Astron. Astrophys. 293, 605–612 (1995)

ASTRONOMY
AND
ASTROPHYSICS

Comparison between two theories of asteroid proper elements

Z. Knežević[1], Ch. Froeschlé[2], A. Lemaitre[3], A. Milani[4], and A. Morbidelli[2]

[1]Astronomical Observatory, Volgina 7, 11050 Belgrade, Yugoslavia
[2]Observatoire de la Côte d'Azur, B.P. 229, F-06304 Nice Cedex 4, France
[3]Department of Mathematics, FUNDP, 8 Rempart de la Vierge, B-5000 Namur, Belgium
[4]Department of Mathematics, University of Pisa, Via Buonarroti 2, I-56127 Pisa, Italy

Received 9 May 1994 / Accepted 5 July 1994

Abstract. A comparison between two theories for the calculation of proper elements is presented in this paper; the first theory, purely analytical, has been developed by Milani & Knežević (1990, 1992, 1994); the second one, a semi-numerical approach, is due to Lemaitre & Morbidelli (1994). The analytical theory, based on series expansions in eccentricity and inclination, is particularly suitable for low inclination and low eccentricity orbits, while the semi-numerical one is devoted to orbits with either large eccentricities or large inclinations. The orbits of the minor planets *3710, 1021, 387, 980, 185* have been computed numerically for 4.5 Myrs, and the proper elements have been derived with both algorithms. The RMS values of the changes with time of the proper elements are taken as a measure of the instability. The results confirm the theoretical predictions, namely the Milani and Knežević proper elements are more stable at low e and I, while the stability of the Lemaitre & Morbidelli proper elements depends very little upon eccentricity and inclination. As a result, the MK elements should be used below about 15° of inclination, and the LM elements should be used above about 17° of inclination. The region between 15° and 17° should be considered as a transition region where both methods have roughly the same stability; therein the use of both data sets and the comparison of the results is recommended. The small values of the instabilities confirm that our long term research program, aimed at providing proper elements for the purpose of asteroid family identification, has been successful, to the point that reliable proper elements are now available for all the regions of the asteroid main belt (and also beyond the main belt, taking into account other available results by ourselves and by others).

Key words: celestial mechanics – minor planets – asteroids

1. Introduction

It is now well known that the identification of dynamical families in the asteroid belt requires the precise determination of proper elements of asteroids. Since the gravitational perturbations due to the planets cause time variations of the osculating orbital elements, if one assumes that asteroid families are the products of the collisional disruption of a parent body, it is necessary to use the so-called proper elements, i. e. quasi constants of motion, in order to identify the objects which have been on very close orbits many Myrs ago (just after the disruption occurred).

The history of the mathematical research for the computation of accurate proper elements is characterized by two main approaches which were developed in parallel: the analytical approach and the semi–numerical approach.

The pioneer of the analytical approach was Hirayama (1923), who computed asteroid proper elements using a linear perturbation theory. Afterwards several authors have determined proper elements using essentially the same method (for an exhaustive list see Valsecchi et al. 1989). However, the linear perturbation theory is not accurate enough for the systematic identification of all the families. Yuasa (1973) developed a more sophisticated theory of asteroid secular perturbations, which takes into account also the terms of higher order (in mass ratio) and degree (in eccentricities and inclinations). Knežević (1988, 1989) corrected and completed Yuasa's theory. Starting from this solution, Milani & Knežević (1990) computed proper elements by developing a new theory based on the Lie series formalism and including an iterative algorithm. Moreover, they published improved versions of their algorithm (Milani & Knežević 1992, 1994), which include the perturbation effects of the four major planets and part of the effects of the inner planets, as well as many other advanced features. The main limitation of analytical theories is that they loose accuracy for large values of eccentricity and inclination of the asteroid, since they use truncated expansions in powers of e and I. Moreover the dynamics is averaged over the argument of perihelion ω assuming it to be a sort of "fast" variable, which is no longer the case at large inclination. Conversely, analytical theories minimize the sources of numerical errors and allow to take into account higher degree planetary perturbation terms (for example terms of degree 2 and 3 in eccentricity e' and inclination I' of the planets).

