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The water bag model and gyrokinetic applications

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Abstract

Predicting turbulent transport in nearly collisionless fusion plasmas requires to solve kinetic (or more precisely *gyrokinetic*) equations. In spite of considerable progress, several pending issues remain; although more accurate, the kinetic calculation of turbulent transport is much more demanding in computer resources than fluid simulations. An alternative approach is based on a water bag representation of the distribution function which is not an approximation but rather a special class of initial conditions allowing to reduce the full kinetic Vlasov equation into a set of hydrodynamic equations while keeping its kinetic character. This model has been applied to gyrokinetic modelling with very encouraging results. The instability threshold for ITG instability is found to be very close to the results obtained from continuous Maxwellian distribution, even for only 10 bags.

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1. Introduction

Low frequency ion-temperature-gradient driven (ITG) instabilities are now commonly held responsible for turbulence giving rise to anomalous radial energy transport in the core of tokamaks. The computation of turbulent thermal diffusivities in fusion plasmas is of prime importance since the energy confinement time is determined by these transport coefficients. During recent years, ion turbulence in tokamaks has been intensively studied both with fluid (see for instance [1–3]) and gyrokinetic simulations using PIC codes [4–6] or Vlasov codes [7–10]. Although more accurate, the kinetic calculation of turbulent transport is much more demanding in computer resources than fluid simulations.

Introduced initially by DePackh [11], Hohl, Feix and Bertrand [12–14] the water bag model was shown to bring the bridge between fluid and kinetic description of a collisionless plasma, allowing to keep the kinetic aspect of the problem with the same complexity as the fluid model. It is the aim of this paper to revisit the water bag model and its possible application to gyrokinetic modelling.

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2. The water bag

Vlasov equation is a difficult one mainly because of its high dimensionality. For each particle species the distribution function $f(\mathbf{r},\mathbf{v},t)$ is defined in a 6D phase space $(\mathbf{r},\mathbf{v})$. The simplest (one spatial dimension, one velocity dimension) implies a 2D (x,v) phase space. Can it be reduced to the sole real space $\mathbf{r}$ as in usual hydrodynamics? In that last case the presence of collisions with frequency much greater than the inverse of all characteristic times implies the existence of a local thermodynamic equilibrium characterised by a density $n(\mathbf{r},t)$, an average velocity $\mathbf{u}(\mathbf{r},t)$ and a temperature $T(\mathbf{r},t)$. A priori in a plasma the distribution function $f(\mathbf{r},\mathbf{v},t)$ is an arbitrary function of $\mathbf{r}$ and $\mathbf{v}$ (and t of course) and phase space is unavoidable.

But consider a 1D plasma (2D phase space) in which at initial time the situation is as depicted in Fig. 1. Between the two curves v_+ and v_- we impose $f(x,v,0)=A$ (A is a constant). For velocities bigger than v_+ and smaller than v_- we have $f(x,v,0)=0$.

According to phase space conservation property of the Vlasov equation, as long as v_+ and v_- remain single valued function, $f(x,v,t)$ remains equal to A for values of v such that $v_-(x,t)<v<v_+(x,t)$. Therefore, the problem is entirely described by the two functions v_+ and v_-. Since a hydrodynamic description involves n, u and P (respectively density, average velocity and pressure) we can predict the possibility of casting the WB model into the hydrodynamic frame with, in addition, an automatically provided state equation.

Remembering that a particle on the contour v_+ (or v_-) remains on this contour the equations for v_+ and v_- are (for instance for an electron plasma, E being the electric field and e the electron charge)

$$\frac{\mathrm{D}v_\pm}{\mathrm{D}t}=\frac{\partial v_\pm(x,t)}{\partial t}+v_\pm\frac{\partial v_\pm}{\partial x}=\frac{e}{m}E(x,t). \tag{1}$$

Now let us introduce the density $n(x,t)=A(v_+-v_-)$ and the average (fluid) velocity $u(x,t)=\frac{1}{2}(v_++v_-)$ into Eq. (1) by adding and subtracting these two equations. We obtain

$$\frac{\partial n}{\partial t}+\frac{\partial}{\partial x}(nu)=0, \tag{2}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=-\frac{1}{mn}\frac{\partial P}{\partial x}+\frac{e}{m}E, \tag{3}$$

$$Pn^{-3}=\frac{m}{12A^2}. \tag{4}$$

Eqs. (2)–(4) are respectively the continuity, Euler and state equation. This hydrodynamic description of the water bag model was pointed out for the first time by Bertrand and Feix [13] but the state equation (4) describes an invariant both in space and time while in the hydrodynamic model we obtain $\frac{\mathrm{D}}{\mathrm{D}t}(Pn^{-\gamma})=0$. It must be noticed that the physics in the two cases is quite different [20].

