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# Lagrangian regularity of the electron magnetohydrodynamics flow on a bounded domain

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A B S T R A C T

In this paper we investigate the regularity in time of the Lagrangian flow associated with the electron magnetohydrodynamics (e-MHD) equations on a bounded domain with a smooth (ultradifferentiable) boundary. This model is widely used in controlled magnetic fusion, in space and astrophysics plasmas and also in physics of solids. We show that initial data with limited smoothness in Sobolev spaces induce a Lagrangian flow-map $X$ and a Lagrangian magnetic vector potential $A$ (viz. the magnetic vector potential evaluated at the Lagrangian spatial point $X$), which are ultradifferentiable in time, with the two particular cases of real analytic and Gevrey time regularity. It turns out that the Lagrangian canonical momentum $P$, the Lagrangian magnetic field $B$, and the Lagrangian electric field $E$ inherit this Lagrangian regularity property. Among others, the proof makes crucial use of a novel Lagrangian formulation of the e-MHD in terms of the Lagrangian fields $(X, A, P, B, E)$. A by-product of this Lagrangian and constructive proof is the design of arbitrary high-order semi-Lagrangian schemes to solve the e-MHD equations on a bounded domain.



## 1. Introduction

The electron magnetohydrodynamics (e-MHD) equations is a fundamental tool of plasma physics for solving problems of pulsed plasmas and controlled magnetic fusion, of space and astrophysics plasmas, and also of physics of solids. The e-MHD equations describe the (hydro-)dynamics of electrons in a plasma where small length and short time scales phenomena are important, and where strong electromagnetic fields and high currents play a crucial role. Based on the quasineutrality assumption, this model retains the Hall effect while the ion motion is neglected. A lot of theoretical and numerical developments with many applications (such as nonlinear skin effects, electron vortices, solitons, electromagnetic instabilities, plasma turbulence,

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magnetic reconnection, ...) concerning the e-MHD equations can be found in the plasma physics literature, and for examples we refer the reader to [86,105,48,27,61,94,69,19,20,49,15,75,6,77,31,62,118,63,42,64] and references therein. For a review of the physics of the e-MHD, the reader can consult references [69,49].

Let $\Omega$ be a bounded and simply-connected domain of $\mathbb{R}^3$ containing the plasma of electrons. The boundary $\partial\Omega$ is smooth and we will come back later on the precise definition of its regularity. Let $\nu$ be the outward pointing unit normal to the boundary $\partial\Omega$. Let $\rho$ and $u$ be respectively the density and the velocity vector of electrons in a plasma. The vectors $b$ and $e$ denote respectively the self-consistent magnetic and electric field. The three-dimensional vectors $(u, b, e)$ are functions of a three-dimension position $x \in \Omega$ and of the time $t > 0$. The so-called e-MHD equations on a bounded domain $\Omega$ read,

$$\partial_t u + u \cdot \nabla u + e + u \times b = 0, \quad x \in \Omega, \quad t \in ]0, T[, \tag{1}$$

$$-\nabla \times b = u, \quad x \in \Omega, \quad t \in ]0, T[, \tag{2}$$

$$\partial_t b + \nabla \times e = 0, \quad x \in \Omega, \quad t \in ]0, T[, \tag{3}$$

$$\nabla \cdot b = 0, \quad \rho = 1, \quad x \in \Omega, \quad t \in ]0, T[, \tag{4}$$

$$u \cdot \nu = 0, \quad b \cdot \nu = 0, \quad e \times \nu = 0, \quad x \in \partial\Omega, \ t \in ]0, T[, \tag{5}$$

$$(u, b, e)_{|t=0} = (u_0, b_0, e_0), \quad x \in \Omega. \tag{6}$$

The boundary conditions (5) mean that the plasma of electrons extends out to an impermeable and perfectly conducting rigid wall [39].

Using the modulated energy method designed in [17] for proving the quasineutral limit of the Vlasov–Poisson system to the incompressible Euler equations, the authors of [18] show that the e-MHD equations in the whole space can be obtained as the quasineutral limit of the Vlasov–Maxwell equations. Using a weighted energy method combined with the curl-div decomposition of the gradient of the velocity vector field to obtain some dissipative structures in the equations, the authors of [91] (see also [92]) established uniform a priori estimates to show the convergence of the compressible Euler–Maxwell system in a periodic box to the e-MHD equations in the quasineutral regime.

Roughly speaking, our result states that in the spatial non-too-smooth regime the time smoothness of the Lagrangian flow of the e-MHD equations (1)-(6), is only limited by the smoothness of the boundary $\partial\Omega$ (see Theorem 2). To described more precisely but still briefly our result, we introduce the initial velocity $u_0$ such that $\nabla \cdot u_0 = 0$, and the initial magnetic vector potential $a_0$ such that $b_0 = \nabla \times a_0$ and $\nabla \cdot a_0 = 0$. In addition, we denote by $X$ the Lagrangian flow-map and by $A$ the Lagrangian magnetic vector potential, i.e. the Eulerian magnetic vector potential evaluated at the Lagrangian (material) point $X$ of the $\overline{\Omega}$-space. We show that initial data $(u_0, a_0)$ with limited smoothness in Sobolev spaces initiate a Lagrangian flow-map $X$ and a Lagrangian magnetic vector potential $A$ whose time regularity is given by the regularity of the boundary $\partial\Omega$. This regularity is described by a broad class of ultradifferentiable functions, which encompasses the real analytic and Gevrey classes. As a consequence, the Lagrangian canonical momentum $P$, the Lagrangian electric field $E$, and the Lagrangian magnetic field $B$ acquire also this Lagrangian regularity property.

