The 1/1 Mean Motion Commensurability: General Perturbation Formalism and Resonant Structure

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Abstract. We describe here a formalism which provides a non-singular evaluation of the disturbing function and its derivatives with respect to canonical variables. In particular, we define canonical variables of a small body which are useful in the restricted model of the 1/1 mean motion commensurability. In this configuration, the perturbed massless body orbits the central body (Sun) with a period equal to that of the perturbing (planetary) body. This situation is usually known as 'co-orbital' motion. The formalism is valid, unlike the classical expansions of the disturbing function, for any value of eccentricities and inclinations of the small body and planet. The adopted procedure is similar to the one of Moons (1993) and gives a complement to her work for a 1/1 mean motion resonance. The perturbation theory is used to compute the main features of the co-orbital dynamics in three approximations of the general three–body model: the *planar–circular*, *planar–elliptic*, and *spatial–circular* models. We develop a new perturbation scheme, which allows us to treat cases where the classical perturbation method fails. We show how the families of tadpole, horseshoe, satellite and compound orbits vary with eccentricity and inclination of the small body, and compute them also for eccentricity of the perturbing body corresponding to an eccentric exoplanet orbit. These and other results are compared with previous, more limited studies.

Key-words: restricted three-body problem - disturbing function - mean motion resonances - Lagrange equilibrium points - tadpole and horseshoe orbits - average motion.

1. Introduction

It is evident that the only true singularities of the Hamiltonian and of its derivatives in the three–body restricted problem correspond to collisions between bodies (when the force function is infinite). For this reason, if the variables are carefully chosen, one expects that no singularity exists at zero eccentricities and/or zero inclinations, simply because this does not correspond to any collision between bodies. There appears, however, a fake singularity when the Hamiltonian is expressed in usual Delaunay elements, for instance. This singularity is similar to the singularity which appears in the transformation from Cartesian to polar coordinates: such transformation has a zero determinant of the matrix of partial derivatives at the coordinates' center. As a consequence it happens that functions that are smooth in Cartesian coordinates have singular derivatives in the polar coordinates.

This work benefits from the algorithm developed by Ferraz-Mello (1987), Ferraz-Mello & Sato (1989), and Roig *et al.* (1998). The so-called *asymmetric expansion*[1] of the disturbing function was originally devised to treat the cases of the commensurable motion at high eccentricities, where the classical Laplacian series are not convergent. Assume a small body and a planet with periods in a $(p+q)/p$ ratio, with p and q integers. In such situation, the small body is said to be in the $(p+q)/p$ mean motion resonance (commensurability) with the planet. Ferraz-Mello (1987) first explicited the formalism of the asymmetric expansion in the planar model –when the small body and the planet move in the same plane– and for $q = 1$. The mean motion resonances with $q = 1$ happen relatively close to the planet where the Laplace expansion is invalid already for a moderate eccentricity of the small body (Sundman 1916, Ferraz-Mello 1994). Such is, for example, the case of Hilda asteroids in the 3/2 Jovian resonance (∼4 AU) where the Laplace series is divergent for eccentricities larger than 0.09. The algorithm was generalized for the mean motion resonances with $q \geq 1$ by Ferraz-Mello & Sato (1989) and to the inclined case by Roig *et al.* (1998).

[1] This terminology recalls the fact that the expansion is centered at any given point in $(e\cos\sigma, e\sin\sigma)$ plane, where e is the eccentricity and σ is the resonant angle (see later).

The asymmetric expansion of the disturbing function is a local expansion around an orbit with non-zero e and/or i. For this reason, a perturbation approach working with this expansion is only locally valid. In particular, the precision of the asymmetric–expansion approximation of the Hamiltonian function degrades for asteroids with large libration amplitudes (Roig *et al.* 1998). For this reason, Moons (1993, 1994) adopted a slightly different procedure which evaluates the Hamiltonian function in resonant canonical variables and in its exact form. In Moons' algorithm, however, the Hamiltonian function is not given as series expansion, which poses a difficulty when a perturbation theory is applied: the averaging over angles can be done only numerically and the results cannot be obtained explicitly by hand-writing (which is, in fact, not a big problem in modern times).

The algorithm developed by Moons (1994) is not valid for the 1/1 mean motion resonance because the canonical variables appropriate to a $(p+q)/p$ resonant condition, defined from the Schubart's (1964) ones, become singular when $q = 0$. The purpose of the present study is the extension of the work of Moons (1994) to the 1/1 resonant condition by a choice of canonical variables specific to this resonance. We will see in the following that the main difficulty is not in the definition of these variables but in the construction of a non–singular expression of the disturbing function, which differs in several aspects from the $q \neq 0$ case.

The developed formalism will be applied in a forthcoming paper to a systematic study of secular dynamics of orbits surrounding the Lagrange equilibrium points (i.e. of the tadpole and horseshoe orbits) of the solar system planets. The pre-requisite for such study, a good knowledge of the behavior of the resonant angle (the resonance structure as we sometimes call it, meaning the number and position of stationary points, their linear stability, shape of separatrices, etc.), will be addressed here, showing a simple application of the new formalism. The secular evolution is computed in this paper only for zero–amplitude tadpole and retrograde satellite orbits of a Jupiter–mass planet. This is done with a new method (inspired by Wisdom's (1985) perturbation treatment of the secular dynamics in the 3/1 jovian resonance), which differs in several aspects from the classical work methods used so far for the 1/1 resonance (Giacaglia and Nacozy 1970, Kozai 1985).

The paper is organized as follows: in section 2, we define the canonical variables appropriate to the 1/1 mean motion commensurability; in section 3, we compute the disturbing function, and in section 4 its derivatives; in section 5, we apply an averaging procedure based on Schubart (1964); and in section 6, we describe the co-orbital dynamics in three models of the restricted three–body problem: planar–circular, planar–elliptic and spatial–circular models. In this last section we compare our results with the ones obtained by Giacaglia & Nacozy (1970) in the elliptic case, and by Kozai (1985) in the spatial case, as well as with several modern works based on the notion of epicyclic motion. The reader interested mainly in the results is encouraged to skip the following sections and to start with section 6.

2. The canonical formalism

Let us assume the heliocentric motion of a massless body perturbed by a planet. The dynamics of the massless body can be described via autonomous Hamiltonian:

$$\mathcal{H} = \mathcal{K} + \mathcal{R} + \mathcal{T}, \tag{1}$$

where (adopting for units of length and mass the Astronomical Unit and the solar mass, and setting the universal gravitational equal to one):

$$\mathcal{K} = -\frac{1}{2a} \quad \text{and} \quad \mathcal{R} = -\mu \left[\frac{1}{|\vec{r} - \vec{r}'|} - \frac{\vec{r} \cdot \vec{r}'}{|\vec{r}'|^3} \right] \ . \tag{2}$$

$\mathcal{K}$ is the Keplerian term which solely depends on the semi-major axis a of the small body, $\mathcal{R}$ is the perturbation which is a function of the heliocentric positions of the small body ($\vec{r}$) and planet ($\vec{r}'$), μ is the planet's mass, and $\mathcal{T}$ is a canonical momentum conjugated to time in order to have the Hamiltonian in an autonomous form. If a fixed orbit of the planet is assumed, $\mathcal{T} = n'\Lambda'$ where n' is the frequency of the mean longitude (λ') and Λ' is the canonical momentum conjugated to λ' (Morbidelli 2001). Here and in the following, the primed variables refer to the planet.

The appropriate canonical variables for the 1/1 commensurability are

$$\begin{aligned} \sigma_{1/1} &= \lambda - \lambda' , & S_{1/1} &= L , \\ \sigma &= -\varpi , & S &= L - G , \\ \sigma_z &= -\Omega , & S_z &= G - H , \\ \lambda' , & & \tilde{\Lambda}' &= \Lambda' + L , \end{aligned} \tag{3}$$

where $L = \sqrt{a}, G = L\sqrt{1-e^2}$ and $H = G\cos i$ are the usual Delaunay's momenta, and $a, e, i, \lambda, \varpi, \Omega$ are the usual orbital elements. The mean and perihelion longitudes are related to the Delaunay's angles (mean anomaly l, pericentric argument ω and node longitude Ω) by: $\lambda = l + \varpi$ and $\varpi = \omega + \Omega$. These resonant variables are, with exception of $\sigma_{1/1}$, identical with those sometimes called the modified Delaunay variables.

In (3), the resonant angle $\sigma_{1/1}$ and the conjugated momentum $S_{1/1}$ will describe the co-orbital resonant motion, σ, S and σ_z, S_z are related to planar and spatial secular evolutions, and λ', Λ' is the degree of freedom associated to short–periodic effects.

In analogy to (3), we also introduce

$$\sigma' = -\varpi' , \qquad \sigma'_z = -\Omega . \tag{4}$$

This notation may seem confusing at the first point because σ, which usually denotes the resonant angle in the case of the $q \neq 0$ commensurability (Schubart 1964), is here –in the 1/1 case– the angle related to secular dynamics. The actual purpose of this notation is to keep close correspondence with the computations done in the works of Ferraz-Mello & Sato (1989), Moons (1993) and Roig *et al.* (1998) for the $q \neq 0$ case. In particular, using (3) and (4), the following relations hold:

$$\begin{aligned} \sigma_z &= \sigma + \omega , \qquad & \sigma' &= \sigma + \varpi - \varpi' , \\ \sigma'_z &= \sigma' + \omega' , \qquad & \sigma'_z &= \sigma_z + \Omega - \Omega' , \end{aligned} \tag{5}$$

which are identical to those found between the variables in the $q \neq 0$ case (Roig *et al.* 1998). These relations will be useful in section 3.1. For this reason, it is important to keep in mind that in our notation the 1/1 resonant angle is $\sigma_{1/1}$ and the associated momentum is $S_{1/1}$.

3. The disturbing function in non–singular variables

In this section, we write the perturbation $\mathcal{R}$ as a function of non–singular variables. Such expression will not have singularity for $e = 0$ and/or $i = 0$, and will be a useful departure point for computing the derivatives.

