from the GONG observations made between 02/26/95 and 02/26/96 (private communication). ($l = 3, 4$) are for 7 months of observations (GONG months 4-10) (communicated by the GONG team)
- GOLF splittings have been estimated by M. Lazrek from the first eight months (04/11/1996-12/31/96) of GOLF observations (private communication)

Fig.2: 2D RLS inversion of LOWL+GOLF data at ten latitudes between the equator and the pole(full lines). Dotted lines are the corresponding 1σ error bars. The crosses represent the results obtained from 1D MOLA inversion in the equatorial plane. The corresponding MOLA averaging kernels are shown at the bottom.

Fig.3: Individual synodic splittings GONG $l = 1, 2$ (Appourchaux & Rabello Soares, private communication) plotted against the turning points of the corresponding modes.

Fig.4: Averaging kernels computed between 0.1 and $0.5R_\odot$ with a step of $0.05R_\odot$. These kernels have been obtained from GONG data and the MOLA method. The regularization parameter is 100 times higher in the lower figure than in the upper one leading respectively to 'large' and 'thin' kernels.

Fig.5: Difference between the sectoral splittings computed from Eq. (2) and from Eq. (4). The rotation rate is assumed to be equal to the surface law in the whole convection zone and constant below. The difference is plotted against the turning points of the modes.