The proof is crucially based on a novel Lagrangian formulation of the e-MHD equations on a bounded domain in terms of the Lagrangian fields $(X, A, P, B, E)$. This Lagrangian formulation uses a generalized Cauchy invariants equation [12,24] for the canonical momentum, the curl of which is Lie-advected by the velocity field $u$. Inserting time-Taylor expansions of $X$ and $A$ in this new Lagrangian formulation, we obtain nonlinear recursion relations among time-Taylor coefficients of $(X, A)$, which allow us to construct recursively the time-Taylor series of $(X, A)$. Contrary to the incompressible Euler equations for which the authors of [13,11] obtain a recursive procedure which is linear in terms of the current time-Taylor coefficient of the Lagrangian flow-map $X$ at a fixed rank, here for the e-MHD, we obtain a recursive procedure which is nonlinear in terms of this time-Taylor coefficient. Moreover the current time-Taylor coefficient of the Lagrangian flow-map $X$ at a fixed rank is here coupled nonlinearly to the current time-Taylor coefficient of the Lagrangian magnetic vector potential $A$ at the same rank (see Remark 5 for more details). From the Lagrangian fields $(X, A)$, we can then build the Lagrangian fields $(P, B, E)$. Finally, we show that this nonlinear recursive procedure converges in a convenient functional framework, which allows us to establish the Lagrangian regularity property of Theorem 2, for the e-MHD equations on a bounded domain.

In the spirit of [58,13,11] (see also [95,121,40] for periodic boundary conditions), this Lagrangian and constructive proof can be very useful to design arbitrary high-order semi-Lagrangian methods for integrating numerically the e-MHD equations on a bounded domain. Indeed, in [58], the authors demonstrate the efficiency of this family of numerical methods of arbitrary high-order to simulate potentially singular Euler flows on bounded domains. Moreover such constructive proof can be extended to non-simply-connected domains. This allows to build similar high-order semi-Lagrangian numerical methods to treat important geometry in plasma physics such as tokamaks, which play a central part in magnetic confinement fusion. Indeed, in such a case we must take into account additional harmonic fields, which constitute the kernels of the elliptic boundary value problems involved in the construction scheme. This time-independent harmonic fields are completely determined by the geometry of the domain, and in particular their regularity is given by the regularity of the domain boundary [11].

Originally this Lagrangian regularity property is exhibited by the incompressible Euler equations in the whole space [25,41,103,28,57], in a periodic box [104,121,40], on a bounded domain [68,106,46,13,57], and on a manifold with boundary [11]. To the best of our knowledge this is the first time that such a Lagrangian regularity property is proven for inviscid and non-resistive magnetized fluids. Indeed, the time analyticity of the Lagrangian flow-map $X$ has been only shown for some inviscid neutral fluids which are governed by 2D incompressible models in the whole plane such as the 2D Boussinesq equations, the 2D incompressible porous media equation and the 2D surface quasi-geostrophic equations [28]. This Lagrangian analyticity property is also shared by the pressureless compressible Euler–Poisson (electrostatic/gravitational) equations in a periodic box [99] and in the whole space [57]. In the latter model we note that the electric/gravitational scalar potential plays the same role as the fluid pressure in the incompressible Euler equations.

Naturally, we can ask the compelling and interesting question whether there are other magneto-hydrodynamics models that support or break this Lagrangian regularity property. Other important fluid models for the electro-magneto-hydrodynamics are ideal incompressible magnetohydrodynamics (IIMHD) [37,39,47,102,9,21,56,114], the extended ideal incompressible magnetohydrodynamics (XIIMHD) [82,112,47,32,81,66,83] including the inertial MHD (IMHD) and the Hall MHD (HMHD) sub-models. There are also various Euler–Maxwell systems such as the incompressible one-fluid Euler–Maxwell system (IEM), the pressureless compressible one-fluid Euler–Maxwell system (PCEM), the compressible one-fluid Euler–Maxwell system [26,65,34,93,110,115,111,116,119,43,60,90,113,53] and the compressible two-fluid Euler–Maxwell equations [120,89,35,51,52]. Observe that the IEM (resp. PCEM) system can be derived from the Vlasov–Maxwell equations by considering mono-kinetic solutions with uniform (resp. non-uniform) charge density for the statistical-distribution function of particles. There are two main obstructions for obtaining the Lagrangian analyticity property for such models, i.e. time analyticity of the corresponding Lagrangian fields. The first obstruction, named $\mathcal{O}_1$, is the presence of several coupled fluids. This concerns two-fluid models and a fortiori multi-fluid models, or models which arise as a derivation from a two-fluid or a multi-fluid theory. Indeed, in a two-fluid transport model the two Lagrangian flow-maps (associated with the velocity field of each fluid) are coupled together through some equations for the electromagnetic fields, which in return determine the velocity fields. Because of this coupling, one Lagrangian flow-map experiences directly the roughness (with respect to Lagrangian variables) of the other Lagrangian flow-map; everything happens as if one Lagrangian flow-map comes across the other one and thus sees its relative roughness. In [57] the author shows that the Vlasov–Poisson equations can not support the Lagrangian analyticity property. This result is consistent with the obstruction $\mathcal{O}_1$ because, by considering multi-kinetic solutions

[7,8] for the Vlasov–Poisson equations we obtain the pressureless compressible multi-fluid Euler–Poisson system. By contrast, for an incompressible one-fluid model or the pressureless compressible one-fluid Euler–Poisson system the Lagrangian flow-map can not run into itself. The second obstruction, named $\mathcal{O}_2$, is the finite speed of propagation property which is not compatible with the Lagrangian analyticity property. A system, in which waves propagate at a finite speed, can not sustain the Lagrangian analyticity property because, for this, some information must propagate at infinite speed. For examples, in the incompressible Euler equations (resp. pressureless compressible one-fluid Euler–Poisson system) this is the pressure (resp. electric scalar potential) which propagates at infinite speed, while for the e-MHD this is the magnetic field or the magnetic vector potential. By contrast it has been shown in [57] that the 2D barotropic (isentropic) compressible Euler equations, where the pressure propagates at a finite speed (property of hyperbolic systems of conservation laws [30]), do not satisfy the Lagrangian analyticity property for its corresponding Lagrangian flow-map $X$.