We define the non–singular non–canonical variables associated to the angles σ, σ', σ_z and σ'_z as:

$$\begin{pmatrix} k \\ h \end{pmatrix} = e \begin{pmatrix} \cos\sigma \\ \sin\sigma \end{pmatrix} , \qquad \begin{pmatrix} \tilde{k} \\ \tilde{h} \end{pmatrix} = \sin\left(\frac{i}{2}\right) \begin{pmatrix} \cos\sigma_z \\ \sin\sigma_z \end{pmatrix} ,$$
$$\begin{pmatrix} k' \\ h' \end{pmatrix} = e' \begin{pmatrix} \cos\sigma' \\ \sin\sigma' \end{pmatrix} , \qquad \begin{pmatrix} \tilde{k}' \\ \tilde{h}' \end{pmatrix} = \sin\left(\frac{i'}{2}\right) \begin{pmatrix} \cos\sigma'_z \\ \sin\sigma'_z \end{pmatrix} . \tag{6}$$

These variables are formally defined as in the $q \neq 0$ case (Ferraz-Mello & Sato 1989, Roig *et al.* 1998), but yet have a different meaning: they are all related to secular dynamics as σ, σ', σ_z and σ'_z are secular angles.

3.1. Basic decomposition of the disturbing function

Following Ferraz-Mello & Sato (1989), Moons (1993) and Roig *et al.* (1998), the disturbing function $\mathcal{R}$ is written as:

$$\mathcal{R} = -\frac{\mu}{a'} \left[\frac{1}{\sqrt{\alpha^2 g_1 + g'_1 - 2\alpha g_4}} - \alpha \frac{g_4}{g_1'^{3/2}} \right] , \tag{7}$$

where

$$g_1 = \left(\frac{r}{a}\right)^2 , \quad g'_1 = \left(\frac{r'}{a'}\right)^2 \quad \text{and} \quad g_4 = \frac{rr'}{aa'} \cos\mathcal{S} , \tag{8}$$

and $\alpha = a/a'$. Note that the definition of g_4 differs from the definition adopted in the above quoted papers. The dependence of the disturbing function on the semi-major axes is explicited in (7). The terms g_1 and g'_1, being adimensional functions, depend only on eccentricities. g_4 is also a function of inclinations (Moons 1993):

$$\begin{aligned}
\cos\mathcal{S} \;=\; & \cos^2\left(\frac{i}{2}\right)\cos^2\left(\frac{i'}{2}\right)\cos(\phi-\phi') \\
+ \; & \sin^2\left(\frac{i}{2}\right)\cos^2\left(\frac{i'}{2}\right)\cos(\phi+\phi'-2\Omega) \\
+ \; & \sin^2\left(\frac{i'}{2}\right)\cos^2\left(\frac{i}{2}\right)\cos(\phi+\phi'-2\Omega') \\
+ \; & \sin^2\left(\frac{i}{2}\right)\sin^2\left(\frac{i'}{2}\right)\cos(\phi-\phi'-2\Omega+2\Omega') \\
+ \; & 2\sin\left(\frac{i}{2}\right)\cos\left(\frac{i}{2}\right)\sin\left(\frac{i'}{2}\right)\cos\left(\frac{i'}{2}\right)\cos(\phi-\phi'-\Omega+\Omega') \\
- \; & 2\sin\left(\frac{i}{2}\right)\cos\left(\frac{i}{2}\right)\sin\left(\frac{i'}{2}\right)\cos\left(\frac{i'}{2}\right)\cos(\phi+\phi'-\Omega-\Omega')\,.
\end{aligned} \tag{9}$$

where $\phi = v + \varpi$ and $\phi' = v' + \varpi'$ are the true longitudes of the small body and planet respectively, and v and v' are the true anomalies.

Using (5), the cosine arguments in (9) can be re-written as:

$$\left\{\begin{array}{lcl}
\phi-\phi' & = & (v-\sigma)-(v'-\sigma') \\
\phi+\phi'-2\Omega & = & (v-\sigma)+(v'-\sigma')+2\sigma_z \\
\phi+\phi'-2\Omega' & = & (v-\sigma)+(v'-\sigma')+2\sigma'_z \\
\phi-\phi'-2\Omega+2\Omega' & = & (v-\sigma)-(v'-\sigma')+2\sigma_z-2\sigma'_z \\
\phi-\phi'-\Omega+\Omega' & = & (v-\sigma)-(v'-\sigma')+\sigma_z-\sigma'_z \\
\phi+\phi'-\Omega-\Omega' & = & (v-\sigma)+(v'-\sigma')+\sigma_z+\sigma'_z\,.
\end{array}\right. \tag{10}$$

At this point, one can appreciate the advantage of the notations in (3) and (4), which through (5), allow us to keep a very close correspondence to the $q \neq 0$ case (compare to eqs. (1) and (5) of Roig *et al.* 1998).

The next step is to write the functions g_1, g'_1 and g_4 in terms of the non–singular variables (6).

3.2. THE TERMS g_1 AND g'_1

Since $r = a(1 - e\cos u)$ and $r' = a'(1 - e'\cos u')$, where u and u' are the eccentric anomalies of the small body and planet, one has immediately that

$$g_1 = (1 - e\cos u)^2 \quad \text{and} \quad g'_1 = (1 - e'\cos u')^2 \ . \tag{11}$$

Considering the relations (10), it is useful to define:

$$\begin{pmatrix} W_1 \\ \\ W_2 \end{pmatrix} = \frac{r}{a} \begin{pmatrix} \cos(v - \sigma) \\ \\ \sin(v - \sigma) \end{pmatrix} , \qquad \begin{pmatrix} W'_1 \\ \\ W'_2 \end{pmatrix} = \frac{r'}{a'} \begin{pmatrix} \cos(v' - \sigma') \\ \\ \sin(v' - \sigma') \end{pmatrix} . \tag{12}$$

These functions will facilitate the evaluation of g_4 and we have trivially that

$$g_1 = W_1^2 + W_2^2 \quad \text{and} \quad g'_1 = W_1'^2 + W_2'^2 \ . \tag{13}$$

Now, using the classical equations of Keplerian motion:

$$\frac{r}{a}\cos v = \cos u - e \ , \quad \frac{r}{a}\sin v = \sqrt{1 - e^2}\sin u \ , \tag{14}$$

W_1 and W_2 are transformed into

$$W_1 = \cos(u - \sigma) - k + hT \ , \quad W_2 = \sin(u - \sigma) + h + kT \ , \tag{15}$$

where

$$T = T_1 e\sin u \quad \text{with} \quad T_1 = \frac{-1}{1 + \sqrt{1 - e^2}} \ . \tag{16}$$

This computation is straightforward and the reader can easily verify it. Similarly, W'_1 and W'_2 are written in the form (15) in planet's variables. T' and T'_1 are defined as in (16) but with e' and u'.

Eventually, eccentric anomalies u and u' can be computed from the Kepler equations:

$$u - e\sin u = l = \sigma_{1/1} + \sigma + \lambda' \tag{17}$$

and

$$u' - e' \sin u' = l' = \sigma' + \lambda' \tag{18}$$

It is easy to see that all previous equations are non–singular for $e = 0$ (and $e' = 0$) and allow for the evaluation of g_1 (and g'_1) as a function of $\sigma_{1/1}, k, h, \lambda'$ (k', h', λ') for any $0 \leq e < 1$ $(0 \leq e' < 1)$.

Equations (15) are fundamental for the non–singular evaluation of the disturbing function and its derivatives. The term T_1 was first introduced by Ferraz-Mello & Sato (1989) for the asymmetric expansion in the $q \neq 0$ case.

3.3. The term g_4

Following Moons (1993), we write:

$$g_4 = \sum_{i=1}^{4} \alpha_i A_i \ , \tag{19}$$

where α_i are the following functions of $\widetilde{k}$, $\widetilde{h}$, $\widetilde{k}'$ and $\widetilde{h}'$:

$$\begin{aligned}
\alpha_1 &= \left(1 - \widetilde{k}^2 - \widetilde{h}^2\right)\left(1 - \widetilde{k}'^2 - \widetilde{h}'^2\right) + \left(\widetilde{k}^2 - \widetilde{h}^2\right)\left(\widetilde{k}'^2 - \widetilde{h}'^2\right) \\
&+ 4\widetilde{k}\widetilde{h}\widetilde{k}'\widetilde{h}' + 2\sqrt{1 - \widetilde{k}^2 - \widetilde{h}^2}\sqrt{1 - \widetilde{k}'^2 - \widetilde{h}'^2}\left(\widetilde{k}\widetilde{k}' + \widetilde{h}\widetilde{h}'\right) \ , \\
\alpha_2 &= 2\widetilde{k}\widetilde{h}\left(\widetilde{k}'^2 - \widetilde{h}'^2\right) - 2\left(\widetilde{k}^2 - \widetilde{h}^2\right)\widetilde{k}'\widetilde{h}' \\
&+ 2\sqrt{1 - \widetilde{k}^2 - \widetilde{h}^2}\sqrt{1 - \widetilde{k}'^2 - \widetilde{h}'^2}\left(\widetilde{h}\widetilde{k}' - \widetilde{k}\widetilde{h}'\right) \ , \\
\alpha_3 &= \left(\widetilde{k}^2 - \widetilde{h}^2\right)\left(1 - \widetilde{k}'^2 - \widetilde{h}'^2\right) + \left(\widetilde{k}'^2 - \widetilde{h}'^2\right)\left(1 - \widetilde{k}^2 - \widetilde{h}^2\right) \\
&- 2\sqrt{1 - \widetilde{k}^2 - \widetilde{h}^2}\sqrt{1 - \widetilde{k}'^2 - \widetilde{h}'^2}\left(\widetilde{k}\widetilde{k}' - \widetilde{h}\widetilde{h}'\right) \ , \\
\alpha_4 &= 2\widetilde{k}\widetilde{h}\left(1 - \widetilde{k}'^2 - \widetilde{h}'^2\right) + 2\left(1 - \widetilde{k}^2 - \widetilde{h}^2\right)\widetilde{k}'\widetilde{h}' \\
&- 2\sqrt{1 - \widetilde{k}^2 - \widetilde{h}^2}\sqrt{1 - \widetilde{k}'^2 - \widetilde{h}'^2}\left(\widetilde{h}\widetilde{k}' + \widetilde{k}\widetilde{h}'\right) \ ;
\end{aligned} \tag{20}$$

and A_i depend on k, h, k' and h' through W–functions:

$$\begin{aligned} A_1 &= W_1 W_1' + W_2 W_2' , \qquad & A_2 &= W_1 W_2' - W_2 W_1' , \\ A_3 &= W_1 W_1' - W_2 W_2' , \qquad & A_4 &= -W_1 W_2' - W_2 W_1' . \end{aligned} \tag{21}$$

These expressions are obtained by the substitution of (10) in (9) and by the expansion of cosine functions into a multiplication of trigonometric functions with arguments: $v - \sigma$, $v' - \sigma'$, σ_z and σ_z'. The true anomalies are then substituted from (14).