Linearising Eq. (1) around and homogeneous equilibrium i.e. $v_\pm(x,t)=\pm a+w_\pm(x,t)$ for an electronic plasma (where $\pm a$ are the constant values for this equilibrium and $w_\pm$ a small perturbation with $|w_\pm|\ll a$) yields the simple dispersion relation for a harmonic perturbation $\omega^2=\omega_p^2+k^2a^2$. Furthermore, computing

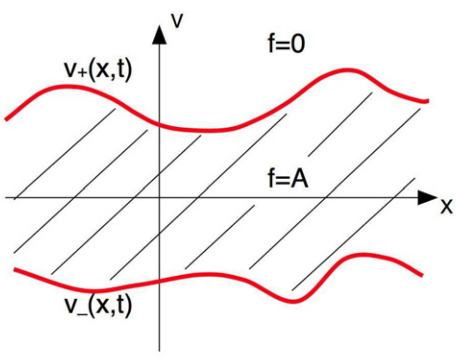

Fig. 1. The water bag model in phase space.

the thermal velocity $v_T^2 = \frac{1}{n_0}\int_{-\infty}^{+\infty} v^2 f_0(v)\mathrm{d}v = a^2/3$ allows to recover exactly the Bohm–Gross dispersion relation $\omega^2 = \omega_p^2 + 3k^2 v_T^2$.

Thus it is very easy to construct the water bag associated to a homogeneous distribution function characterised by a density n_0 and a thermal velocity v_T: we just have to choose the water bag parameters (a and A) as follows: $a = \sqrt{3}v_T$ and $A = n_0/2a$. Of course there is no Landau resonance since the phase velocity $v_\varphi = (a^2 + \omega_p^2/k^2)^{1/2}$ is greater than a and thus lies in a region of velocity space where there are *no particles*. To recover the Landau damping the water bag has to be generalised into the multiple water bag.

3. Multifluids and multiple water bag models (MWB)

This generalisation was straightforward [15–17]; Berk and Roberts [18] and Finzi [19] used a double WB model to study the two stream instability in a computer simulation including the filamentation of the contours and their multivalued nature (a highly difficult problem from a programming point of view).

Let us consider $2N$ contours in phase space labelled v_j^+ and v_j^- (where $j = 1, \ldots, N$). Fig. 2 shows a three-bag system ($N = 3$) where the distribution function takes on three different constant values F_1, F_2 and F_3.

Introducing the *bag* heights A_1, A_2 and A_3 as shown also in Fig. 2 the distribution function writes

$$f(x, v, t) = \sum_{j=1}^{N} A_j(\Upsilon(v - v_j^-(x,t)) - \Upsilon(v - v_j^+(x,t))), \tag{5}$$

where Υ is the Heaviside unit step function. Notice that some of the A_j can be negative.

Let us now introduce for each bag j the density n_j, average velocity u_j and pressure P_j as done above for the one-bag case i.e. $n_j = A_j(v_j^+ - v_j^-)$, $u_j = (1/2)(v_j^+ + v_j^-)$ and $P_j n_j^{-3} = m/(12A_j^2)$. For each bag j we recover the continuity and Euler equation as written in (2) and (3) namely

$$\frac{\partial n_j}{\partial t} + \frac{\partial}{\partial x}(n_j u_j) = 0, \tag{6}$$

$$\frac{\partial u_j}{\partial t} + u_j\frac{\partial u_j}{\partial x} = -\frac{1}{mn_j}\frac{\partial P_j}{\partial x} + \frac{e}{m}E. \tag{7}$$

The coupling between the bags is given by the total density $\sum_j n_j$ in the Poisson equation

$$\frac{\partial E}{\partial x} = \frac{e}{\varepsilon_0}\left(\sum_{j=1}^{N} n_j - n_0\right). \tag{8}$$