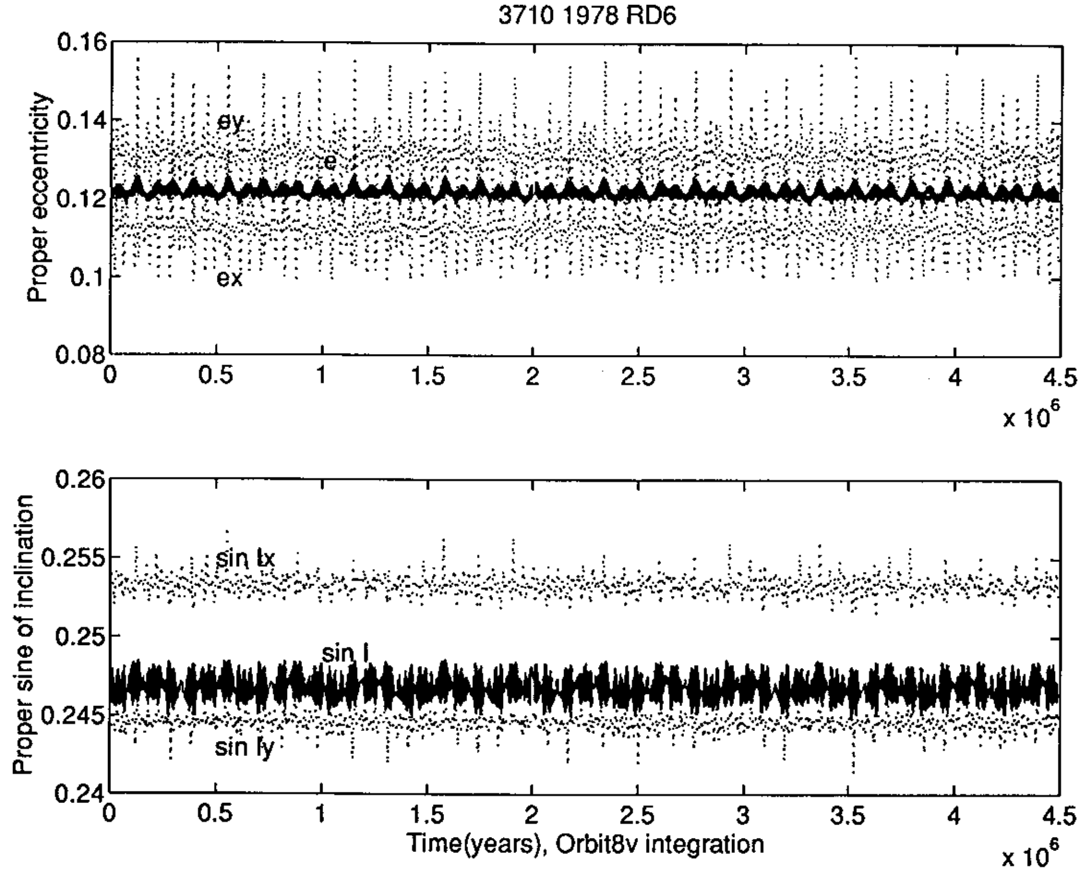

Fig. 1. Minor planet *3710* : proper eccentricities (e_{MK}, e_x and e_y) and proper sine of inclination ($\sin I_{MK}$, $\sin I_x$, $\sin I_y$) as functions of time over 4.5 Myrs

The pioneer of the semi-numerical approach was Williams (1969) who, using Gauss numerical averaging, developed a method which did not involve any expansion and truncation in terms of powers of the asteroid eccentricity and inclination, so that precise proper elements can be computed also for high inclination objects. Moreover, his theory takes completely into account the strongly non-linear dynamics related to the motion of the argument of perihelion of the small body which is dominant at high inclination, as shown by Kozai (1962). However Williams' proper elements are computed taking only account of the terms of the first order in the planetary masses, so that near the mean motion resonances for example, where higher order terms become large, their accuracy is seriously degraded. Lemaitre & Morbidelli (1994) revisited the use of semi-numerical methods; they employed the semi-analytical averaging method by Henrard (1990), and took into account the quadratic terms in the mass ratio. Following Milani and Knežević, they also used an iterative procedure and the improved values of amplitudes, phases and frequencies of the secular planetary motion, computed numerically by Nobili et al. (1989). A problem with semi-numerical theories is that they are computationally expensive and sensitive to numerical errors. Moreover, only the linear terms in the eccentricity and the inclination of the perturbing planets can be taken into account in a consistent way; on the other hand they are very suitable to study asteroids with large eccentricity and/or inclination.

It is therefore intuitive that the analytical approach and the numerical approach are complementary, the first one producing more accurate results for asteroids with moderate inclination and eccentricity (the majority of the population), the second one being preferable for highly inclined or eccentric objects (a minority of nevertheless great importance).

The main purpose of this paper is to provide a quantitative comparison between Milani and Knežević results (MK, analytical approach) and Lemaitre and Morbidelli results (LM, semi-numerical approach). In particular we want to determine which value of the inclination is the threshold below which MK theory should be used and above which LM theory is preferable, and possibly, determine also a region of *overlapping* where both theories are equivalent, in the sense that their accuracy is comparable. This will be done by numerically integrating for 4.5 million years some real asteroids with different inclinations and testing the stability of MK and LM proper elements over the full time span. The paper is organized as follows: in Sect. 2, we briefly recall the main properties of the two theories, giving comparative advantages and limitations for each of them. The description of quantitative experiments on a set of asteroids and their results are given in Sect. 3. Based on these results, we