It is obvious that (19) evaluated through (20) and (21) is non–singular for zero values of e, e', i, i' and, together with (13) allows for the computation of the disturbing function (7) from α_i and W–functions.

3.4. Non–singular canonical variables

Following Moons (1993), we define the non–singular canonical variables by a canonical transformation from (3) as:

$$\begin{pmatrix} x \\ \\ y \end{pmatrix} = \sqrt{2S} \begin{pmatrix} \cos\sigma \\ \\ \sin\sigma \end{pmatrix} , \quad \begin{pmatrix} \xi \\ \\ \eta \end{pmatrix} = \sqrt{2S_z} \begin{pmatrix} \cos\sigma_z \\ \\ \sin\sigma_z \end{pmatrix} . \tag{22}$$

The Hamiltonian (1) can be easily evaluated in these variables using the formulae developed in the previous sections and

$$a = L^2 , \quad e = \sqrt{1 - \left(\frac{G}{L}\right)^2} \quad \text{and} \quad \sin\frac{i}{2} = \sqrt{\frac{1}{2}\left(1 - \frac{H}{G}\right)} , \tag{23}$$

where

$$L = S_{1/1} , \quad G = S_{1/1} - S , \quad H = S_{1/1} - S - S_z , \tag{24}$$

and $2S = x^2 + y^2$, $2S_z = \xi^2 + \eta^2$.

By this, we conclude the exposition of formalism which generalizes the evaluation of the disturbing function to the case of co-orbital motion. Next, we proceed with the derivatives.

4. Non–singular calculation of the derivatives of the disturbing function

The purpose of this section is to calculate the derivatives of the Hamiltonian (1) with respect to non–singular canonical variables $S_{1/1}, \sigma_{1/1}, x, y, \xi, \eta$ (eq. 22). In fact, as the derivatives of the Keplerian part are trivial, attention will be payed only to the disturbing function. We divide the computation into two steps: first (section 4.1), we calculate the derivatives with respect to the non–singular non–canonical variables $a, \sigma_{1/1}, k, h, \widetilde{k}, \widetilde{h}$ and second, we evaluate the needed derivatives with respect to β –generically denoting one of $S_{1/1}, \sigma_{1/1}, x, y, \xi, \eta$– as:

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial \beta} &= \frac{\partial \mathcal{H}}{\partial a}\frac{\partial a}{\partial \beta} + \frac{\partial \mathcal{H}}{\partial \sigma_{1/1}}\frac{\partial \sigma_{1/1}}{\partial \beta} \\ &+ \frac{\partial \mathcal{H}}{\partial k}\frac{\partial k}{\partial \beta} + \frac{\partial \mathcal{H}}{\partial h}\frac{\partial h}{\partial \beta} + \frac{\partial \mathcal{H}}{\partial \widetilde{k}}\frac{\partial \widetilde{k}}{\partial \beta} + \frac{\partial \mathcal{H}}{\partial \widetilde{h}}\frac{\partial \widetilde{h}}{\partial \beta} \end{aligned} \tag{25}$$

Evaluation of the above expression needs derivatives of non–canonical elements with respect to the canonical ones, which we explicit in section 4.2.

The derivative with respect to λ' does not interests us much at the moment. As we will see later, this variable can be removed from the Hamiltonian by averaging, and the corresponding degree of freedom vanishes from the resulting Hamiltonian, which describes the resonant and secular dynamics. Conversely, we will compute the derivatives with respect to planet's elements $a', k', h', \widetilde{k}', \widetilde{h}'$ for completeness.

4.1. Derivatives with respect to the non–singular non–canonical variables

The direct $\mathcal{D}$ and indirect $\mathcal{I}$ parts of the disturbing function (7):

$$\mathcal{D} = \frac{1}{\sqrt{\alpha^2 g_1 + g_1' - 2\alpha g_4}} \qquad \text{and} \qquad \mathcal{I} = \alpha \frac{g_4}{g_1'^{3/2}} \ , \tag{26}$$

will be treated separately. Since none of the g functions depend on semi-major axes a and a', the derivatives of $\mathcal{D}$ and $\mathcal{I}$ with respect to these quantities can be trivially obtained from:[2]

$$\frac{\partial \mathcal{D}}{\partial \alpha} = -\mathcal{D}^3\left(\alpha g_1 - g_4\right) \qquad \text{and} \qquad \frac{\partial \mathcal{I}}{\partial \alpha} = \frac{g_4}{g_1'^{3/2}}\ . \tag{27}$$

The derivatives with respect to $(\sigma_{1/1}, k, h)$ are:

$$\begin{aligned} \frac{\partial \mathcal{D}}{\partial(\sigma_{1/1}, k, h)} &= -\alpha\mathcal{D}^3\left[\frac{1}{2}\frac{\partial g_1}{\partial(\sigma_{1/1}, k, h)} - \frac{\partial g_4}{\partial(\sigma_{1/1}, k, h)}\right], \\ \frac{\partial \mathcal{I}}{\partial(\sigma_{1/1}, k, h)} &= \frac{\alpha}{g_1'^{3/2}}\frac{\partial g_4}{\partial(\sigma_{1/1}, k, h)}\ . \end{aligned} \tag{28}$$

The derivatives with respect to (k', h') are:

$$\begin{aligned} \frac{\partial \mathcal{D}}{\partial(k', h')} &= -\mathcal{D}^3\left[\frac{1}{2}\frac{\partial g_1'}{\partial(k', h')} - \alpha\frac{\partial g_4}{\partial(k', h')}\right], \\ \frac{\partial \mathcal{I}}{\partial(k', h')} &= \frac{-3\alpha g_4}{2g_1'^{5/2}}\left[\frac{\partial g_1'}{\partial(k', h')}\right] + \frac{\alpha}{g_1'^{3/2}}\left[\frac{\partial g_4}{\partial(k', h')}\right]\ . \end{aligned} \tag{29}$$

The derivatives with respect to $\widetilde{k}, \widetilde{h}, \widetilde{k}', \widetilde{h}'$ are:

$$\begin{aligned} \frac{\partial \mathcal{D}}{\partial(\widetilde{k}, \widetilde{h}, \widetilde{k}', \widetilde{h}')} &= \alpha\mathcal{D}^3\left[\frac{\partial g_4}{\partial(\widetilde{k}, \widetilde{h}, \widetilde{k}', \widetilde{h}')}\right], \\ \frac{\partial \mathcal{I}}{\partial(\widetilde{k}, \widetilde{h}, \widetilde{k}', \widetilde{h}')} &= \frac{\alpha}{g_1'^{3/2}}\left[\frac{\partial g_4}{\partial(\widetilde{k}, \widetilde{h}, \widetilde{k}', \widetilde{h}')}\right], \end{aligned} \tag{30}$$

What rests are the derivatives of g_1, g_1' and g_4.

The function g_1 depends on W_1 and W_2 (eq. 13) which are explicited in (15). In (15) and (16), the eccentric anomaly u must be thought as a function of $\sigma_{1/1}, k, h$ from Kepler equation (17). Similarly, the eccentric anomaly u' in W_1', W_2' is a function of k', h' from the Kepler equation of the planet's orbit (18). Consequently, the only non-trivial derivatives

[2] Caution should be payed here in not forgetting the $1/a'$ factor, which scales the disturbing function with the planet's semi-major axis in (7), when derivating with respect to a'.

are $\partial u/\partial\sigma_{1/1}$, $\partial(u-\sigma)/\partial(k,h)$ and $\partial T/\partial(\sigma_{1/1},k,h)$ for the small body, and $\partial(u'-\sigma')/\partial(k',h')$ and $\partial T'/\partial(k',h')$ for the planet. Let us compute in the following: $\partial u/\partial\sigma_{1/1}$, $\partial(u-\sigma)/\partial(k,h)$ and $\partial T/\partial(\sigma_{1/1},k,h)$; the derivatives of primed quantities are obtained by an identical procedure.

The reader can easily verify that:

$$\begin{aligned}\frac{\partial W_1}{\partial\sigma_{1/1}} &= \frac{\partial u}{\partial\sigma_{1/1}}\left[-\sin(u-\sigma)+hT_1e\cos u\right] \ ,\\ \frac{\partial W_2}{\partial\sigma_{1/1}} &= \frac{\partial u}{\partial\sigma_{1/1}}\left[\cos(u-\sigma)+kT_1e\cos u\right] \ ,\end{aligned} \tag{31}$$

where

$$\frac{\partial u}{\partial\sigma_{1/1}} = (1-e\cos u)^{-1} \ . \tag{32}$$

The last equation has been obtained by derivating the Kepler equation (17) with respect to $\sigma_{1/1}$. Similarly:

$$\begin{aligned}\frac{\partial W_1}{\partial k} &= -\frac{\partial(u-\sigma)}{\partial k}\sin(u-\sigma)-1+h\frac{\partial T}{\partial k} \ ,\\ \frac{\partial W_1}{\partial h} &= -\frac{\partial(u-\sigma)}{\partial k}\sin(u-\sigma)+T+h\frac{\partial T}{\partial h} \ ,\\ \frac{\partial W_2}{\partial k} &= \frac{\partial(u-\sigma)}{\partial k}\cos(u-\sigma)+T+k\frac{\partial T}{\partial k} \ ,\\ \frac{\partial W_2}{\partial h} &= \frac{\partial(u-\sigma)}{\partial h}\cos(u-\sigma)+1+k\frac{\partial T}{\partial h}\end{aligned} \tag{33}$$

with

$$\begin{aligned}\frac{\partial T}{\partial k} &= \frac{\partial T_1}{\partial k}e\sin u+T_1\cos\sigma\sin u+T_1e\frac{\partial u}{\partial k}\cos u \ ,\\ \frac{\partial T}{\partial h} &= \frac{\partial T_1}{\partial h}e\sin u+T_1\sin\sigma\sin u+T_1e\frac{\partial u}{\partial h}\cos u \ ,\end{aligned} \tag{34}$$

where, from the Kepler equation (17):

$$\frac{\partial(u-\sigma)}{\partial k} = D\sin(u-\sigma) \quad \text{and} \quad \frac{\partial(u-\sigma)}{\partial h} = D\cos(u-\sigma) \ , \tag{35}$$

and

$$e\frac{\partial u}{\partial k} = D(e\sin u\cos\sigma - \sin\sigma) \ ,$$
$$e\frac{\partial u}{\partial h} = D(e\sin u\sin\sigma + \cos\sigma) \ , \tag{36}$$

with $D = (1 - e\cos u)^{-1}$. Moreover, $\partial T_1/\partial(k,h)$ can be easily obtained from:

$$\frac{\partial T_1}{\partial e} = \left(\frac{-e}{\sqrt{1-e^2}}\right) T_1^2 \ . \tag{37}$$

The above equations allow for the evaluation of the g_1 function derivatives which are clearly non-singular for $0 \leq e < 1$, which was our primary motivation. The derivatives of g_1' with respect to k', h' are identical to the derivatives of g_1 with respect to k, h, substituting the unprimed quantities for the primed ones.