Linearising Eqs. (6)–(8) for an electronic plasma around an homogeneous (density n_0) equilibrium i.e. $v_j^\pm(x,t) = \pm a_j + w_j^\pm(x,t)$ with $|w_j^\pm| \ll a_j$ yields the dispersion relation

$$1 - \frac{\omega_p^2}{n_0}\sum_{j=1}^{N}\frac{2a_jA_j}{\omega^2 - k^2a_j^2} = 0. \tag{9}$$

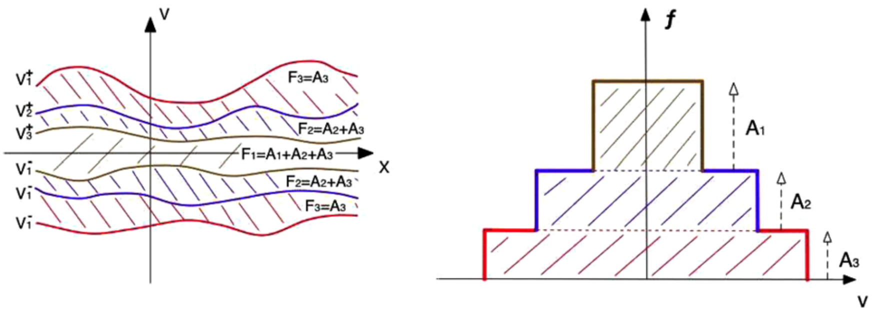

Fig. 2. Multiple water bag: phase space plot for a three-bag model (left) and corresponding MWB distribution function (right).

If all A_j's are positive (single hump distribution function) this equation has $2N$ real frequencies located between $\pm a_j$ and $\pm a_{j+1}$. The Landau damping is recovered as a phase mixing process of real frequencies [15,21] which is reminiscent of the Van Kampen–Case treatment of the electronic plasma oscillations [22,23].

The connection with a multifluid model is more illuminating if we consider the equivalence *in the fluid momentum sense* of a multiple water bag distribution and a continuous distribution.

Let us consider an homogeneous equilibrium distribution function $f_0(v)$. For simplicity reason we suppose f_0 is an even function of v (odd momenta are zero). In the water bag formalism it means symmetrical equilibrium contours $\pm a_j$. Let us define the ℓ-momentum of f_0 (ℓ even only):

$$\mathscr{M}_\ell(f_0) = \int_{-\infty}^{\infty} v^\ell f_0(v)\mathrm{d}v \tag{10}$$

and the ℓ-momentum of the corresponding water bag

$$\mathscr{M}_\ell(\mathrm{WB}) = \frac{1}{\ell+1}\sum_j 2A_j a_j^{\ell+1}. \tag{11}$$

Let us now sample the v-axis with appropriate a_j's. Thus equating Eqs. (10) and (11) for $\ell = 0, 2, \ldots, 2(N-1)$ yields a system of N linear equations for the N unknown $A_j, j = 1, \ldots, N$. Using an integration by parts we get

$$\sum_j 2A_j a_j^{\ell+1} = -\int_{-\infty}^{\infty} v^{\ell+1}\frac{\mathrm{d}f_0}{\mathrm{d}v}\mathrm{d}v, \quad \ell = 0, 2, \ldots, 2(N-1). \tag{12}$$

A water bag model with N bags is equivalent to a continuous distribution function for momenta up to $\ell_{\max} = 2(N-1)$. Nevertheless Eq. (12) has the form of a Vandermonde system which becomes ill-conditioned for higher values of the number of bags N (for instance for $N = 15$) and a cut-off in velocity space $a_N = 5v_T$ the matrix elements vary from 1 to 5^{28}!