Now, we rapidly examine whether we find such obstructions in the models mentioned above. We start with the XIIMHD whose the mathematical structure is extremely close to the incompressible Euler equations. Indeed this model can be seen as a two-fluid model where each fluid satisfies an incompressible Euler equation written in terms of a generalized vorticity. The coupling between the two incompressible Euler equations arises from the determination of two velocity fields, which are defined through Biot–Savart-type laws involving the two generalized vorticities as source terms. Therefore the well-posedness theory for such model is the same as the incompressible Euler equations [66,83], and the XIIMHD can not sustain the Lagrangian analyticity property because it meets the obstruction $\mathcal{O}_1$. Due to the strong coupling of the two characteristic curves sets through Biot–Savart-type laws, characteristic curves of one set feel or see directly the roughness (with respect to Lagrangian variables) of characteristic curves of the second set, when characteristics of the first set cross those of the second one. By contrast, for the incompressible Euler equations of a single fluid, a Lagrangian particle stays on its characteristic curve, which never crosses and feels directly the others, because of the incompressibility property. The interaction of a characteristic curve with the others is always indirect, through the pressure field given instantaneously. We also have the same conclusion for the compressible version of the XIIMHD. As far as it concerns the IIMHD, it is well-known that this model is derived from a two-fluid theory [39,47], and that the IIHMD can be recast as another two-fluid model by using the Elsasser variables [37]. Moreover the Lagrangian formulation of the IIHMD equations, in terms of Lagrangian flow-map $X$, can be recast as a quasilinear or nonlinear system of wave equations [107,117,1,10], which satisfies the finite speed propagation property [59]. Therefore the IIMHD meets the obstructions $\mathcal{O}_1$ and $\mathcal{O}_2$, and thus it can not support the Lagrangian analyticity property. The conclusion will be the same for the compressible version of the IIMHD. The compressible one-fluid and two-fluid Euler-Maxwell equations can be seen as systems of nonlinear hyperbolic conservation laws with no dissipation (such as viscosity or resistivity effects). It is also well-known that such systems exhibit the finite speed propagation property [30]. Then, the compressible two-fluid Euler-Maxwell equations meet the obstructions $\mathcal{O}_1$ and $\mathcal{O}_2$, while the compressible one-fluid Euler-Maxwell equations meet only the obstruction $\mathcal{O}_1$. The obstruction $\mathcal{O}_2$ is even more striking by assuming the hypothesis of generalized irrotational flow (namely $b = \nabla \times u$) since with this assumption the compressible one-fluid and two-fluid Euler–Maxwell equations can be recast as quasilinear systems of wave and Klein–Gordon-type equations [43,51,60,52,53]. Finally, since Maxwell equations can be classified as a hyperbolic system of first or second order in time, they satisfy the finite speed propagation property and thus the IEM and PCEM models meet the obstruction $\mathcal{O}_2$. Therefore all the Euler–Maxwell systems considered above can not support the Lagrangian analyticity property. Finally, we remark that the feature of incompressibility versus compressibility is not a criterion to determine whether a model satisfies or not the Lagrangian analyticity property, even if from the above considerations incompressible models are more able to verify this time regularity property than compressible models. The mathematical proofs of all these claims will be the matter of future work.

The outline of the paper is as follows. In Section 2, we first recast the e-MHD equations in more suitable Eulerian forms (Section 2.1), especially in terms of the magnetic vector potential, to obtain a well-posedness theory for the e-MHD equations on a bounded domain (Sections 2.2). Then, in Section 2.3, after introducing our notation and defining the functional framework, we state our main result concerning the Lagrangian regularity of the e-MHD flow, namely Theorem 2. In Section 3, we present the proof of Theorem 2 in three steps. First, in Section 3.1, we derive a novel Lagrangian formulation of the e-MHD equations on a bounded domain. Then, in Section 3.2, we use this Lagrangian formulation to derive a nonlinear recursive procedure to construct the solution of the e-MHD equations. Finally, in Section 3.3, we study the convergence of this nonlinear recursive procedure and we obtain regularity estimates for the Lagrangian fields $(X, A, P, B, E)$.

## 2. The e-MHD equations on a bounded domain and main result

This section is divided in three subsections. In Section 2.1 we rewrite the e-MHD equations in two more convenient Eulerian forms. In particular we derive an Eulerian formulation which involves the magnetic vector potential. Using these Eulerian reformulations we state a well-posedness result in Sobolev spaces for the e-MHD in Section 2.2. Finally, after recalling the functional framework of ultradifferentiable functions, we present our main result about the Lagrangian regularity of the e-MHD flow in Section 2.3.

### *2.1. Eulerian reformulations of the e-MHD equations on a bounded domain*

We first introduce the magnetic vector potential $a$ such that

$$b = \nabla \times a, \quad \nabla \cdot a = 0. \tag{7}$$

Following e.g. [61], we introduce the canonical momentum $p$ and its corresponding generalized vorticity $\omega_*$ defined by

$$p = u - a, \quad \text{and} \quad \omega_* = \nabla \times p = \omega - b, \tag{8}$$

where

$$\omega = \nabla \times u \tag{9}$$

is the standard fluid-vorticity. We note that $\nabla \cdot p = 0$, since from (2) we obtain $\nabla \cdot u = 0$ and from (7) we have $\nabla \cdot a = 0$. Using definitions (7)-(8), and subtracting the Maxwell–Faraday equation (3) to the curl of the momentum equation (1), we obtain the following incompressible Euler equation for the generalized vorticity $\omega_*$,

$$\partial_t \omega_* = \nabla \times (u \times \omega_*) \quad \text{or} \quad \partial_t \omega_* + u \cdot \nabla \omega_* - \omega_* \cdot \nabla u = 0 \quad \text{or} \quad \omega_* = \nabla^T X \, \omega_{*0}. \tag{10}$$

The last equation of (10), the so-called Cauchy or vorticity-transport formula (see, e.g., [24,83]) $\omega_*(t, X(t,\alpha)) = \nabla_\alpha^T X(t,\alpha) \, \omega_{*0}(\alpha)$, corresponds to the integration of the two first equations of (10) along the characteristic curves $t \mapsto X(t,\alpha)$, which are defined by the following ordinary differential equation,

$$\dot{X}(t,\alpha) \equiv \partial_t X(t,\alpha) = u(t, X(t,\alpha)), \quad X(0,\alpha) = \alpha \in \overline{\Omega}. \tag{11}$$