Let us proceed with $g_4 = \sum \alpha_i A_i$, where $\alpha_i = \alpha_i(\widetilde{k}, \widetilde{h}, \widetilde{k}', \widetilde{h}')$. The 'planar' part of g_4 is a trivial function of W_1, W_2, W_1', W_2' through coefficients A_i (21), while the derivatives of W_1, W_2, W_1' and W_2' with respect to $\sigma_{1/1}, k, h, k', h'$ were given in (31) and (33), plugging in (32) and (34-37).

What rests are the derivatives of α_i with respect to $\widetilde{k}, \widetilde{h}, \widetilde{k}', \widetilde{h}'$, which are obtained by trivial derivation of (20). This computation is not explicited here because it constitutes 16 expressions. It can be seen at a first glance in (20) that if ($i' \neq 0$ and $i = \pi$) or ($i \neq 0$ and $i' = \pi$), these derivatives are singular because of the terms $\cos i/2$ or $\cos i'/2$ which appear in the denominator when taking the derivative of $\sqrt{1 - \widetilde{k}^2 - \widetilde{h}^2}$ or $\sqrt{1 - \widetilde{k}'^2 - \widetilde{h}'^2}$. This is however a very particular case of no importance for our present needs. Apart from this case, there are no singularities in all the expressions given above for $0 \leq e < 1$, $0 \leq i < \pi$, $0 \leq e' < 1$ and $0 \leq i' < \pi$.

4.2. Derivatives with respect to the non–singular canonical variables

The next step is to evaluate derivatives of the disturbing function with respect to non–singular canonical variables (22). For this we need, according to (25), to calculate $\partial(a, \sigma_{1/1}, k, h, \widetilde{k}, \widetilde{h})/\partial(S_{1/1}, \sigma_{1/1}, x, y, \xi, \eta)$. The key point of the following calculation is that although $\partial\mathcal{H}/\partial S$ and $\partial\sigma/\partial(x, y)$ go to infinity for $e \to 0$, both the $\partial\mathcal{H}/\partial(k, h)$ (as shown in section 4.1) and $\partial(x, y)/\partial(k, h)$ are well defined for $e = 0$. Similarly, $\partial\mathcal{H}/\partial(\widetilde{k}, \widetilde{h})$ and $\partial(\xi, \eta)/\partial(\widetilde{k}, \widetilde{h})$ are also well defined for $e = 0$ and/or $i = 0$.

Note first that, from (23-24):

$$
\begin{aligned}
\frac{\partial e}{\partial S_{1/1}} &= \frac{G}{eL^2}\left(\frac{G}{L} - 1\right) , & \frac{\partial \sin(i/2)}{\partial S_{1/1}} &= \frac{1}{4G\sin(i/2)}\left(\frac{H}{G} - 1\right) , \\
\frac{\partial e}{\partial S} &= \frac{G}{eL^2} , & \frac{\partial \sin(i/2)}{\partial S} &= \frac{1}{4G\sin(i/2)}\left(1 - \frac{H}{G}\right) , \\
\frac{\partial e}{\partial S_z} &= 0 , & \frac{\partial \sin(i/2)}{\partial S_z} &= \frac{1}{4G\sin(i/2)} .
\end{aligned}
\tag{38}
$$

Next, let us proceed with $\partial\mathcal{H}/\partial S_{1/1}$. From (25):

$$
\frac{\partial \mathcal{H}}{\partial S_{1/1}} = \frac{\partial \mathcal{H}}{\partial a}\frac{\partial a}{\partial S_{1/1}} + \frac{\partial \mathcal{H}}{\partial k}\frac{\partial k}{\partial S_{1/1}} + \frac{\partial \mathcal{H}}{\partial h}\frac{\partial h}{\partial S_{1/1}} + \frac{\partial \mathcal{H}}{\partial \widetilde{k}}\frac{\partial \widetilde{k}}{\partial S_{1/1}} + \frac{\partial \mathcal{H}}{\partial \widetilde{h}}\frac{\partial \widetilde{h}}{\partial S_{1/1}} , \tag{39}
$$

where trivially:

$$
\begin{aligned}
\frac{\partial k}{\partial S_{1/1}} &= \frac{\partial e}{\partial S_{1/1}}\cos\sigma , & \frac{\partial h}{\partial S_{1/1}} &= \frac{\partial e}{\partial S_{1/1}}\sin\sigma \\
\frac{\partial \widetilde{k}}{\partial S_{1/1}} &= \frac{\partial \sin(i/2)}{\partial S_{1/1}}\cos\sigma_z , & \frac{\partial \widetilde{h}}{\partial S_{1/1}} &= \frac{\partial \sin(i/2)}{\partial S_{1/1}}\sin\sigma_z ,
\end{aligned}
\tag{40}
$$

and $\partial a/\partial S_{1/1} = 2L$ from (23). The derivatives $\partial(e, \sin i/2)/\partial S_{1/1}$ were already given in (38). The final expression is obtained by substituting (38) and (40) in (39).

The remaining derivatives are little more involved. Let us show in detail how to compute $\partial\mathcal{H}/\partial x$. The derivatives with respect to y, ξ, η are found by a similar recipe. We note that:

$$\frac{\partial\mathcal{H}}{\partial x} = x\frac{\partial\mathcal{H}}{\partial S} - \frac{y}{2S}\frac{\partial\mathcal{H}}{\partial\sigma} \ , \tag{41}$$

where $\partial\mathcal{H}/\partial S$ and $\partial\mathcal{H}/\partial\sigma$ are to be worked out separately. The latter one is simple:

$$\frac{\partial\mathcal{H}}{\partial\sigma} = -h\frac{\partial\mathcal{H}}{\partial k} + k\frac{\partial\mathcal{H}}{\partial h} \ , \tag{42}$$

while the former one requires additional substitutions. By using the chain rule of derivation:

$$\begin{aligned}\frac{\partial\mathcal{H}}{\partial S} &= \left(\frac{\partial\mathcal{H}}{\partial k}\frac{\partial k}{\partial e} + \frac{\partial\mathcal{H}}{\partial h}\frac{\partial h}{\partial e}\right)\frac{\partial e}{\partial S} \\ &+ \left(\frac{\partial\mathcal{H}}{\partial\tilde{k}}\frac{\partial\tilde{k}}{\partial\sin(i/2)} + \frac{\partial\mathcal{H}}{\partial\tilde{h}}\frac{\partial\tilde{h}}{\partial\sin(i/2)}\right)\frac{\partial\sin(i/2)}{\partial S} \ ,\end{aligned} \tag{43}$$

where $\partial(e, \sin i/2)/\partial S$ were given in (38). To get the expression of $\partial\mathcal{H}/\partial x$, the above pieces must be put together and simplified.

In the following, we show the final form of derivatives of the Hamiltonian with respect to non–singular canonical variables:

$$
\begin{aligned}
\frac{\partial \mathcal{H}}{\partial S_{1/1}} &= 2L\frac{\partial \mathcal{H}}{\partial a} - \left[\left(\frac{\partial \mathcal{H}}{\partial k}\right) k + \left(\frac{\partial \mathcal{H}}{\partial h}\right) h\right] \frac{G}{L(L+G)} \\
&- \left[\left(\frac{\partial \mathcal{H}}{\partial \tilde{k}}\right) \tilde{k} + \left(\frac{\partial \mathcal{H}}{\partial \tilde{h}}\right) \tilde{h}\right] \frac{1}{2G} \\
\frac{\partial \mathcal{H}}{\partial x} &= \left[\left(\frac{\partial \mathcal{H}}{\partial k}\right) \cos\sigma + \left(\frac{\partial \mathcal{H}}{\partial h}\right) \sin\sigma\right] \frac{G}{L}\sqrt{\frac{2}{L+G}} \cos\sigma \\
&- \left[\left(\frac{\partial \mathcal{H}}{\partial h}\right) \cos\sigma - \left(\frac{\partial \mathcal{H}}{\partial k}\right) \sin\sigma\right] \frac{1}{L}\sqrt{\frac{L+G}{2}} \sin\sigma \\
&+ \left[\left(\frac{\partial \mathcal{H}}{\partial \tilde{k}}\right) \tilde{k} + \left(\frac{\partial \mathcal{H}}{\partial \tilde{h}}\right) \tilde{h}\right] \frac{x}{2G} \\
\frac{\partial \mathcal{H}}{\partial y} &= \left[\left(\frac{\partial \mathcal{H}}{\partial k}\right) \cos\sigma + \left(\frac{\partial \mathcal{H}}{\partial h}\right) \sin\sigma\right] \frac{G}{L}\sqrt{\frac{2}{L+G}} \sin\sigma \\
&+ \left[\left(\frac{\partial \mathcal{H}}{\partial h}\right) \cos\sigma - \left(\frac{\partial \mathcal{H}}{\partial k}\right) \sin\sigma\right] \frac{1}{L}\sqrt{\frac{L+G}{2}} \cos\sigma \\
&+ \left[\left(\frac{\partial \mathcal{H}}{\partial \tilde{k}}\right) \tilde{k} + \left(\frac{\partial \mathcal{H}}{\partial \tilde{h}}\right) \tilde{h}\right] \frac{y}{2G} \\
\frac{\partial \mathcal{H}}{\partial \xi} &= \left(\frac{\partial \mathcal{H}}{\partial \tilde{k}}\right) \frac{1}{2\sqrt{G}} \\
\frac{\partial \mathcal{H}}{\partial \eta} &= \left(\frac{\partial \mathcal{H}}{\partial \tilde{h}}\right) \frac{1}{2\sqrt{G}}
\end{aligned}
\tag{44}
$$

It is clear that none of the above expressions are singular for $e = 0$ and/or $i = 0$ ($G = L = \sqrt{a}$ for $e = 0$). Consequently, the Hamiltonian equations:

$$\begin{aligned}
\frac{\mathrm{d}S_{1/1}}{\mathrm{d}t} &= -\frac{\partial\mathcal{H}}{\partial\sigma_{1/1}} \,, & \frac{\mathrm{d}\sigma_{1/1}}{\mathrm{d}t} &= \frac{\partial\mathcal{H}}{\partial S_{1/1}} \,, \\
\frac{\mathrm{d}x}{\mathrm{d}t} &= -\frac{\partial\mathcal{H}}{\partial y} \,, & \frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{\partial\mathcal{H}}{\partial x} \,, \\
\frac{\mathrm{d}\xi}{\mathrm{d}t} &= -\frac{\partial\mathcal{H}}{\partial\eta} \,, & \frac{\mathrm{d}\eta}{\mathrm{d}t} &= \frac{\partial\mathcal{H}}{\partial\xi}
\end{aligned} \tag{45}$$

are definite for whatever eccentricities and inclinations of the perturbed and perturbing body. We will solve them in several approximations using the perturbation theory.