A more convenient solution can be found for a regular sampling $a_j = \left(j - \frac{1}{2}\right)\Delta a$ and is explained in Fig. 3: consider F_j at the middle of the interval $\Delta a = \frac{2a_N}{2N-1}$ and compute $F_j = f_0\left(a_j - \frac{\Delta a}{2}\right)$. An approximate (up to second order in Δa) solution for Eq. (12) is obtained from

$$\frac{1}{\Delta a}\left(f_0\left(a_j + \frac{\Delta a}{2}\right) - f_0\left(a_j - \frac{\Delta a}{2}\right)\right) = \frac{F_{j+1} - F_j}{\Delta a} = \frac{-A_j}{\Delta a} = \frac{\mathrm{d}f_0}{\mathrm{d}v} + \mathscr{O}(\Delta a^2). \tag{13}$$

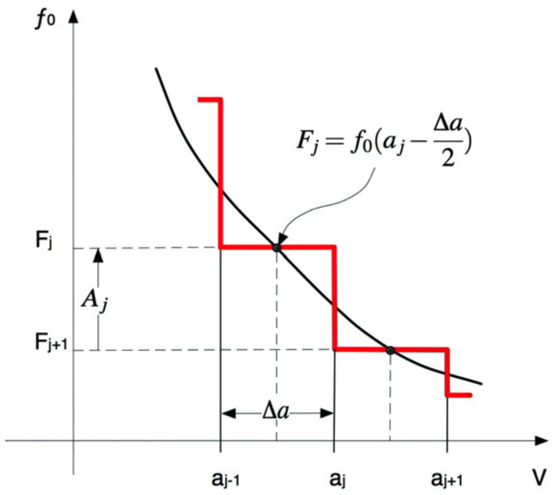

Fig. 3. Constructing the bags from a continuous distribution.

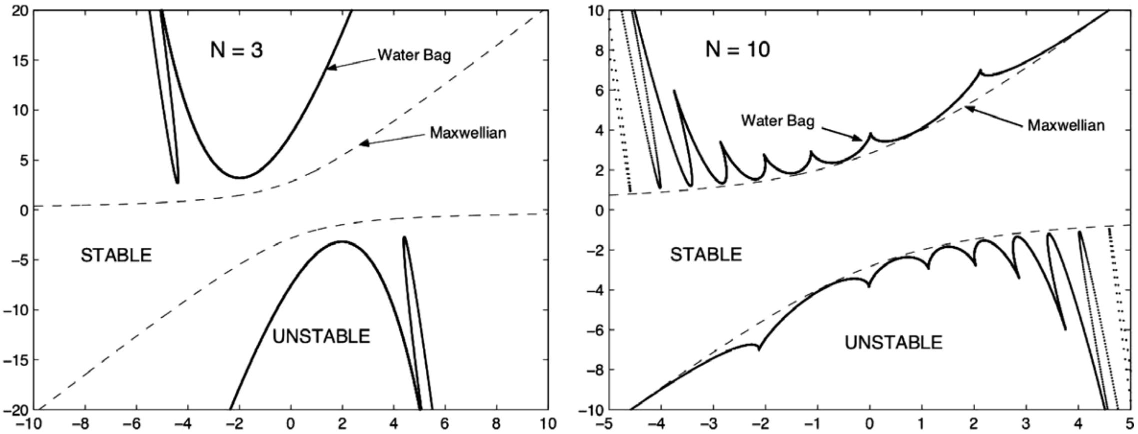

Fig. 4. Threshold in the (κ_T, κ_n) plane (arbitrary units) for $N = 3$ and $N = 10$. The dashed and solid lines correspond respectively to continuous and water bag distribution functions.

For N larger than 10–15 this approximate solution gives rather good results for a Maxwellian distribution function in the sense that the MWB-momenta (11) are close to the Maxwellian values (10). Of course for N smaller than 10 the exact A_j's can be obtained solving directly the linear system (12).

As a final remark it is important to point out that the use of MWB appears as a reduction of the phase space dimension (elimination of the velocity variable). Of course the eliminated velocity reappears in the various bags j $(j = 1, \ldots, N)$ and if we want a precise description of a continuous distribution we need a large N. But from a physical point of view, many interesting results can be obtained with a small N, sometimes $N = 2$ or 3 (see Fig. 4) or even 1 (see Section 2). In the Vlasov phase space (x, v), the exchange of velocity is described by a differential operator. From a numerical point of view this operator has to be approximated by finite differences, and a minimum size for the mesh in the velocity space has to be required: a too rough mesh would bring a numerical catastrophe. On the contrary, the MWB is *not* an approximation but an exact model with a *choice of special initial conditions*.