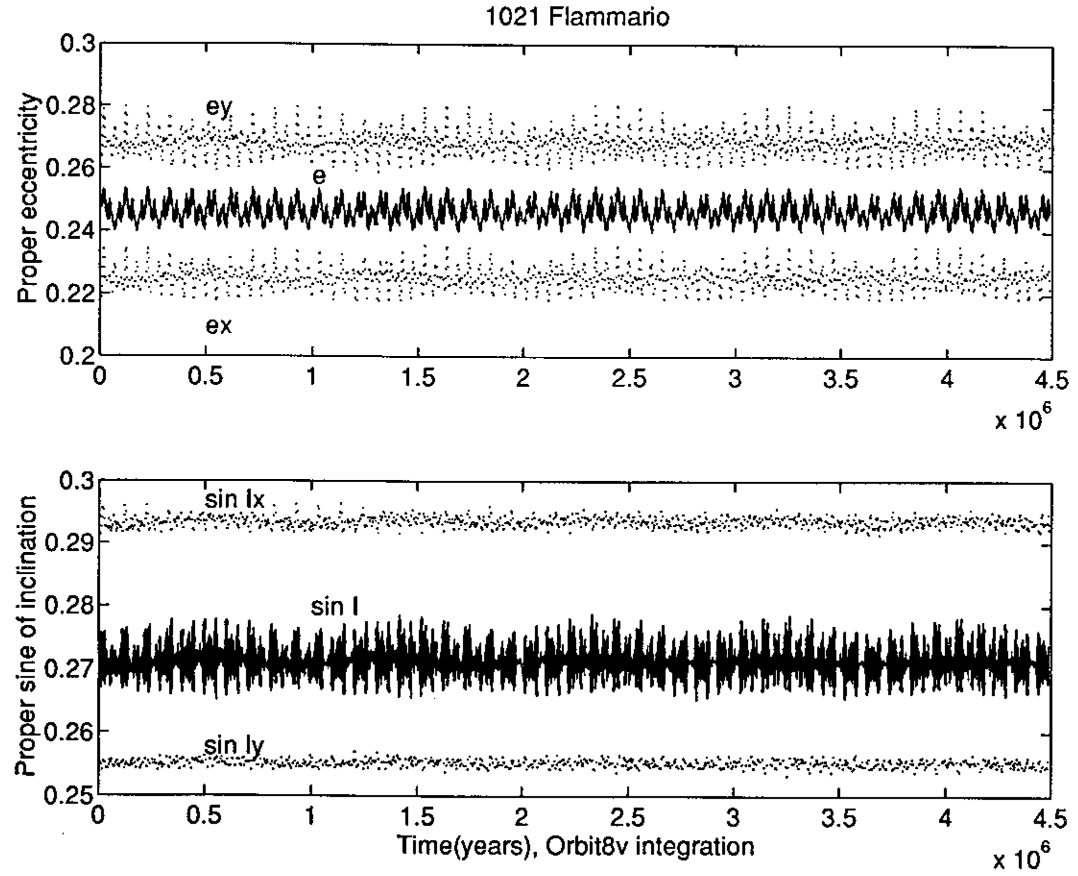

Fig. 2. Minor planet *1021* : proper eccentricities (e_{MK}, e_x and e_y) and proper sine of inclination ($\sin I_{MK}$, $\sin I_x$, $\sin I_y$) as functions of time over 4.5 Myrs

give in the conclusions some suggestions for users who want to identify and study asteroid families, starting from our proper elements catalogues.

2. Overview of the methods

In this part we will recall the main features of the two theories; for more details refer to Milani & Knežević (1990, 1992, 1994) and Lemaitre & Morbidelli (1994).

2.1. Milani – Knežević (MK) theory

In its present form the MK theory takes into account terms of up to the degree 4 in eccentricity and inclination of the asteroid and the perturbing planet(s) in the first order with respect to perturbing masses (and some of even higher degree), and terms of degree 2 in the second order (plus terms of degree 4 with the square of the small divisor 3:1). The improvements introduced by Milani and Knežević with respect to the original theory by Yuasa are many: let us mention here just a few most significant ones, like: rearrangement of the perturbing Hamiltonian in such a way to include degree 2 second order terms into the integrable part (the second order linear theory), introduction of an iterative procedure to handle the variable frequency problem, designing of an indirect method to estimate the instability of resulting mean and proper elements, switching to an adapted theory close to the secular resonances, numerous technical improvements (better major planet secular theory, more realistic dynamical model, generic term technique) etc. The version of the theory used in this paper accounts for the short-periodic perturbations by Jupiter and Saturn only, since this proved to be enough (Knežević et al. 1988) for the derivation of accurate mean elements. For the long periodic perturbations, in accordance with a dynamical model used in the integrations (see Sect. 3), the contribution of major planets Jupiter to Neptune was used for the computation of forced perturbations and linear frequencies, Jupiter and Saturn for the degree 4 terms, and Jupiter only for the second order terms. In practice, one performs a canonical transformation to remove first the short-periodic perturbations and obtain the mean elements. Then, the resulting Hamiltonian is split in the integrable part and the long-periodic perturbation, and the solution of the integrable part is computed. By substituting this solution in the long-periodic perturbation and performing another canonical transformation, one finally obtains the proper elements. Due to the fact that the transformation from mean to proper elements is an implicit one (variable frequency problem), an iterative procedure is introduced, and the convergence of this procedure serves as an indicator of the quality of results. The divergence occurs in the vicinity of resonances.