5. Averaging

The starting point of the perturbation approach is the Hamiltonian (1) written in the non–singular canonical variables $S_{1/1}, \sigma_{1/1}, x, y, \xi, \eta$ (eq. 22), and $\widetilde{\Lambda}', \lambda'$. Assuming that the planet's orbit is fixed in the space (i.e. constant $a', e', i', \varpi', \Omega'$)[3], the Hamiltonian is:

$$\mathcal{H} = -\frac{1}{2S_{1/1}^2} - n'S_{1/1} + n'\widetilde{\Lambda}' + \mathcal{R}(S_{1/1}, \sigma_{1/1}, x, y, \xi, \eta, \lambda') \,. \tag{46}$$

Here we identify the integrable part

$$\mathcal{H}_0 = n'\widetilde{\Lambda}' - \frac{1}{2S_{1/1}^2} - n'S_{1/1} \tag{47}$$

and its small perturbation $\mathcal{R}$. The angle $\sigma_{1/1}$ cannot be eliminated by the perturbation transformation (Morbidelli 2001), because of the presence of resonance:

$$\dot{\sigma}_{1/1} = \frac{\partial}{\partial S_{1/1}}\left[-\frac{1}{2S_{1/1}^2} - n'S_{1/1}\right] = n' - n \sim 0 \,, \tag{48}$$

[3] Such assumption is not necessary and the averaging can easily be done in a more general case. We prefer to make this assumption here to keep the notation simple.

which would inhibit the series convergence due to the small divisors problem.

The perturbation theory (Hori 1966) shows us how to construct a canonical transformation which eliminates the short period terms and reduces the number of freedom degrees. This transformation is at the first order in μ (mass of the planet) formally equivalent to averaging of $\mathcal{R}$ over λ'. This is accomplished by a numerical averaging procedure, first proposed by Schubart (1964) in the celestial mechanics context: "The value of the considered function is compute for equidistant points in the periodicity interval of a short–period argument, while the critical and other long–period arguments remain at a constant value. No series development is necessary." Since this procedure requires no series expansions, it is valid for whatever eccentric and/or inclined orbit.

Neglecting $\mathcal{H}_0$ and all terms of order larger than one in the small parameter μ, the average Hamiltonian writes:

$$\langle\mathcal{H}\rangle = -\frac{1}{2S_{1/1}^2} - n'S_{1/1} + \langle\mathcal{R}\rangle \quad \text{with} \quad \langle\mathcal{R}\rangle = \frac{1}{2\pi}\int\limits_0^{2\pi} \mathcal{R}\,\mathrm{d}\lambda' \; . \tag{49}$$

The above averaged Hamiltonian is a function of the mean non–singular canonical variables $(S_{1/1}, \sigma_{1/1}, x, y, \xi, \eta)$ and parametrically depends on the planetary elements $(a', e', i', \varpi', \Omega')$.

From the Kepler equation (17), we have that $\mathrm{d}\lambda' = (1 - e\cos u)\mathrm{d}u$ ($\sigma_{1/1}$ and σ are assumed fixed), which allows us to compute the integral (49) over the eccentric anomaly instead of λ', and which avoids the need to solve the Kepler equation for the small body by iterations. Let $\mathcal{F}$ be the disturbing function or one of its derivatives. The averaged value of $\mathcal{F}$ is then computed as:

$$\langle\mathcal{F}\rangle = \frac{1}{N}\sum_{j=1}^{N}(1 - e\cos u_j)\mathcal{F}(u_j) \; , \tag{50}$$

where $u_j = 2\pi j/N$ and N is chosen in order to reach the desired accuracy[4]. Solving the Kepler equation for the planet (18) cannot be avoided. First, one calculates $\lambda' = u - e \sin u - \sigma_{1/1} - \sigma$ and then solves $u' - e' \sin u' = \lambda' - \varpi'$ for u' by iterations (we use the Newton–Ralphson algorithm, Press *et al.* 1986).

We end up with the averaged Hamiltonian (49), which have three degrees of freedom (DOF) and which is autonomous when $(a', e', i', \varpi', \Omega')$ are assumed fixed. In the following, we will consider three special configurations of the small body and planetary orbits, where the number of DOF of (49) is less than three: $e' = i = i' = 0$ in section 6.1 (1 DOF); $i = i' = 0$ in section 6.2 (2 DOF); and $e' = i' = 0$ in section 6.3 (2 DOF).

6. Structure of the co-orbital motion

Schubart (1964) studied the 1/1 resonance without making an expansion of the disturbing function. His study was limited to the circular problem ($e' = 0$). He adapted the resonant canonical formalism of Poincaré (1902) to any ratio $(p + q)/p$ between the mean motions of the small body and planet computed several diagrams showing the resonant structure of the 1/1 resonance. Schubart was the first to note the displacement with e and i of the fixed points associated to the center of the tadpole libration from 60°.

Giacaglia & Nacozy (1970) computed the coupled evolution of the longitude of perihelion and eccentricity in the planar–elliptic model with Jupiter as a perturbing planet. Their work suffered from low resolution dictated by the computer resources available to them, and

[4] We typically use $100 \leq N \leq 300$, which proves to be enough in most cases.

more importantly, from non–validity of their perturbation approach for $e' >\sim 0.2$. We develop a different perturbation approach and compare to the results of Giacaglia & Nacozy (1970) in section 6.2.

Kozai (1985) treated the spatial–circular problem by a very rough approximation, fixing the center of L4 tadpole orbits at $\sigma_{1/1} = 60$ deg and $a = a'$. We show in section 6.3, that the libration island of the argument of perihelion found by Kozai (1985) disappears when the computation is done rigorously, i.e. by computing the actual position of the stationary point.

Namouni (1999) studied the co-orbital dynamics with the terrestrial planets using the Hill's equation (Hénon and Petit 1986) and by numerical integrations. He demonstrated that in the spatial case, the families of tadpole, horseshoe and retrograde satellite orbits are not separated. The secular modulation of the resonant motion causes transitions between the orbital families. Namouni showed that in certain configurations the transiting orbits can avoid close approaches to the perturbing planet for long time intervals. We reproduce some of his results by the perturbation method in section 6.3.

6.1. The planar–circular model

The force function in planar ($i = i' = 0$) and circular ($e' = 0$) configuration of the co-orbital motion of two bodies is invariant by rotation of the reference system about the Z-axis (assuming the positions and velocity vectors of the small body and planet in (X, Y) plane, and the Sun in the origin). This tells us that the Hamiltonian cannot depend on the perihelion longitude of the small body. Hence, the average Hamiltonian has a single DOF associated to the resonant variables $(S_{1/1}, \sigma_{1/1})$ and parametrically depends on $S = \sqrt{a}(1 - \sqrt{1 - e^2})$.

We reproduce, in Fig. 1, the classical plot of the level curves $\mathcal{H} = const.$ for several different values of the parameter S, assuming a Jupiter–mass perturbing planet at $a' = 5.2026$ AU. For easier interpretation, we represent the trajectories in a $(a, \sigma_{1/1})$ plane parameterized by the value of e at L4. The phase diagram at small e (Fig. 1a) is similar to the well–known diagram of the zero–velocity curves (Hill's curves) of the non–averaged restricted circular and planar problem in the reference system with the origin at the barycenter of the Sun–planet system and co-rotating with the planet. In Fig. 1a, it is easy to identify the classical Lagrange equilibrium points and distinguish the domains of tadpole (around L4 or L5) and horseshoe orbits (around L3).

Six stationary points exist in Fig. 1a: one saddle point at $\sigma_{1/1} = \pi$, two stable equilibrium points at $\sigma_{1/1} = \pm\sigma_{1/1}^{\rm asym}$ with $\pi/3 \leq \sigma_{1/1}^{\rm asym} < \pi$ (asymmetric centers), and the stable equilibrium point $\sigma_{1/1} = 0$ (this last one is easier to see in Figs. 1b-d). The two other unstable points corresponding to L1 and L2 are difficult to see in Fig. 1a because of the resolution. The separatrix originating from the saddle point L3 surrounds the asymmetric centers L4 and L5, and separates domains of tadpole and horseshoe orbits. The horseshoe orbits are enclosed by the separatrix originating at L2, while the separatrix connecting L1 to itself encloses a small region of the prograde satellite motion (see Morbidelli 2001). This resonant structure is similar to the one which characterizes the mean motion resonances of the type $1/r$, r integer. These exterior resonances were extensively studied by Message (1958, 1959), Schubart (1964), Beaugé (1994) and Morbidelli *et al.* (1995).

One can note in Fig. 1 that while the semi-major axis of the stationary point is almost independent of e, while $\sigma_{1/1}^{\rm asym}$ changes. The center of asymmetric tadpole librations shifts towards π with increasing e

(from Fig. 1a to Fig. 1d) Conversely, the saddle point at the center of horseshoe orbits is always at $\sigma_{1/1} = \pi$.

The orbits of the small body and of the planet in the 1/1 resonance are close to each other and can easily cross. In Fig. 1b-d, the thick line shows the places in the $a, \sigma_{1/1}$ where such crossings happen (crossing curve). Assuming the co-planar orbits and the circular orbit of the planet, the crossing curve was computed from the conditions of a collision, namely $r = a'$ and $\phi = \lambda'$. Fixing a (and e), we compute the resonant angle corresponding to the collision as $\sigma_{1/1} = l - v$, where $l = u - e \sin u$ and $\tan(v/2) = \sqrt{(1-e)/(1+e)} \tan(u/2)$, and where the eccentric anomaly u is extracted from $a(1 - e \cos u) = a'$.