From equations (2) and (4), we observe that the magnetic field $b$ plays the role of the standard fluid stream function. Taking the curl of the Maxwell–Ampère equation (2) and subtracting to it the magnetic field $b$, we obtain from definition (8) for $\omega_*$,

$$\begin{cases} -(1-\Delta)b = \omega_* \quad \text{on} \ \ \Omega, \\ \nabla \cdot b = 0 \quad \text{on} \ \ \overline{\Omega}, \\ \nu \cdot \nabla \times b = 0, \quad \text{and} \quad b \cdot \nu = 0 \quad \text{on} \ \ \partial\Omega, \end{cases} \tag{12}$$

and

$$u = -\nabla \times b \quad \text{on} \ \ \overline{\Omega}. \tag{13}$$

The boundary conditions in (12) come from the boundary conditions (5), i.e. $b \cdot \nu = 0$ and $\nu \cdot \nabla \times b = -u \cdot \nu = 0$ on $\partial\Omega$. We note that the boundary value problem (12) for the magnetic field $b$ is well-posed in Sobolev spaces (see, e.g., Theorem 3.5 in Chapter 1 of [45]). Finally the electric field $e$ is given by the momentum equation (1), i.e.

$$e = -(\partial_t + u \cdot \nabla)u - u \times b \quad \text{on} \ \ \overline{\Omega}, \tag{14}$$

while initial conditions

$$(u, b, e)_{|t=0} = (u_0, b_0, e_0), \tag{15}$$

keep the same. Therefore the e-MHD equations (1)-(6) are equivalent to the system constituted by equations (10)-(15) from which we will deduce in the next section the well-posedness of the e-MHD equations in Sobolev spaces.

The proof of our main result (see Theorem 2 in Section 2.3) relies on a new Lagrangian formulation of the e-MHD (see Section 3.1) in which the magnetic vector potential $a(t, x)$ (more precisely its Lagrangian counterpart, i.e. the Lagrangian magnetic vector potential $A(t, \alpha) = a(t, X(t, \alpha))$) plays a central role. Indeed we will see in Section 3.1 (Propositions 1 and 2) that the determination of the magnetic field $b(t, x)$ (more precisely its Lagrangian counterpart $B(t, \alpha) = b(t, X(t, \alpha))$) will be just a straightforward and explicit computational consequence of a self-consistent and nonlinear determination of the characteristic curves $t \mapsto X(t)$ and the Lagrangian magnetic vector potential $A$. Therefore, it is important to determine here the boundary value problem satisfied by the magnetic vector potential $a$. We must be careful that the boundary value problem for the magnetic vector potential $a$ must be consistent with equations (1)-(6) or (10)-(15). Combining the Maxwell–Ampère equation (2) and definition (7) for the magnetic field $b$, we obtain

$$\begin{cases} \Delta a = u \quad \text{on} \ \ \Omega, \\ \nabla \cdot a = 0 \quad \text{on} \ \ \overline{\Omega}, \\ a \times \nu = 0 \quad \text{on} \ \ \partial\Omega. \end{cases} \tag{16}$$

The boundary condition in (16) comes from the boundary condition $\nu \cdot \nabla \times a = b \cdot \nu = 0$ on $\partial\Omega$ (see equation (5)), and the vector analysis formula

$$\nabla \cdot (\psi \times \varphi) = \varphi \cdot (\nabla \times \psi) - \psi \cdot (\nabla \times \varphi), \quad \text{and} \quad \nabla \times \nu = 0, \tag{17}$$

for any three-dimensional vector $\psi$ and $\varphi$. We note that the boundary value problem (16) is well-posed in Sobolev spaces (see, e.g., Theorem 3.6 in Chapter 1 of [45]). We also note that the boundary condition $a \times \nu = 0$ on $\partial\Omega$ is consistent with the boundary condition $e \times \nu = 0$ on $\partial\Omega$. Indeed, the electric field is always given by $e = -\nabla\phi - \partial_t a$, where $\phi$ is the electric scalar potential subjected to the boundary condition $\phi = 0$ on $\partial\Omega$, and satisfying $-\Delta\phi = \nabla \cdot e$ on $\Omega$. Since $\phi = 0$ on $\partial\Omega$ implies that the tangential derivative of $\phi$ on $\partial\Omega$ vanishes, i.e. $\nu \times \nabla\phi = 0$ on $\partial\Omega$, then $e \times \nu = -\partial_t(a \times \nu) + \nu \times \nabla\phi = 0$ on $\partial\Omega$. Therefore the e-MHD equations (1)-(6) or (10)-(15) are also equivalent to the system constituted by equations (7)-(11) and (13)-(16).

### *2.2. Well-posedness of the e-MHD equations on a bounded domain*

Here, we state the local-in-time well-posedness of classical solutions in Sobolev spaces $H^s$ ($s \geq 0$) for the e-MHD equations on a bounded domain.

**Theorem 1** *(Well-posedness of e-MHD equations on a bounded domain). Let $\Omega$ be a bounded and simply-connected domain of $\mathbb{R}^3$ with $\mathscr{C}^\infty$ boundary $\partial\Omega$. Let $s > 3/2 + 1$. Let $u_0 \in H^s(\Omega)$ (initial fluid vorticity $\omega_0 = \nabla \times u_0 \in H^{s-1}(\Omega)$) such that $\nabla \cdot u_0 = 0$ on $\Omega$ and $u_0 \cdot \nu = 0$ on $\partial\Omega$. Let the initial fields $(a_0, b_0)$ be the unique solutions of the following boundary value problems,*

$$\begin{cases} \Delta b_0 = \nabla \times u_0 \quad \text{on } \Omega, \\ \nabla \cdot b_0 = 0 \quad \text{on } \overline{\Omega}, \\ \nu \cdot \nabla \times b_0 = 0, \quad \text{and } \; b_0 \cdot \nu = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{18}$$