4. Application to gyrokinetic modelling

In principle, one has to solve a 6D kinetic equation to determine the distribution function. However, for strongly magnetized plasmas, gyrokinetics allows to recast the Vlasov equation into a 5D equation in which the fast gyroangle does not appear explicitly but in which the particle information is not lost. The physical model is based on the gyrokinetic equation for the ions with an adiabatic response for the electrons [24]. Moreover, to study slab ion temperature gradients (ITG) instabilities a simplified drift-kinetic model in cylindrical geometry can be considered [8]: the uniform and constant magnetic field $\mathbf{B}$ is along the axis of the column (z coordinate); electron inertia is ignored (adiabatical response to the low frequency fluctuations) and ion finite Larmor radius effects are neglected as well so that only the guiding-center trajectories are taken into account. There is no equilibrium radial electric field. The plasma quasi-neutrality approximation $\delta n_i = \delta n_e$ is sufficient for low frequency electrostatic perturbations. With these assumptions the evolution of the ion guiding-center distribution function $f(\mathbf{r}_\perp, z, v_\parallel, t)$ is described by the drift-kinetic Vlasov equation (see [8] for more details).

The most important and interesting feature is that *f depends only on the velocity component* $v_\parallel$ parallel to $\mathbf{B}$. Consequently, a water bag description can be simply obtained of the form (5) where the contours $v_j^\pm$ are now function of (r, θ, z, t) and obey equations similar to Eq. (1) with the additional guiding-center term, supplemented by the quasi-neutrality equation:

$$\frac{\partial v_j^\pm}{\partial t} - \frac{1}{r\mathbf{B}}\left[\frac{\partial \phi}{\partial \theta}\frac{\partial v_j^\pm}{\partial r} - \frac{\partial \phi}{\partial r}\frac{\partial v_j^\pm}{\partial \theta}\right] + v_j^\pm \frac{\partial v_j^\pm}{\partial z} + \frac{q_i}{m_i}\frac{\partial \phi}{\partial z} = 0, \tag{14}$$

$$Z_i \sum_{j=1}^{M} A_j(v_j^+ - v_j^-) - n_e(r) = \frac{en_e(r)}{T_e(r)}\phi, \tag{15}$$

where $T_e(r)$ and $n_e(r)$ are respectively the electron temperature and density radial profiles.

We call this model the gyro water bag. In order to know how to build such a gyro water bag that is suited for gyrokinetic purpose, we shall concentrate on the linear analysis, in the same spirit as Section 3. Let us consider the following expansion around an equilibrium depending only on the radial variable r:

$$\phi = 0 + \delta\phi(r)\mathrm{e}^{\mathrm{i}(k_\parallel z + m\theta - \omega t)}, \tag{16}$$

$$v_j^\pm = \pm a_j(r) + w_j^\pm(r)\mathrm{e}^{\mathrm{i}(k_\parallel z + m\theta - \omega t)}. \tag{17}$$

A little algebra yields the dispersion relation $\epsilon(\omega, k_\parallel) = 0$ with the following gyro water bag dispersion function:

$$\epsilon(\omega, k_\parallel) = 1 - Z_i\frac{k_\parallel^2 c_S^2}{n_i}\sum_{j=1}^{M}\frac{2a_jA_j}{\omega^2 - k_\parallel^2 a_j^2} + Z_i\frac{k_\theta \rho_S c_S}{n_i}\omega\sum_{j=1}^{M}\frac{2a_jA_j\kappa_j}{\omega^2 - k_\parallel^2 a_j^2}, \tag{18}$$

where $n_i(r) = \sum_\ell 2a_\ell A_\ell$ is the ion density, $c_S(r)$ is the local ion acoustic velocity defined by $c_S^2 = T_e(r)/m_i$, $\rho_S(r) = c_S/\Omega_c$ with Ω_c being the ion cyclotron frequency. The wavenumber k_θ is defined by $k_\theta = m/r$. Finally, $\kappa_j(r)$ is the radial density gradient of bag j i.e. $\kappa_j(r) = \frac{\mathrm{d}}{\mathrm{d}r}\ln a_j(r)$.