The Hamiltonian function is not analytical at the collision curve: while the average of the disturbing function (49) is well defined, the derivatives computed from (50) with $N \to \infty$ are indefinite. The crossing curve divides the phase plane of the averaged model in two distinct regions. The one centered at $\sigma_{1/1} = 0$ and $a = n'^{-2/3} \sim 5.2$ AU is characterized by the libration of $\sigma_{1/1}$ around 0. In this regime, the small body retrogradely moves around the resonant planet as retrograde satellites. For this reason, these orbits are usually called the retrograde satellites orbits (or pseudo–satellite orbits). In the second domain, centered at $\sigma_{1/1} = \pi$ and $a = n'^{-2/3}$ in Fig. 1b-d, the tadpole orbits near the asymmetric centers are protected from close encounters with the resonant planet. It is the collision curve which determines the angular position of the asymmetric center and delimits the size of the horseshoe domain.

What happens if a real body moves on the trajectory which hits the crossing curve? In most cases, such situation corresponds to a close encounter where the perturbation method behind averaging fails.

The only invariant of motion is the Jacoby constant. The Jacoby constant can be approximated at a large distance from the planet by the Tisserand parameter:

$$T = \frac{a'}{a} + \frac{2}{\sqrt{a'}}\sqrt{a(1-e^2)}\cos i \;, \tag{51}$$

which, for $i = 0$, translates to the conservation of

$$\frac{1}{S_{1/1}^2} + \frac{2}{n'}(S_{1/1} - S) \;. \tag{52}$$

Consequently, one sees that S changes at close encounters and the small body can pass between different levels of this otherwise conserved (in averaged model) quantity. It is instructive to plot the curves of $T(a,e,i=0) = const.$ and of $S = \sqrt{a}(1-\sqrt{1-e^2}) = const.$ in the a,e space (Fig. 2). The $S = const.$ and $T = const.$ curves are almost parallel near the resonant semi-major axis ($\sim$ 5.2 AU in this case), which means that an encounter of a small body with an instantaneous semi-major axis near that of the resonant planet does not necessarily change S significantly.

Coming back to Fig. 1, we numerically determine the value of $\sigma_{1/1}$ corresponding to the stable equilibrium point in the center of the asymmetric libration (no. 1 in Fig. 3a). The width of the tadpole island is then conveniently measured by the extent of tadpole separatrices at $\sigma_{1/1}^{\rm asym}$ (no. 2 in Fig. 3a). Similarly, the extent of horseshoe orbits can be measured at $\sigma_{1/1} = \sigma_{1/1}^{\rm asym}$ (no. 3 in Fig. 3a) or along $\sigma_{1/1} = \pi$ (no. 4 in Fig. 3a). The width of the retrograde satellite region is computed at $\sigma_{1/1} = 0$ (no. 5 in Fig. 3a).

Figures 3 and 4 show the locations and widths of the tadpole, horseshoe and retrograde satellite orbits for Jovian and terrestrial planets, respectively. The horseshoe domain is always larger than the tadpole one. The tadpole and horseshoe domains shrink with increasing eccentricity. Conversely, the size of the retrograde satellite domain is almost independent of eccentricity. The orbits located above the curve $e = \pm(a_{\rm planet} - a)/a$ (the sign is + for $a > a_{\rm planet}$ and − for $a < a_{\rm planet}$) do cross a non–resonant planet and are unstable.

6.2. The planar–elliptic model

The averaged disturbing function of this model ($i = i' = 0$) depends on two angles: $\sigma_{1/1}$ and $\varpi - \varpi'$. Fixing $\varpi' = 0$, the average Hamiltonian has two degrees of freedom represented by the couples of canonical variables $(S_{1/1}, \sigma_{1/1})$ and $(S, \sigma = -\varpi)$:

$$\mathcal{H} = \mathcal{H}_0(S_{1/1}, \sigma_{1/1}, S) + \mathcal{H}_1(S_{1/1}, \sigma_{1/1}, S, \sigma; e') \ , \tag{53}$$

where $\mathcal{H}_0$ is the planar–circular part and $\mathcal{H}_1$ parametrically depends on the planet's eccentricity. While $\mathcal{H}_0$ is of the zero order in eccentricities, $\mathcal{H}_1 \sim \mathcal{O}(ee')$. Consequently, if the eccentricities are reasonably small, $\epsilon = ee'$ plays the role of a small parameter in the above quasi–integrable Hamiltonian. Moreover, we note the separation of time scales of $\sigma_{1/1}$ and σ evolutions dictated by $\mathcal{H}_0$: the frequency of $\sigma_{1/1}$ is at least a factor of 20 faster than the one of the secular angle σ for all solar system planets. This means, that $\sigma_{1/1}$ performs numerous oscillations while σ stays almost constant and in this sense, we can treat the Hamiltonian (53) in the frame of the adiabatic invariant approximation.

Figures 5 and 6 show the resonant structure of (53) with $e' = 0.05, 0.2, 0.5$ and $e(L4) = 0.2$ assuming fixed σ ($\varpi = 0, \pi/2, \pi, 3\pi/2$). The thick line is the crossing curve computed from the condition of a collision, assuming elliptic orbits of the small body and planet and $\varpi - \varpi'$ fixed.

Comparing Figs. 5 and 6 with Fig. 1b, we can see how the non–zero eccentricity of the perturbing planet modifies the resonant orbits. If $e' = 0.05$, the change with respect to circular model is small, but already for $e' = 0.2$, the separatrices and the centers of tadpole orbits are modulated by the phase of ϖ. The phase portrait is symmetrical with respect to $\sigma_{1/1} = \pi$ for $\varpi = 0$ and π (Fig. 5) and is asymmetrical for $\varpi = \pi/2$ and $3\pi/2$ (Fig. 6). This is determined by the position of the crossing curve. When $\varpi = \pi/2$, the tadpole and horseshoe orbits shift towards smaller $\sigma_{1/1}$ and the domain associated with L4 becomes larger than the one around L5 (Figs. 6c and 6e). When $\varpi = 3\pi/2$, the opposite happens (Figs. 6d and 6f).

The ϖ–modulated pulsation of separatrices means in practice that an object initially near the separatrix between tadpole and horseshoe domains may undergoes transitions between domains. These transitions create a chaotic layer around the separatrix as in the case of a modulated pendulum.

The secular dynamics, that is the evolution of the eccentricity with the precession of ϖ in the planar–elliptic model can be computed by the following method. Having the Hamiltonian written as (53), and assuming ϵ small, the Arnold action–angle variables for the integrable part $\mathcal{H}_0$ can be computed by the method of Henrard (1990) and introduced into the perturbation $\mathcal{H}_1$. The resulting Hamiltonian can then be averaged over the new angle associated with $\sigma_{1/1}$, which is a much faster variable than ϖ in the whole resonant domain.

This procedure is simple in the limit of zero amplitude libration around the stable equilibrium points at the center of tadpole and retrograde satellite domains. In this case, the averaging transformation is an identity, and the evolution of S, σ can be computed as level curves of the Hamiltonian:

$$\mathcal{H} = \mathcal{H}_0(S^0_{1/1}, \sigma^0_{1/1}, S) + \mathcal{H}_1(S^0_{1/1}, \sigma^0_{1/1}, S, \sigma; e') \ , \tag{54}$$

where $S^0_{1/1}$ and $\sigma^0_{1/1}$ correspond to the asymmetric or satellite centers (which are functions of S).

In Figs. 7a, 7c and 7e , we show the level curves resulting from this calculation for $e' = 0.05, 0.2, 0.5$ of a Jupiter–mass planet. Similar figures has been obtained by an equivalent procedure by Giacaglia and Nacozy (1970). For $e' = 0.05$ and 0.2, (Figs. 7a and 7c) the correspondence with this work is reasonable (Giacaglia and Nacozy figures have low resolution because they computed the Hamiltonian on a grid 10×16; here 100×100). We identify two libration islands, one at small and the other at large eccentricities. The center of the small eccentricity libration island is roughly at $\varpi = 60$ deg and the eccentricity is roughly equal to the eccentricity of the perturbing planet (Lagrange periodic orbit).

The trajectories are strangely distorted for $e' = 0.5$ (Fig. 7e). The center of the left libration island is now at $\varpi \sim 70$ deg and at much smaller eccentricity ($e \sim 0.35$) than the planet's one. This is an artifact of the computational method because the $\mathcal{H}_1$ becomes large with large e' and $\mathcal{H}_0$ is no longer a good approximation of the full Hamiltonian. We propose here another approach inspired by Wisdom (1985).

This algorithm is motivated by the separation of time scales of two degrees of freedom. Typically, $f_{1/1} \gg \dot{\sigma}$, where $f_{1/1}$ is the libration frequency of $\sigma_{1/1}$ and $\dot{\sigma}$ is the time derivative of σ.[5] Intuitively, we may assume S, σ as fixed parameters in (53) so that $\mathcal{H}$ becomes integrable and one can introduce Arnold action-angle variables (J, ψ)

[5] In the case of Jupiter, the period of tadpole libration of $\sigma_{1/1}$ is typically 140–250 years, while the period of ϖ is 3,000–10,000 years.

for the remaining degree of freedom $(S_{1/1}, \sigma_{1/1})$. The definition of the action is:

$$J = \frac{1}{2\pi}\oint S_{1/1}\mathrm{d}\sigma_{1/1} = J(S_{1/1}, \sigma_{1/1}, S, \sigma) \ , \tag{55}$$

where the integral is calculated over a closed trajectory of $\mathcal{H}(S_{1/1}, \sigma_{1/1}, S, \sigma)$ (and not of $\mathcal{H}_0$!) assuming S, σ as constant parameters. The variables J, ψ are then inserted in (53), giving $\mathcal{F}_0(J, S, \sigma)$, which we assume to be the new Hamiltonian. If so, the action J becomes an integral of motion and the secular dynamics in S, σ is then described by $\mathcal{F}_0(J, S, \sigma)$, where J is a parameter. The phase diagrams in (S, σ) can be obtained as level curves of $\mathcal{F}_0$.