*and*

$$\begin{cases} \Delta a_0 = u_0 \quad \text{on } \Omega, \\ \nabla \cdot a_0 = 0 \quad \text{on } \overline{\Omega}, \\ a_0 \times \nu = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{19}$$

*Consequently $a_0 \in H^{s+2}(\Omega)$ (initial magnetic field $b_0 = \nabla \times a_0 \in H^{s+1}(\Omega)$) and the initial generalized vorticity $\omega_{*0} = \omega_0 - b_0 \in H^{s-1}(\Omega)$. Let $e_0 \in H^{s-1}(\Omega)$ with $e_0 \times \nu = 0$.*

*Then there exist a time $T > 0$ and a unique solution to the e-MHD equations (1)-(6), or (10)-(15), or (7)-(11) and (13)-(16), such that*

$$u \in \mathscr{C}(0, T; H^s(\Omega)) \cap W^{1,\infty}(0, T; H^{s-1}(\Omega)) \cap \mathscr{C}^1([0, T) \times \Omega), \tag{20}$$

$$a \in \mathscr{C}(0, T; H^{s+2}(\Omega)) \cap W^{1,\infty}(0, T; H^{s+1}(\Omega)) \cap \mathscr{C}^1(0, T; \mathscr{C}^{2,\gamma}(\Omega)), \quad 0 < \gamma < 1, \tag{21}$$

$$b \in \mathscr{C}(0, T; H^{s+1}(\Omega)) \cap W^{1,\infty}(0, T; H^s(\Omega)) \cap \mathscr{C}^1(0, T; \mathscr{C}^{1,\gamma}(\Omega)), \quad 0 < \gamma < 1, \tag{22}$$

$$e \in L^\infty(0, T; H^{s-1}(\Omega)), \tag{23}$$

$$\omega_* \in \mathscr{C}(0, T; H^{s-1}(\Omega)). \tag{24}$$

**Remark 1.** In the boundary value problem (19) we can equivalently replace $u_0$ by $-\nabla \times b_0$, where $b_0$ is the solution of the boundary value problem (18).

**Proof of Theorem 1.** We start with the well-posedness of the boundary value problems which determine the initial conditions $(a_0, b_0)$ from the initial velocity field $u_0$. The well-posedness of the boundary value problem for $b_0$ (resp. $a_0$) in Sobolev spaces $H^s$ is ensured by the Theorem 3.5 (resp. Theorem 3.6) of Chapter 1 of [45] (see also Section 3 of Chapter 1 of [45] for more information on regularity results for the Helmholtz–Hogde decomposition). We next use reformulation (10)-(15) of the e-MHD equations (1)-(6). As it is pointing out in [18], we observe that equations (10)-(13) form a closed set of equations which have the same mathematical structure as the incompressible Euler equations. Therefore the existence and uniqueness results are the same as the incompressible Euler equations on a regular bounded domain (see [36,16,108,67,45,83]), and we obtain the existence of a unique velocity field $u$ satisfying the regularity result (20) and solving (1)-(6) or (10)-(15) in the classical sense. Moreover we obtain the regularity result (24) for the generalized vorticity $\omega_*$. Using standard elliptic regularity estimates [76,79,45,44], continuous Sobolev embeddings theorems and the regularity estimate (20) for $u$, we obtain from the boundary value problems (16) and (12) the regularity results (21)-(22) for the magnetic vector potential $a$ and the magnetic field $b$ respectively. Finally using the regularity result (20) for $u$ and the regularity result (22) for $b$ we obtain from the momentum equation (14) or (1) the regularity result (23) for the electric field $e$. This ends the proof of Theorem 1. $\square$

### 2.3. Lagrangian regularity of the e-MHD flow on a bounded domain

In order to present the main result, we must first define some functional spaces that we use to describe time regularity and boundary smoothness. Let $\mathscr{D}$ be a domain in $\mathbb{R}^d$ and let $\mathfrak{B}$ be a Banach space endowed with the norm $\|\cdot\|_{\mathfrak{B}}$. Let $\mathcal{M} := \{\mathcal{M}_\sigma\}_{\sigma\geq 0}$ be a sequence of positive numbers. The ultradifferentiable class $\mathcal{C}\{\mathcal{M}\}(\mathscr{D};\mathfrak{B})$ is defined as the set of functions $\psi : \mathscr{D} \longrightarrow \mathfrak{B}$ such that for any compact set $K \subset \mathscr{D}$ there exist constants (depending on $\psi$) $R_\psi$, $C_\psi$ such that for all $\sigma \in \mathbb{N}$,

$$\sup_{x\in K} |||D^\sigma \psi(x)||| \leq C_\psi R_\psi^{-\sigma} \mathcal{M}_\sigma. \tag{25}$$

The map $x \mapsto D^\sigma\psi(x)$ is a function defined on $\mathscr{D}$ with values in the set of symmetric $\sigma$-linear operators, which is endowed with the standard induced operator-norm $|||\cdot|||$. The class $\mathcal{C}\{\mathcal{M}\}$ is invariant under multiplication by a constant, i.e. $\mathcal{C}\{\lambda\mathcal{M}\}(\mathscr{D};\mathfrak{B}) = \mathcal{C}\{\mathcal{M}\}(\mathscr{D};\mathfrak{B})$ for $\lambda > 0$. As in [13], we choose the "log-superlinear Faà-di-Bruno" (LSL–FdB in short) class. For such a class the sequence of weights $\{\mathcal{M}_\sigma/\sigma!\}_{\sigma\geq 0}$ (and $M_0 = \mathcal{M}_0$) satisfies