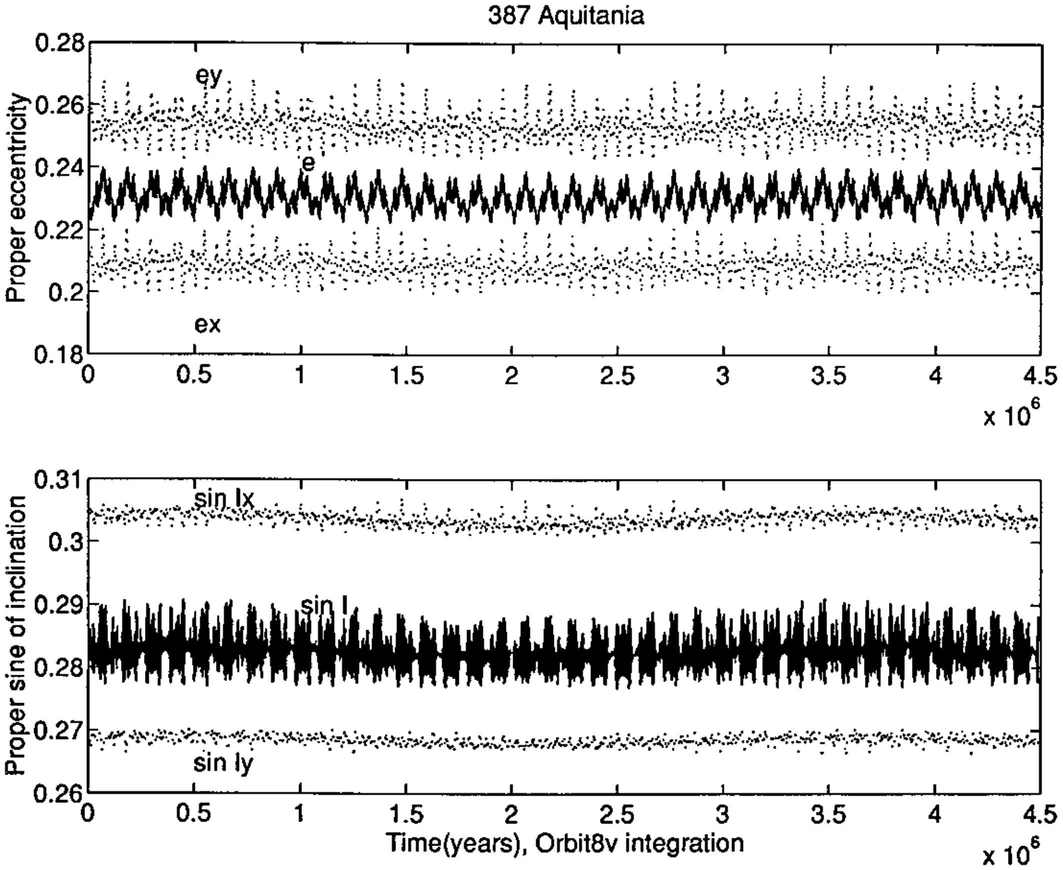

Fig. 3. Minor planet *387* : proper eccentricities (e_{MK}, e_x and e_y) and proper sine of inclination ($\sin I_{MK}$, $\sin I_x$, $\sin I_y$) as functions of time over 4.5 Myrs

2.2. *Lemaitre – Morbidelli (LM) theory*

The LM theory is a semi-numerical one which does not involve any truncation of high powers of the asteroid eccentricity and inclination. The Hamiltonian is split in two parts, an integrable one (2-body problem) and a perturbation part due to the planets. All the short periodic terms are eliminated, using a Lie algorithm to average the Hamiltonian up to order 2 in planetary masses. Again the resulting mean Hamiltonian is separated into two parts: an integrable part and a "perturbation". The integrable part is the mean Hamiltonian where the planets are assumed to be on circular and coplanar orbits; then this Hamiltonian depends only of one angle, the argument of perihelion ω. This is the Hamiltonian first studied by Kozai (1962) which gives a strongly non-linear dynamics at large inclination. The perturbation part contains only the terms which are linear in the planetary eccentricities e' and inclinations I'. Only Jupiter and Saturn are taken into account in the model, which includes 7 forcing frequencies ($g_5, g_6, g_7, 2g_6 - g_5, s_6, s_7, s_8$).

Introducing Arnold action angles variables the integrable part of the Hamiltonian (Morbidelli & Henrard 1991) is transformed into a new one which is function of the actions only and is characterized by two constant frequencies which are the proper free frequencies of the longitude and of the argument of perihelion.

Following Henrard (1990) a second averaging over the long periodic angles is performed. The proper action angles variables are then calculated by solving the implicit relation which connects the proper variables to mean ones, by an iterative process.

The proper actions are defined with reference to the integrable Kozai approximation, which is geometrically very different from the classical linear secular perturbation theory. As a result, a given value of the proper actions identifies a "proper orbit" on which e and I oscillate with the motion of the perihelion argument ω, within a range which can be wide in the case of large inclination. The proper e and I can be defined as the values occurring when ω is either 0° (e_x, I_x) or 90° (e_y, I_y); they correspond to the minimum of e and maximum of I in the first case and conversely to a maximum of e and a minimum of I in the second case.

2.3. *The comparative advantages*

Although the two theories have some common features (two averaging processes to derive asteroid proper elements, iterations, Hamiltonian formalism, use of the same major planet secular theory, poor handling of the resonant cases, etc.), even from their above brief description it is already clear what are their relative advantages and disadvantages with respect to each other. As already mentioned, the theory by Milani and Knežević