This procedure is appealing, but the above description invokes a non-rigorous step from because the transformation $(S_{1/1}, \sigma_{1/1}, S, \sigma) \rightarrow (J, \psi, S, \sigma)$, considered in the 4-dimensional space, is not canonical. An algorithm inspired by the above procedure but with rigorous justification has been suggested to us by A. Giorgilli. The recipe is the following. To construct a canonical transformation $S_{1/1}, \sigma_{1/1}, S, \sigma \rightarrow J, \psi, S^*, \sigma^*$, we introduce the generating function:

$$W(J, \sigma_{1/1}, S^*, \nu) = \int S_{1/1}(J, \sigma_{1/1}, S^*, \sigma)\, \mathrm{d}\sigma_{1/1} + S^*\sigma \ , \tag{56}$$

where the function $S_{1/1}(J, \sigma_{1/1}, S^*, \sigma)$ is the inversion of (55), in which S has been formally substituted by S^*. The integral in (55) is calculated over a closed level curve of $\mathcal{H}(S_{1/1}, \sigma_{1/1}, S, \sigma)$ (eq. 53) in the plane $S_{1/1}, \sigma_{1/1}$, with S and σ as parameters.

The canonical transformation generated by (56) is

$$\begin{aligned} S_{1/1} &= \frac{\partial W}{\partial \sigma_{1/1}} \ , \qquad & S &= \frac{\partial W}{\partial \sigma} = S^* + \rho_1(J, \sigma_{1/1}, S^*, \nu) \ , \\ \psi &= \frac{\partial W}{\partial J} \ , \qquad & \sigma^* &= \frac{\partial W}{\partial S^*} = \sigma + \rho_2(J, \sigma_{1/1}, S^*, \nu) \ , \end{aligned} \tag{57}$$

where $\rho_1 = \frac{\partial}{\partial \sigma} \int S_{1/1}\,\mathrm{d}\sigma_{1/1}$ and $\rho_2 = \frac{\partial}{\partial S^*} \int S_{1/1}\,\mathrm{d}\sigma_{1/1}$. The new Hamiltonian becomes:

$$\mathcal{F}_0(J, S^*, \sigma^*) + \mathcal{F}_1(J, S^*, \psi, \sigma^*) \ , \tag{58}$$

where $\mathcal{F}_0$ is a zero order term of the Taylor expansion in ρ_1, ρ_2 and $\mathcal{F}_1 \sim \mathcal{O}(\max(\rho_1, \rho_2))$. The functions ρ_1 and ρ_2 are small for small J, because $S_{1/1} = \sqrt{a}$ and the semi-major axis of the level curves with small amplitudes is insensitive to both S and σ. Moreover, up to the first order in $\max(\rho_1, \rho_2)$:

$$\mathcal{F}_1 \sim \left(\frac{\partial \mathcal{H}}{\partial S}\right) \frac{\partial}{\partial \sigma} \int S_{1/1}\,\mathrm{d}\sigma_{1/1} - \left(\frac{\partial \mathcal{H}}{\partial \sigma}\right) \frac{\partial}{\partial S^*} \int S_{1/1}\,\mathrm{d}\sigma_{1/1} \ , \tag{59}$$

where $\mathcal{H}$ is the one of eq. (53), $\partial \mathcal{H}/\partial S = \dot{\sigma}$, and $\partial \mathcal{H}/\partial \sigma = -\dot{S}$, where $\dot{\sigma}$ and $\dot{S}$ are the time derivatives corresponding to the slow degree of freedom. As $\dot{\sigma}$ and $\dot{S}$ are small, the coefficients appearing in the Taylor expansion (58) are small, and $\mathcal{F}_1$ may be neglected. Consequently, $\mathcal{F}_0(J, S, \sigma)$ becomes the new Hamiltonian. This justifies the heuristic approach based on the time-scales separation of the two degrees of freedom described in the paragraph of eq. (55).

The above algorithm makes sense only in a domain in the phase space where J is an analytic function of S, σ. It happens that due to a large secular modulation of the 1/1 resonant structure, the cycle corresponding to the action J for which the calculation is done may topologically change breaking the differentiability of (55). An example of this is shown in Fig. 7f for the trajectories at large eccentricity.

The whole scheme simplifies when we limit our analysis to $J = 0$. In this case, $S^* = S$, $\sigma^* = \sigma$, and $\mathcal{F}_1 = 0$. To obtain the Hamiltonian, one simply fixes $S_{1/1}, \sigma_{1/1}$ at the stationary point $S^0_{1/1}, \sigma^0_{1/1}$ of $H(S_{1/1}, \sigma_{1/1}; S, \sigma)$, computed with S and σ as fixed parameters. In fact, the only difference with respect to the procedure based on (54) is that now the stationary point is not the one of $\mathcal{H}_0(S_{1/1}, \sigma_{1/1}, S)$ (Fig. 1) but of $\mathcal{H}(S_{1/1}, \sigma_{1/1}; S, \sigma)$ (Figs. 5 and 6).

We compute the averaged Hamiltonian for the locus of L4 tadpole island (Figs. 7b, 7d and 7f)[6]. For $e' = 0.05$ (Fig. 7b), we recover the phase space portrait computed by the classical method of Giacaglia & Nacozy (1970) (Fig. 7a). For $e' = 0.2$ and 0.5, there are significant differences between our and classical computations. The large central island, dominating the secular dynamics in Fig. 7e is shifted to higher eccentricities (Fig. 7f) and is largely modified. The center of the island near $\varpi = 60$ deg is also shifted to larger eccentricities (equal to e' of the perturbing planet) and its deformation is removed.

The thick lines in Figs. 7b, 7d and 7f show the orbital evolution of several test particles initially placed in a non–averaged three–body problem at the center of tadpole libration. These integrations has been performed using *Snail* integrator (programmed by D.N.). The orbital elements were digitally filtered to assure better correspondence with the averaged problem. The correspondence is good and shows an excellent validity of our approach.

The void region in high eccentricities in Figs. 7b, 7d and 7f, where we do not plot the level curves, is the place where the L4 tadpole center does not exist. The trajectory evolving along a level curve and hitting this region suffers a transition to horseshoe or compound orbit. Our perturbation method cannot follow it over this transition. Such trajectories probably suffer sooner or later a close encounter to the planet. Examples of this evolution are the two trajectories of test particles integrated in high eccentricities in a non–averaged problem (Fig.

[6] The level curves computed for L5 tadpole center are mirrored equivalents of level curves of L5 with respect to $\varpi = \pi$.

7f), which escape from the $1/1$ resonance after entering the void region, suffering transition from the center of tadpole domain to the retrograde satellite domain and having a close encounter with the planet.

Figure 8 shows the secular evolution of e and ϖ at the retrograde satellites center for $e' = 0.01, 0.05, 0.1, 0.2, 0.3$ and 0.5. The phase space portrait is dominated by two islands at $\varpi = 0$ and π which increase with increasing e'. At $e' = 0.5$, all retrograde satellite orbits starting with zero eccentricity attain the Sun–grazing orbits as $e \to 1$.

6.3. The spatial–circular case

Due to the invariance of the disturbing function by rotation of the coordinate system and by inversion of the Z axis, the Hamiltonian of the averaged spatial–circular model has the form:

$$\mathcal{H} = \mathcal{H}_0(S_{1/1}, \sigma_{1/1}, S) + \mathcal{H}_1(S_{1/1}, \sigma_{1/1}, S, S_z, \sigma_z - \sigma = \omega) \ , \qquad (60)$$

where $\mathcal{H}_0$ is the planar–circular approximation and $\mathcal{H}_1$ is its perturbation depending on the inclination. The canonical transformation:

$$S, \sigma, S_z, \sigma_z \to S, \sigma^* = -\omega, S_z^* = S + S_z, \sigma_z \qquad (61)$$

transforms the above Hamiltonian into

$$\mathcal{H} = \mathcal{H}_0(S_{1/1}, \sigma_{1/1}, S) + \mathcal{H}_1(S_{1/1}, \sigma_{1/1}, S, \sigma^*, S_z^*) \ , \qquad (62)$$

where $S_z^* = \sqrt{a}(1 - \sqrt{1-e^2}\cos i)$ being constant, plays a role of a parameter.

The classical treatment of the above Hamiltonian is similar to the one discussed in the previous section: after the substitution of the Arnold action–angle variables for $\mathcal{H}_0$, the Hamiltonian is averaged over the new angle associated to $\sigma_{1/1}$, which results in an integrable system in the (S, σ^*) degree of freedom parameterized by S_z^* and by the action associated with the averaged $(S_{1/1}, \sigma_{1/1})$ degree of freedom.

Kozai (1985) computed the level curves of the above Hamiltonian for Jupiter, fixing $\sigma_{1/1} = 60$ deg and $a = n'^{-2/3}$. In Figs. 9a, 9c and 9e, we reproduce his result. The shown surfaces of $S_z^* = const.$ are the ones characterized by $e_{\max} = 0.4$ (a), 0.6 (c) and 0.8 (e), where the corresponding S_z^* is computed from $S_z^* = \sqrt{a}(1 - \sqrt{1 - e_{\max}^2})$. The quantity $e_{\max}$ is the largest eccentricity which can be obtained on $S_z^* = const.$ and is related to the largest possible inclination on $S_z^* = const.$ through $e_{\max} = \sin i_{\max}$. The shape of curves $S_z^* = const.$ in (e, i) space makes clear that the evolutions of e and i are anti–correlated: when e increases, i decreases and vice versa.

In Figs. 9b, 9d and 9f, some cases are shown, but now not assuming the position of the asymmetric stationary point of $\mathcal{H}_0$ at $\sigma_{1/1} = 60$ deg and $a = n'^{-2/3}$ as in Kozai (1985) and computing the dependence of the asymmetric stationary point of $\mathcal{H}_0$ as a function of S. The difference between our and Kozai's results is evident. This comparison reveals that the libration islands found by Kozai (1985) in the 1/1 mean motion resonance were an artifact of his calculation. Our computation indicates that even at large e, i of the asymmetric stationary point no ω–libration exists. But this is only the first part of the story.

Figure 10 shows the resonant structure of (62) computed for a Jupiter-mass planet and the $S = const$ level corresponding to $e = 0.2$ at the locus of tadpole trajectories. We assume a fixed inclination ($i = 23.6$ peg in (ab), 36.8 deg in (cd) and 53.1 in (ef)) and fixed argument of perihelion ($\omega = 0$ in (ace) and $\pi/2$ in (bdf)). The thick lines in (bdf) are the collision curves. Here, as the motion happens in space, it is not only $a, e, \sigma_{1/1}$ which define a collision event, but also ω. In fact, ω must verify

$$\cos\omega = \pm\frac{1}{e}\left[\frac{a}{a'}(1-e^2)-1\right] \tag{63}$$

to have an intersection of the small body and planet orbits in the ascending (+) or descending node (−) of the small body orbit. For $e \sim 0.2$ and $a \sim a'$ as in Fig. 10, the intersection of orbits happen for $\omega \sim 80$ deg. This is the reason for which we plot the collision curve (computed through the eccentric anomaly and the Kepler equation as in the planar–circular case) only in right panels of Fig. 10, where $\omega = \pi/2$ and the orbits are close to an intersection. Conversely, the orbits are well separated in space when $\omega = 0$. Figure 10 shows how the structure of the co-orbital motion changes with the appearance and disappearance of the collision curve.