**Definition 1.** The log-superlinear Faà-di-Bruno class is the set of functions satisfying (25), where the weights $M_\sigma = \mathcal{M}_\sigma/\sigma!$ verify the following properties,

i) differentiation stability:

$$\exists\, C_{\mathrm{D}} > 0 : \quad M_{\sigma+1} \leq C_{\mathrm{D}}^\sigma M_\sigma, \quad \forall \sigma \in \mathbb{N}. \tag{26}$$

ii) log-superlinearity:

$$M_\sigma M_\ell \leq M_0 M_{\sigma+\ell}, \quad \forall \sigma,\, \ell \in \mathbb{N}. \tag{27}$$

iii) (FdB)-stability:

$$\forall \mu_i \in \mathbb{N}^*, \ \text{ such that } \ \mu_1 + \ldots + \mu_\ell = \sigma, \ \text{ we have } \ M_\ell M_{\mu_1} \ldots M_{\mu_\ell} \leq M_\sigma. \tag{28}$$

**Remark 2.** Using the Leibniz differentiation rules, log-superlinearity implies that the class $\mathcal{C}\{\mathcal{M}\}(\mathscr{D};\mathfrak{B})$ is an algebra with respect to pointwise multiplication. Using the Faà di Bruno formula [38,54,29], (FdB)-stability implies stability under composition in the class $\mathcal{C}\{\mathcal{M}\}(\mathscr{D};\mathfrak{B})$ (see the proof of Proposition 3.1 in [97], or Proposition 1.4.2 in [72]). Finally, the differentiability stability property implies closure under differentiation in $\mathcal{C}\{\mathcal{M}\}(\mathscr{D};\mathfrak{B})$ [96,80,70].

**Remark 3.** Some well-known classes of functions belong to the LSL–FdB class. The first one is the real analytic functions class, which corresponds to $M_\sigma = 1$ (e.g. [84,23]). The second one, widely discussed in the literature [33,22,14,80,100,109,74,98], is the log-convex class which corresponds to $M_\sigma^2 \leq M_{\sigma-1}M_{\sigma+1}$, with $M_0 = M_1 = 1$ (see Lemma 2.9 of [73]). A particular case of the latter is the Gevrey class (see, e.g., [84,80]), which corresponds to $M_\sigma = (\sigma!)^r$, with $r \geq 0$.

We introduce $\alpha \mapsto X_t(\alpha) = X(t,\alpha)$ the Lagrangian flow-map tracking at time $t$ the position of a particle starting at $\alpha \in \overline{\Omega}$. The Lagrangian flow-map $X$ satisfies the following ordinary differential equation,

$$\partial_t X(t,\alpha) = u(t, X(t,\alpha)), \quad X(0,\alpha) = \alpha \in \overline{\Omega}. \tag{29}$$

Using regularity property (20) for the velocity field $u$, and the Cauchy–Lipschitz–Picard theorem for ordinary differential equations (see, e.g., [55]), we already know that $X \in \mathscr{C}^1([0,T) \times \overline{\Omega})$. We also introduce the Lagrangian magnetic vector potential $A = A(t,\alpha)$, the Lagrangian magnetic field $B = B(t,\alpha)$, the Lagrangian electric field $E = E(t,\alpha)$, the Lagrangian velocity field $U = U(t,\alpha)$, and the Lagrangian canonical momentum $P = P(t,\alpha)$, which are defined as follows,

$$\begin{aligned} &A(t,\alpha) = a(t, X(t,\alpha)), \quad B(t,\alpha) = b(t, X(t,\alpha)), \quad E(t,\alpha) = e(t, X(t,\alpha)), \\ &U(t,\alpha) = u(t, X(t,\alpha)), \quad P(t,\alpha) = p(t, X(t,\alpha)). \end{aligned} \tag{30}$$

Using the previous definitions and notation, the main result of this paper is

**Theorem 2** *(Lagrangian regularity of the e-MHD flow on a bounded domain). Assume that the hypotheses of Theorem 1 hold, and in addition that the boundary $\partial\Omega$ belongs to $\mathcal{C}\{\mathcal{M}\}$, where $\mathcal{M} := \{\sigma! M_\sigma\}_{\sigma \geq 0}$, with the sequence $\{M_\sigma\}_{\sigma \geq 0}$ satisfying Definition 1 (log-superlinear Faà-di-Bruno class). Then there exists a time $T = C(\Omega, \|u_0\|_{H^s(\Omega)}, \|a_0\|_{H^s(\Omega)})$ such that the Lagrangian fields $(X, A, B, E)$ satisfy*

$$X,\ A \in \mathcal{C}\{\mathcal{M}\}\big(]0,T[; H^s(\Omega)\big), \quad \textit{and} \quad B, E \in \mathcal{C}\{\mathcal{M}\}\big(]0,T[; H^{s-1}(\Omega)\big).$$

**Remark 4.** Since $U = \dot{X}$ and $P = U - A$, we directly obtain from Theorem 2 that

$$U, P \in \mathcal{C}\{\mathcal{M}\}\big(]0,T[; H^s(\Omega)\big).$$

## 3. Proof of Theorem 2

Here, we give a proof of Theorem 2, which is divided in three steps. In Section 3.1 we derive a novel Lagrangian formulation of the e-MHD equations by using among others the invariants of the equations, known as the Cauchy invariants equation [24,121,40,12,13,11]. In Section 3.2, using this new Lagrangian formulation, we obtain novel recursion relations among time-Taylor coefficients to construct, through a nonlinear recursive procedure, formal time-Taylor series for the Lagrangian flow-map $X$ and the Lagrangian magnetic vector potential $A$. These two time series allow to determine, via some explicit formula, the Lagrangian magnetic field $B$, the Lagrangian electric field $E$, the Lagrangian velocity field $U$ and the Lagrangian canonical momentum $P$. Section 3.3 is devoted to the convergence analysis of the construction method of Section 3.2 by proving that such formal time series converge in a suitable functional framework. In the sequel, we use the standard convention that an index variable appearing twice in a single term, implies the summation of that term over all the values of the index.