At a given point $r = r_0$ the A_j's can be computed using the same trick as explained in Section 3, Fig. 3. But as compared to Eq. (9), new unknown κ_j appear in Eq. (18) which measure the local density gradient of the corresponding bag j. To determine these unknowns the knowledge of the equilibrium gradients at $r = r_0$ is needed:

(i) the ion temperature gradient $\kappa_T = \frac{\mathrm{d}}{\mathrm{d}r}\ln T_i$,
(ii) the ion density gradient $\kappa_n = \frac{\mathrm{d}}{\mathrm{d}r}\ln n_i$.

Computing the κ_j's as a function of κ_n and κ_T is done in the following way. We first write the equilibrium ion distribution function of the form

$$f_i(r, v) = \frac{n_i(r)}{v_{T_i}(r)}\mathscr{G}\left(\frac{v}{v_{T_i}(r)}\right), \tag{19}$$

where $\mathscr{G}$ is a normalised function $\int_{-\infty}^{\infty}\mathscr{G}(x)\mathrm{d}x = 1$. Like Section 3, we suppose $\mathscr{G}$ an even function so that odd momenta are zero and we define the ℓ-momentum of $\mathscr{G}$ by $\mathscr{M}_\ell(\mathscr{G}) = \int_{-\infty}^{\infty}x^\ell\mathscr{G}(x)\mathrm{d}x$. Thus in the momentum sense we have

$$\sum_{j=1}^{N} 2A_ja_j^{\ell+1} = (\ell+1)n_iv_{T_i}^\ell\mathscr{M}_\ell(\mathscr{G}) \tag{20}$$

for $\ell = 0, 2, \ldots, 2(N-1)$. This equation must hold not only at $r = r_0$ (which is equivalent to solve Eq. (12) to get the A_j's) but also in the vicinity of $r = r_0$. Taking the derivative of Eq. (20) yields

$$\sum_{j=1}^{N} 2A_ja_j^{\ell+1}\kappa_j = \left(\kappa_n + \frac{\ell}{2}\kappa_T\right)n_iv_{T_i}^\ell\mathscr{M}_\ell(\mathscr{G}). \tag{21}$$

Again Eq. (21) is a linear system of N equations for the N unknown κ_j providing the a_j's are given. It is convenient to look for solutions of Eq. (21) of the form

$$2a_jA_j\kappa_j = \chi_j\kappa_n + \frac{1}{2}(2a_jA_j - \chi_j)\kappa_T \tag{22}$$

by introducing an auxiliary variable χ_j. Putting (22) into (21) yields the following system:

$$\sum_{j=1}^{N} \chi_j a_j^\ell = \int_{-\infty}^{\infty} v^\ell f_i(v)\mathrm{d}v. \tag{23}$$

A convenient solution for the χ_j's can be found in the case of a regular sampling $a_j = (j - \frac{1}{2})\Delta a$ (see Fig. 3). Computing $F_j = f_0\left(a_j - \frac{\Delta a}{2}\right)$ and performing a Taylor expansion yields the very simple approximate solution (with the same remarks as for Eq. (13))

$$\frac{\chi_j}{\Delta a} = \left(f_0\left(a_j - \frac{\Delta a}{2}\right) + f_0\left(a_j + \frac{\Delta a}{2}\right)\right) + \mathcal{O}(\Delta a^2).$$

Once the quantities A_j and κ_j (for $j = 1, \ldots, N$) in the gyro water bag dispersion function (18) have been computed from (13), (22) and (23), the discrete dispersion relation for N bags fits the continuous one in the momentum sense up to order $\ell_{\max} = 2(N-1)$. For instance, for $N = 3$ only a qualitative agreement is obtained, while for $N = 10$ the instability threshold for ITG instability is found to be very close to the results obtained from continuous Maxwellian distribution function [25] (see Fig. 4).

These results are very interesting and tend to prove that the gyro water bag model is able to depicting and resolving kinetic effects in the nonlinear regime with the numerical cost of a (multi)fluid simulation. Preliminary results obtained with a 2D code $(r, z, v_\parallel)$ based on discontinuous-Galerkin and Lax–Wendroff type methods are very encouraging. Furthermore, finite Larmor radius effects and toroidal geometry, which have been neglected in the present paper, are now under consideration without any further difficulties, and will be published in a forthcoming paper.

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