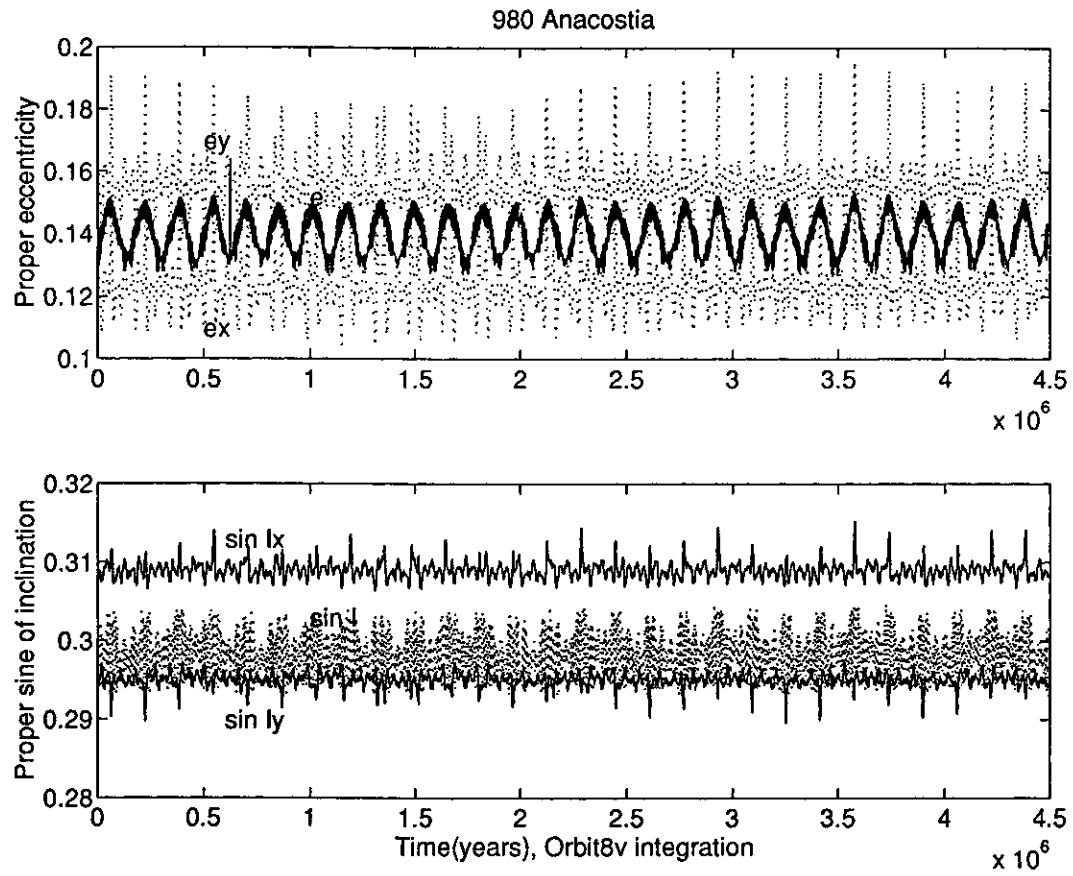

Fig. 4. Minor planet *980* : proper eccentricities (e_{MK}, e_x and e_y) and proper sine of inclination (sin I_{MK}, sin I_x, sin I_y) as functions of time over 4.5 Myrs

is based on a truncated expansion of the disturbing function, and thus can be successfully applied only to objects with low to moderate eccentricity and/or inclination; the theory of Lemaitre and Morbidelli, not using any expansion with respect to the asteroid elements, is applicable to any e and I. The MK theory splits the perturbing Hamiltonian, in accordance with the truncation hypothesis, in such a way that the important fourth degree term $e^2 I^2 \cos(2\omega)$ belongs to the "perturbation", while LM one successfully handles it within the integrable part. Conversely, the analytical theory uses planetary elements to the same order/degree as these of the asteroid, while the semi-analytical theory keeps only corresponding linear terms.

For these reasons, it is obvious that the stability of the MK proper elements is good at low eccentricities and inclinations, but degrades for increasing e, I, while for the LM proper elements one expects essentially the same stability for any value of the asteroid elements, although somewhat worse than the stability of MK at low e and I. However, we need to confirm these qualitative arguments with stability tests over long time spans, and determine the boundary at which the users should switch from one set of data to the others.

3. Stability tests

We have designed an experiment which consisted in the following steps. First we compute the orbits of a number of asteroids, by numerical integration, for a time span longer than the typical periods of the secular perturbations (4.5 million years). We have used the numerical integration software Orbit8v written by one of us (A.M.), and which is distributed with the public domain proper elements data base; the dynamical model includes the perturbations from the four major outer planets. Second, we remove the short periodic perturbations from the output of the numerical integration by digital filtering, with the methods developed by Carpino et al. (1987); this generates a time series of mean elements, one value every 1,000 years. As an alternative, we could have used the analytical procedure to generate mean elements by Milani & Knežević (1990, 1994); we preferred the numerical filtering procedure, because the analytical calculation of mean elements is also truncated to degree four in e, I and therefore could introduce some instability which is a separate problem from the stability of the proper elements, and which we are not discussing here. Third, we compute proper elements time series with both the analytical and the semi-analytical theory; we take into account the need to use secular forcing frequencies adapted to the dynamical model of the numerical integration, as discussed in Milani & Knežević (1992, Sect. 3). Finally, we