### *3.1. A Lagrangian formulation of the e-MHD equations on a bounded domain*

Let us introduce the symmetric matrix $G$ defined by

$$G := G(X) = \mathcal{A}\mathcal{A}^T, \tag{31}$$

where the inverse Jacobian matrix $\mathcal{A}$ is defined by

$$\mathcal{A} := \mathcal{A}(X) = \left(\frac{\partial X}{\partial \alpha}\right)^{-1} = \frac{\partial \alpha}{\partial X}. \tag{32}$$

In terms of the Lagrangian flow-map $X$ the matrix $\mathcal{A}$ is given by

$$(\mathcal{A})_{ij} = \left(\frac{\partial \alpha^i}{\partial X^j}\right)_{ij} = \frac{1}{2J}\varepsilon_{ii_1i_2}\varepsilon_{jj_1j_2}\frac{\partial X^{j_1}}{\partial \alpha^{i_1}}\frac{\partial X^{j_2}}{\partial \alpha^{i_2}}, \tag{33}$$

where $J := \det(\partial X/\partial \alpha)$ is the Jacobian of the Lagrangian flow-map $X$ and $\varepsilon_{ijk}$ is the Levi–Civita symbol. Formula (33) can be obtained from the well-known formula $M^{-1} = \mathrm{Cof}(M)/\det(M)$ for the inverse of any square matrix $M$, where $\mathrm{Cof}(M)$ is the cofactor matrix associated with $M$, i.e. the matrix of cofactors of $M$. Indeed the cofactor matrix $\mathrm{Cof}(M)$ is rewritten in tensorial form (right-hand side of (33)) by expressing every cofactor, i.e. a determinant, in tensorial form. Since $\nabla \cdot u = 0$, the flow is incompressible, i.e. the Lagrangian flow-map $X$ is a volume-preserving map. Therefore,

$$J := \det\left(\frac{\partial X}{\partial \alpha}\right) = 1. \tag{34}$$

We also introduce the operator ":" which denotes the scalar product between matrices, i.e. $M : N = \sum_{i,j} M_{ij}N_{ij}$. With this notation, we have

**Proposition 1** *(Lagrangian formulation of the e-MHD equations on a bounded domain). A Lagrangian formulation in the variables $(t,\alpha) \in [0,T[\times\overline{\Omega}$ of the e-MHD equations on the bounded domain $\Omega$ is given by*

$$\nabla \dot{X}^k \times \nabla X^k + b_0 = \nabla A^k \times \nabla X^k + \omega_0, \quad \alpha \in \Omega, \quad t \in [0,T[, \tag{35}$$

$$\det\left(\frac{\partial X}{\partial \alpha}\right) = 1, \quad \alpha \in \Omega, \quad t \in [0,T[, \tag{36}$$

$$\nabla \cdot (G\,\nabla A) = \dot{X}, \quad \alpha \in \Omega, \quad t \in [0,T[, \tag{37}$$

$$\nabla \cdot (\mathcal{A}\,A) = \mathcal{A} : \nabla A = 0, \quad \alpha \in \overline{\Omega}, \quad t \in [0,T[, \tag{38}$$

$$B^i = (\nabla X^k \times \nabla X^i) \cdot \nabla A^k, \quad \forall i, \quad \alpha \in \Omega, \quad t \in [0,T[, \tag{39}$$

$$E = -\ddot{X} - \dot{X} \times B, \quad \alpha \in \Omega, \quad t \in [0,T[, \tag{40}$$

$$U = \dot{X}, \quad P = U - A = \dot{X} - A, \quad \alpha \in \Omega, \quad t \in [0,T[, \tag{41}$$

$$\dot{X} \cdot \nu(X) = 0, \quad A \times \nu(X) = 0, \quad \alpha \in \partial\Omega, \quad t \in [0,T[. \tag{42}$$

**Proof.** We begin with equation (35). In the geometric language [12], the 2-form associated with the generalized vorticity vector $\omega_*$ is exact and equal to the exterior derivative of the 1-form associated with the canonical momentum $p$. This is the meaning of equation (8) translated in terms of differential forms. Moreover equation (10) means that this 2-form is Lie-advected by the vector field $u$. Therefore, using Theorem 1 of [12] on the existence of generalized Cauchy invariants equations for Lie-advected differential forms which are exact, we deduce the following Cauchy invariants equation,

$$\nabla_\alpha P^k(t,\alpha) \times \nabla_\alpha X^k(t,\alpha) = \omega_{*0}(\alpha).$$

This equation can be rewritten as

$$\nabla_\alpha \dot{X}^k(t,\alpha) \times \nabla_\alpha X^k(t,\alpha) + b_0(\alpha) = \nabla_\alpha A^k(t,\alpha) \times \nabla_\alpha X^k(t,\alpha) + \omega_0(\alpha),$$

which is equation (35). The Cauchy invariants equation (35) has to be seen as an integrated form of the Lie-advection equation (10) for the generalized vorticity $\omega_*$ along the characteristic curves $t \mapsto X(t)$. The Eulerian incompressibility condition $\nabla \cdot u = 0$ becomes, in terms of the Lagrangian variable $\alpha$, equation (36), which means that the Lagrangian flow-map $X$ is volume-preserving. Equations (35)-(36) determine the Lagrangian flow-map $X$ knowing the Lagrangian magnetic vector potential $A$. Hence, it remains to obtain an equation for the determination of the Lagrangian magnetic vector potential $A$ knowing the Lagrangian flow-map $X$. For this we rewrite, in the Lagrangian variable $\alpha$, the Laplace equation in (16) for the magnetic vector potential $a$. Let $\{x^i\}_i$ be a Cartesian orthonormal coordinate system of the Euclidean space $\mathbb{R}^3$, equipped with the orthonormal frame $\{e_i\}_i$, i.e. such that $e_i \cdot e_j = \delta_{ij}$. A point $M$ of $\mathbb{R}^3$ can be written as $M = x^i e_i$. We then consider the change of variables $x \leftrightarrow \alpha$, with $x := X(t,\alpha)$. A natural frame $\{\widehat{e}_i\}_i$ associated with the coordinate system $\{\alpha^i\}_i$ is given by

$$\widehat{e}_i = \frac{\partial M}{\partial \alpha^i} = \frac{\partial X^k}{\partial \alpha^i}\frac{\partial M}{\partial x^k} = \frac{\partial X^k}{\partial \alpha^i} e_k.$$