This modulation of the phase space portrait with ω is even more pronounced for larger eccentricity of the small body. Figures 11 and 12 show this for $e = 0.5$ at the locus of the tadpole trajectories. In this case, the intersection of orbits happens at $\omega \sim 60, 120, 240$ and 300 deg. We show the crossing curve only for $\omega = \pi/4$ and $3\pi/4$, which are not exactly the crossing values but are only 15 deg away from them.

Apart from the known domains of motion (tadpole, horseshoe and satellite), there appear also orbits enclosing both asymmetric and satellite domains (Figs. 11c, 11d and 11f); orbits similar to the retrograde satellites, but having the almost full 2π amplitude in $\sigma_{1/1}$ (Fig. 11e); orbits enclosing one asymmetric and satellite domains (either with –Figs. 12c and 12d– or without the preservation of the other asymmetric island –Figs. 12d and 12f); and orbits enclosing the horseshoe and satellite domains (Fig. 12a and 12b). This complex structure of the spatial co-orbital motion modulated by ω was noted by Namouni (1999).

Following his terminology, we refer to the new types of orbits as the compound orbits.

In order to take into account this changing structure in the computation of the secular evolution of ω we use the new perturbation treatment of section 6.2. Restricting the study only to equilibrium point $S^0_{1/1}, \sigma^0_{1/1}$ of

$$\widetilde{\mathcal{H}}_0 = \mathcal{H}_0(S_{1/1}, \sigma_{1/1}, S) + \mathcal{H}_1(S_{1/1}, \sigma_{1/1}, S, \sigma_0^*, S_z^*) \ , \qquad (64)$$

which is parameterized by S and S_z^*, the perturbation procedure simplifies in a similar manner as described in section 6.2.

Figure 13 shows the result for the L4 asymmetric center, a Jupiter–mass planet and six different inclinations. The void region is where the asymmetric equilibrium point of (64) does not exist. A trajectory evolving along one of the level curves and hitting the void region suffers transitions and is generally unstable. This can be seen in Figs. 13d and 13e, where three trajectories have been numerically integrated in the non–averaged model with initial conditions at the L4 asymmetric center and $e = 0.05, 0.2$ and 0.4. For this comparison, the orbital elements were smoothed by a digital filter. All trajectories evolve in good correspondence with the level curves increasing their e and ω until they reach the void domain and suffer transitions (generally to near–separatrix compound orbits). Some of them later come back to the asymmetric domain (such as the trajectory started at $e = 0.2$ in Fig. 13d) and some of them are ejected from the 1/1 resonance due to a close approach to the planet.

In the right column of Fig. 13, where the motion at higher inclinations is shown, the computed trajectories are limited to small eccentricities. This is due to the fact that the L4 tadpole libration does not exist in larger eccentricities.

The new perturbation approach gives an image of the co-orbital dynamics which is more complete than the classical approach (compare Figs. 13b, 13d and 13f with Fig. 9b, 9d and 9f). While for $i_{\max} =$ 23.6 and 36.8 deg both methods give basically the same result (one can however notice that the 'waves' in Fig. 13c are somewhat more pronounced than in Fig. 9d), the new approach correctly identifies the maximum domain in ω, where the secular evolution can be computed for higher e and i (Fig. 13bdef).

Figure 14 shows the secular dynamics at the center of retrograde satellite orbits at several inclinations. The trajectories touching the void region are forced by the secular dynamics to transit from the center of the satellite region to a compound orbit. One can see that satellites with the heliocentric eccentricity superior to ~ 0.4 are protected from transitions

Namouni (1999) has numerically computed the secular evolution of several tadpole, horseshoe and satellite orbits for terrestrial–mass planets, showing the existence of ω–librations for trajectories transiting between horseshoe and satellite regimes. Our results are complementary to his, showing the secular evolution on the loci of tadpole and satellite orbits for a Jupiter–mass planet. Basically, we recover by analytic means the same complex structure of the co-orbital domain at large inclinations, but shifted to larger e and i. Apparently, this is the effect of the larger mass of the perturbing planet used here.

7. Conclusions

We developed in this paper a non–singular formalism to compute the Hamiltonian function and its derivatives for whatever eccentricities and inclinations of the small body and planet, having in mind the problem of co-orbital motion. We generalized the classical perturbation treatment in order to account for the secular motion inside the 1/1 mean motion resonance, in the case of eccentric planetary orbits or high–inclined orbits of small bodies. This method has an excellent validity, which was demonstrated by the comparisons with numerical simulations.

The resonant and secular dynamics of a small body in the 1/1 mean motion resonance with a Jupiter–mass planet has been computed. Accounting for an eccentric orbit of the perturbing planet or for the mutual inclination, results in a number of new types of orbits. The secular dynamics at the tadpole and satellite centers was studied. The new perturbation approach allowed us to compute the global picture of the secular dynamics in the space where the tadpole or satellite domains exist. We have understood the way how the resonant structure changes in secular evolution and the rule of transitions imposed by this evolution. The trajectories having transitions are chaotic and generally unstable.

In the next paper, we will use the developed technique in a systematic exploration of the secular dynamics of co-orbital regime in the solar system. Moreover, we envisage the use of our formalism in an algorithm for the proper elements computation of Trojan asteroids located around the L_4 and L_5 Lagrangian points of the Sun–Jupiter system.

8. Acknowledgments

The authors wish to thank Alessandro Simula, Fernando Roig and Helena Morais for their stimulating suggestions. The work of Fabrice Thomas was supported by the Research Foundation of the State of

São Paulo (FAPESP). The work of David Nesvorný was sponsored by the 'Centre National de la Recherche Scientifique' in the frame of the Poincaré postdoctoral fellowship.

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FIGURE CAPTIONS:

Figure 1: The resonant structure of the co-orbital regime in the planar–circular model with a Jupiter–mass planet. Four $S = const.$ surfaces are shown corresponding to $e = 0, 0.2, 0.4$ and 0.8 at the locus of L4 tadpole librations. The bold lines are the collision curves.

Figure 2: The curves of $T = const.$ (full lines) and $S = const.$ (dashed lines). The phase portraits shown in Fig. 1 correspond to the 2nd (Fig. 1b), 4th (Fig. 1c) and 8th dashed curves of $S = const.$ counted from the bottom.

Figure 3: The sizes of the tadpole, horseshoe and satellite domains for the jovian planets. Here, $e' = 0$ and co-planar motion are assumed. The dashed lines are the curves above which collisions with other–than–resonant planets occur.

Figure 4: The same as Fig. 3 but for the terrestrial planets.

Figure 5: The resonant structure of the co-orbital regime in the planar–elliptic model with a Jupiter–mass planet. Three values of the planets' eccentricity ($e' = 0.05, 0.2, 0.5$) and two perihelion longitudes ($\varpi = 0, \pi$) are assumed. The bold lines are the collision curves.

Figure 6: The same as Fig. 5 but with $\varpi = \pi/2$ (left) and $3\pi/2$ (right).

Figure 7: The secular evolutions of e and ϖ at the locus of L4 tadpole librations in the planar–elliptic model with a Jupiter–mass planet. Three values of the planets' eccentricity are assumed ($e' = 0.05, 0.2, 0.5$). Left: the level curves computed by the Giacaglia and Nacozy (1970) method identifying the H_0 with the Hamiltonian of the planar–circular model. Right: the level curves computed by the new algorithm. The dots show the evolutions of several test particles numerically integrated in the non-averaged planar–elliptic model. We show 6 test particles per figure. The test particles started at $e = 0.4, 0.6$ and $\varpi = 0$ in (f) were ejected from the 1/1 resonance within the integration time span (100,000 yr).

Figure 8: The same as Fig. 7 but for the locus of retrograde satellites orbits. The level curves are computed by the new algorithm for six eccentricities of the perturbing planet ($e' = 0.01, 0.05, 0.1, 0.2, 0.3$ and 0.5)

Figure 9: The secular evolutions of e and ω at the locus of L4 tadpole librations in the spatial–circular model with a Jupiter–mass planet. Three values of $S_z^* = const.$ are assumed and the figures are labeled by the maximum inclination possible at each level ($i_{\max} = 23.6, 36.8, 53.1$ deg)

Left: the level curves computed as in Kozai (1985) assuming $\sigma^0_{1/1} = 60$ deg and $a^0 = n'^{-2/3}$. Right: the level curves computed by the classical perturbation algorithm, identifying H_0 with the Hamiltonian of the planar–circular model, but numerically computing $\sigma^0_{1/1}$ and a^0, which differ from the values assumed by Kozai for orbits with moderate to large eccentricities.

Figure 10: The resonant structure of the co-orbital regime in the spatial–circular model with a Jupiter–mass planet. The $S = const.$ is fixed so that $e(L4) = 0.2$. Three inclinations ($i = 23.6, 36.8, 53.1$ deg) and two perihelion arguments ($\omega = 0, \pi/2$) are assumed. The bold lines are the collision curves.

Figure 11: The same as Fig. 10 but with $e(L4) = 0.5$.

Figure 12: The same as Fig. 11 but with $\omega = 0$ (left) and $\pi/2$ (right).

Figure 13: The secular evolutions of e and ω at the locus of L4 tadpole librations in the spatial–circular model with a Jupiter–mass planet. Six values of $S^*_z = const.$ are assumed and the figures are labeled by the maximum possible inclination ($i_{\max} = 23.6, 36.8, 53.1, 64.2, 71.8, 87.4$ deg). The level curves are computed by the new perturbation algorithm, locally identifying H_0 with the Hamiltonian of the spatial–circular model. We do not plot the level curves in the region where L4 tadpole orbits do not exist. The dots show the evolutions of several test particles numerically integrated in the non-averaged spatial–circular model. We show three test particles per figure. The test particles are started at $e = 0.05, 0.2, 0.4$ and $\varpi = 2$. Two integrated trajectories in (e) and all trajectories in (d) undergo transitions to the retrograde satellite domain.

Figure 14: The same as Fig. 13 but for the locus of the retrograde satellite domain.