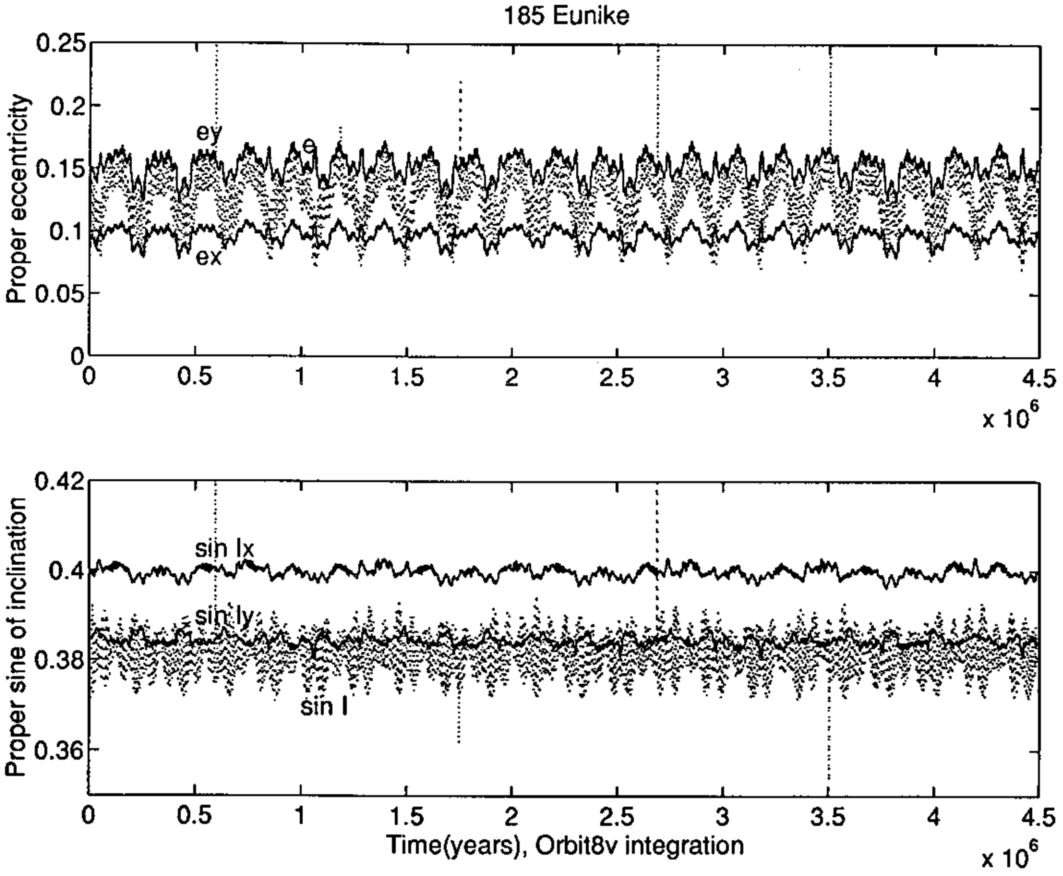

Fig. 5. Minor planet *185* : proper eccentricities (e_{MK}, e_x and e_y) and proper sine of inclination ($\sin I_{MK}$, $\sin I_x$, $\sin I_y$) as functions of time over 4.5 Myrs

use as measure of the instability of each proper element the root mean square of the dispersion with respect to the average over the entire time span. For the proper frequencies g, s we can also compare the values computed by both theories with the value obtained by a linear fit to the proper angles, which is –within a very good approximation– a "true" value.

The choice of the test asteroids was made in such a way to cover a range in proper inclination below and above the $g - g_6$ secular resonance, where we could expect the transition region to occur, while keeping at the same time the proper semimajor axes of the selected bodies as close to each other, and as far from low-order mean motion resonances, as possible. Among the asteroids which fulfill these conditions we have selected the five asteroids *3710, 1021, 387, 980, 185* (in order of increasing inclination). The proper eccentricities are moderate-to-high; no special selection arrangement was made with respect to e. The reason for giving a priority to inclination is that the frequency $g - s$ of the argument of perihelion ω depends more upon I than upon e, and the critical parameter for the quality of the MK proper elements is the distance from the Kozai resonance ($g - s = 0$). We excluded some other test cases because they were in the main $g - g_6$ secular resonance, where proper elements should be computed by a different algorithm (Morbidelli 1993). We have also avoided asteroids affected by the nearby $13/5$ mean motion resonance, because they are strongly chaotic and the instability of the proper elements for these objects does not measure the quality of the algorithms.

The results of our tests are summarized in Tables 1 and 2. In Table 1 we give for each asteroid the mean semimajor axis, the average values of MK proper e and $\sin I$ with the RMS of the dispersion, the average values of the LM proper elements $e_x, \sin I_x, e_y, \sin I_y$ with the RMS of the dispersion. Table 2 contains the average values of the proper frequencies g and s as computed by both algorithms, together with the values obtained by fit to the proper angles. The same results are presented graphically in Figs. 1–5.

The results of this test can be summarized as follows. The theoretical predictions (see Sect. 2) are confirmed: for lower I the MK proper elements are more stable, while at high I the LM ones are more stable. The eccentricity is always somewhat more stable for MK, but the difference becomes irrelevant for *387, 980*; the inclination is always somewhat more stable for LM, but the difference is irrelevant for *3710*. This can be understood taking into account that the four frequency terms, which are not accounted for by LM, contain higher powers of e' and I', but I' is much smaller than e'; thus the unremoved long periodic perturbations affecting the inclination are much smaller than the ones affecting the eccentricity. The instability of the LM

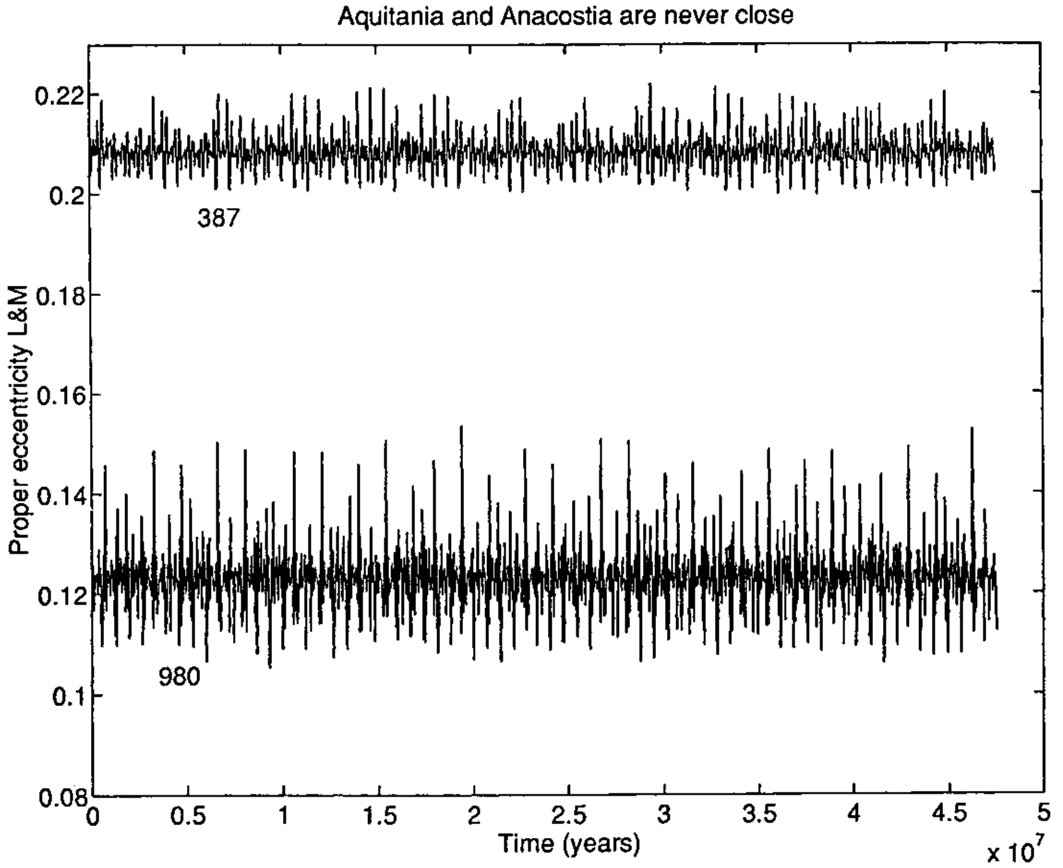

Fig. 6. Minor planets *387* and *980* : proper eccentricities e_x as functions of time over 50 Myrs

proper elements does not increase in a systematic way with the inclination, as expected, while the degradation of the MK proper elements due to the truncation in I becomes apparent above 15°.

The proper frequencies computed by the LM algorithm are in all tests very close to the fit values, while the MK values for s are significantly less accurate in two cases (*1021, 387*). This could be interpreted as an effect of the computation of the second order (in the perturbing mass) contribution, since the MK truncation does not allow to account for some of the main second order corrections in s. This comes from the D'Alembert rule by which there are no terms in the perturbing function of degree one in the inclinations (see Milani & Knežević 1994, Sect. 5.1).

The case of *185 Eunike* illustrates the expected inconsistency of the MK type truncation at inclinations above the $g - g_6$ resonance; actually the MK proper elements are not defined, since the iterative procedure is divergent in most cases. For the proper eccentricity, it is apparent from Fig. 5 that the divergent MK iteration oscillates between values close to the maximum value e_y and the minimum value e_x computed by the LM algorithm; this confirms that the MK algorithm cannot handle correctly the 2ω main term of the Kozai approximation at such a high inclination. On the contrary the LM proper elements are stable enough for the identification of asteroid families even at very high inclination, as confirmed also by other tests in the Hungaria region (Lemaitre 1994).

The two test cases *387 Aquitania* and *980 Anacostia* have been proposed as a genetically related pair (Burbine et al. 1992) on the basis of the similarity of their spectra; they also have comparatively small differences in proper a and I. However, given the results of our stability tests over several million years, there is no way to explain the difference in proper e by 0.09. This difference would imply for the relative velocity of two fragments from the same parent body a value of at least $1,600\ m/s$; within this distance in the velocity space there are so many asteroids that even the correlation between two spectra is not statistically significant. We have performed longer tests (50 Myrs) for these two minor planets, to confirm the stability of our results and the absence of drift on very long periods of time. The result for the eccentricity is shown in figure 6 and the stability is exactly the same for 50 Myrs than it was for 5 Myrs.

4. Conclusions

The purpose of this work is to show that there are now proper elements stable enough for asteroid family identification in essentially all the regions of the asteroid main belt (for the regions outside the main belt, see Milani 1993, 1994; Schubart 1982, 1991; Morbidelli 1993). However, the users of proper elements catalogues should preferably use the MK proper elements be-

Table 1. Tests results : for each minor planet the first line represents e_{MK}, $\sin I_{MK}$ and the corresponding RMS, the second lines corresponds to e_x and $\sin I_x$, the third one to e_y and $\sin I_y$ taken from LM results, with the associated RMS values

N°	a (AU)	Proper e	RMS	Proper $\sin I$	RMS
3710	2.738944	.12182	.0013	.24680	.0007
		.11228	.0054	.25328	.0006
		.13099	.0063	.24456	.0005
1021	2.737802	.24655	.0030	.27138	.0023
		.22508	.0032	.29354	.0008
		.26834	.0037	.25523	.0006
387	2.739151	.23071	.0037	.28268	.0028
		.20843	.0038	.30383	.0010
		.25348	.0045	.26855	.0007
980	2.741016	.13995	.0064	.29785	.0026
		.12311	.0074	.30905	.0011
		.15578	.0092	.29502	.0009
185	2.738547	.12942	.0204	.38128	.0045
		.09723	.0062	.39970	.0012
		.15252	.0095	.38449	.0011

Table 2. Tests results : for each minor planet three values of the frequency g (longitude of the pericenter) and of the frequency s (longitude of the node) are given, the first one calculated by MK theory, the second one calculated by LM theory and the third one is numerically fit to the proper angles; they are all given in arcseconds per year

N°	g_{MK}	g_{LM}	g_{fit}	s_{MK}	s_{LM}	s_{fit}
3710	42.805	42.984	43.462	-53.707	-53.614	-54.001
1021	40.748	40.431	41.064	-66.296	-64.126	-64.184
387	38.571	38.577	39.303	-63.256	-61.280	-61.381
980	34.445	35.524	36.342	-51.789	-51.757	-52.390
185	16.356	20.377	22.039	-43.640	-44.968	-45.993

low a boundary which can be approximately determined to be at 16°, and the LM elements above that boundary. Since it is not correct to compute differences between proper elements computed with different theories, the users need to use both methods on an overlap region which must have a width larger than the size of the largest family. We recommend to use both methods and to compare the results at least between 15° and 17°; as an example of this comparison, see the Maria family shown in Fig. 3 in Knežević & Milani (1994).

The proper elements derived by both the MK algorithm (for low to moderate e, I) and by the LM algorithm (for high e and/or I) are public domain, and can be obtained by `anonymous ftp` on the `internet`, from the server `gauss.dm.unipi.it` (directory `pub/propel`). Future upgrades will be available in the same way; please consult the README files for the description of the current versions.

References

Burbine, T.H., Gaffey, M.J., and Bell, J.F.: 1992, S–asteroids 387 Aquitania and 980 Anacostia: possible fragments of the breakup of a spinel–bearing parent body with CO3/CV3 affinities. *Meteoritics* **27**, 424–434.

Carpino, M., Milani, A., and Nobili, A.M.: 1987, Long-term numerical integrations and synthetic theories for the motion of the outer planets. *Astron. Astrophys.* **181**, 182–194.

Henrard, J.: 1990, A semi-numerical perturbation method for separable hamiltonian systems. *Celestial Mechanics* **49**, 43–67.

Hirayama, K: 1923, Families of asteroids. *Japan J. Astron. Geophys.* **1**, 55–93.

Knežević, Z.: 1988, Asteroid mean orbital elements. *Bull. Astron. Obs. Belgrade* **139**, 1–6.

Knežević, Z.: 1989, Asteroid long–periodic perturbations: The second order Hamiltonian. *Celestial Mech.*, **46**, 147–158.

Knežević, Z., and Milani, A.: 1994, Asteroid proper elements: the big picture. In *Asteroids Comets Meteors 1993* (A. MIlani, M. Di MArtino, and A. Cellino, Eds.), pp. 143–157.

Knežević, Z., Carpino, M., Farinella, P., Froeschlé, Ch., Froeschlé, Cl., Gonczi, R., Jovanović, B., Paolicchi, P., and Zappalà, V.: 1988, Asteroid short-periodic perturbations and the accuracy of mean orbital elements. *Astron. Astrophys.* **192**, 360–369.

Kozai, Y.: 1962, Secular perturbations of asteroids with high inclination and eccentricities. *Astron. J.*, **67**, 591–598.

Lemaitre, A.: 1994, Hungaria: a potential new family. In *Seventy-five Years of HIRAYAMA Asteroid Families: The Role of Collisions in the Solar System History*, (Y. Kozai, R.P. Binzel, and Th. Hirayama, Eds.), PASP Conf. Proc., in press.

Lemaitre, A., and Morbidelli, A.: 1994, Calculation of proper elements for high inclined asteroidal orbits. *Celestial Mechanics*, in press.

Milani, A.: 1993, The Trojan asteroid belt: proper elements, stability, chaos and families. *Celestial Mechanics* **57**, 59–94.

Milani, A.: 1994, The dynamics of the Trojan asteroids. In *Asteroids Comets Meteors 1993* (A. Milani, M. Di Martino, and A. Cellino, Eds.), pp. 158–174.

Milani, A., and Knežević, Z.: 1990, Secular perturbation theory and computation of asteroid proper elements. *Celestial Mech.* **49**, 247–411.

Milani, A., and Knežević, Z.: 1992, Asteroid proper elements and secular resonances. *Icarus* **98**, 211–232.

Milani, A., and Knežević, Z.: 1994, Asteroid proper elements and the dynamical structure of the asteroid main belt. *Icarus*, in press.

Morbidelli, A.: 1993, Asteroid secular resonant proper elements. *Icarus* **105**, 48–66.

Morbidelli, A., and Henrard, J.: 1991, Secular resonances in the asteroid belt: Theoretical perturbation approach and the problem of their location, *Celestial Mech.*, **51**, 131–168.

Nobili, A. M., Milani, A., and Carpino, M.: 1989, Fundamental frequencies and small divisors in the orbits of the outer planets. *Astron. Astrophys.*, **210**, 313–336.

Schubart, J.: 1982, Three characteristic parameters of orbits of Hilda-type asteroids. *Astron. Astrophys.* **114**, 200–204.

Schubart, J.: 1991, Additional results on orbits of Hilda-type asteroids. *Astron. Astrophys.* **241**, 297–302.

Valsecchi, G. B., Carusi, A., Knežević, Z., Kresák, Ľ., and Williams, J.G.: 1989, Identification of asteroid dynamical families. In *Asteroids II* (M. Binzel, T. Gehrels and M. S. Matthews, Eds), pp. 368–385.

Williams, J.G.: 1969, *Secular perturbations in the Solar System*, Ph.D. Dissertation, Univ. of California, Los Angeles.

Yuasa, M.: 1973, Theory of secular perturbations of asteroids including terms of higher order and higher degree. *Publ. Astron. Soc. Japan*, **25**, 399-445.