We then obtain the metric tensor

$$g_{ij} := \widehat{e}_i \cdot \widehat{e}_j = \frac{\partial X^k}{\partial \alpha^i}\frac{\partial X^k}{\partial \alpha^j} \quad \text{and its inverse} \quad g^{ij} = \frac{\partial \alpha^i}{\partial X^k}\frac{\partial \alpha^j}{\partial X^k} = \mathcal{A}\mathcal{A}^T = G.$$

Using standard expressions for usual differential operators in curvilinear coordinate systems (see, e.g., [78]), we obtain

$$\begin{aligned}\Delta_x a(t,x) &= \Delta_x a(t, X(t,a)) = \Delta_x A \\ &= \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial \alpha^i}\left(g^{ij}\sqrt{|g|}\frac{\partial A}{\partial \alpha^j}\right) = \frac{\partial}{\partial \alpha^i}\left(\frac{\partial \alpha^i}{\partial X^k}\frac{\partial \alpha^j}{\partial X^k}\frac{\partial A}{\partial \alpha^j}\right) \\ &= \nabla \cdot (\mathcal{A}\mathcal{A}^T \nabla A) = \nabla \cdot (G \nabla A), \end{aligned} \tag{43}$$

where $|g| := \det(g^{ij}) = \det(g_{ij}) = 1$, because the Lagrangian flow-map $X$ preserves the volume (incompressibility of the flow satisfying $\nabla \cdot u = 0$). From (43) and using the ordinary differential equation (29), i.e. $\dot{X} = u(t, X(t,\alpha)) = U(t,\alpha)$, the Laplace equation $\Delta a(t,x) = u(t,x)$, written in the Lagrangian variable $\alpha$, becomes equation (37). We continue by showing equation (38), which is the Lagrangian form (in the variable $\alpha$) of the Eulerian constraint $\nabla \cdot a = 0$. Indeed, using (33) and $J = 1$, after some algebra we obtain

$$\frac{\partial}{\partial \alpha^k}\left(\frac{\partial \alpha^k}{\partial X^i}\right) = 0, \quad \forall i. \tag{44}$$

Using (44) and the chain rule, we obtain from $\nabla \cdot a = 0$,

$$0 = \nabla_x \cdot a(t,x) = \frac{\partial a^j}{\partial X^j}(t, X(t,\alpha)) = \frac{\partial A^j}{\partial X^j}(t,\alpha) = \frac{\partial \alpha^i}{\partial X^j}\frac{\partial A^j}{\partial \alpha^i} = \frac{\partial}{\partial \alpha^i}\left(\frac{\partial \alpha^i}{\partial X^j} A^j\right),$$

which is (38). We now prove equation (39). Using the relation $b = \nabla \times a$, equation (33) and the chain rule we obtain

$$\begin{aligned}B^i(t,\alpha) = b(t, X(t,\alpha)) = (\nabla \times a)(t, X(t,\alpha)) &= \varepsilon_{ijk}\frac{\partial a^k}{\partial x^j}(t, X(t,\alpha)) = \varepsilon_{ijk}\frac{\partial \alpha^l}{\partial x^j}\frac{\partial a^k}{\partial \alpha^l}(t, X(t,\alpha)) \\ &= \varepsilon_{ijk}\frac{\partial \alpha^l}{\partial x^j}\frac{\partial A^k}{\partial \alpha^l}(t,\alpha) = \frac{1}{2}\varepsilon_{ijk}\varepsilon_{ll_1l_2}\varepsilon_{jj_1j_2}\frac{\partial X^{j_1}}{\partial \alpha^{l_1}}\frac{\partial X^{j_2}}{\partial \alpha^{l_2}}\frac{\partial A^k}{\partial \alpha^l}\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}(\delta_{ij_2}\delta_{kj_1} - \delta_{ij_1}\delta_{kj_2})\varepsilon_{ll_1l_2}\frac{\partial X^{j_1}}{\partial \alpha^{l_1}}\frac{\partial X^{j_2}}{\partial \alpha^{l_2}}\frac{\partial A^k}{\partial \alpha^l}\\
&= \frac{1}{2}\varepsilon_{ll_1l_2}\left(\frac{\partial X^k}{\partial \alpha^{l_1}}\frac{\partial X^i}{\partial \alpha^{l_2}} - \frac{\partial X^i}{\partial \alpha^{l_1}}\frac{\partial X^k}{\partial \alpha^{l_2}}\right)\frac{\partial A^k}{\partial \alpha^l}\\
&= \varepsilon_{ll_1l_2}\frac{\partial X^k}{\partial \alpha^{l_1}}\frac{\partial X^i}{\partial \alpha^{l_2}}\frac{\partial A^k}{\partial \alpha^l} = (\nabla X^k \times \nabla X^i)^l\frac{\partial A^k}{\partial \alpha^l},
\end{aligned}
$$

which is equation (39). Equations (41) are straightforward, while equation (40) is just the momentum equation (14) evaluated on the Lagrangian flow-map $X$. Finally, using the invariance of the boundary under the Lagrangian flow-map $X$ (a particle being initially on the boundary remains on it forever) equation (42) is obtained from the evaluation of the boundary conditions (5) at the spatial point $x = X(t,\alpha) \in \partial\Omega$, $\alpha \in \partial\Omega$. □

We now introduce the following decomposition for the Lagrangian flow-map $X$ and the Lagrangian magnetic vector potential $A$,

$$
X(t,\alpha) = \alpha + \xi(t,\alpha), \quad \text{and} \quad A(t,\alpha) = a_0(\alpha) + \Psi(t,\alpha), \tag{45}
$$

with $\xi(0,\alpha) = 0$, and $\Psi(0,\alpha) = 0$. Rewriting equations (35)-(42) in terms of the new unknowns $(\xi,\Psi)$, we have

**Proposition 2** *(Lagrangian formulation of the e-MHD equations on a bounded domain). Let $(\xi,\Psi)$ be defined by (45). The Lagrangian formulation (35)-(42) of the e-MHD is equivalent to the following set of equations,*

$$
\nabla \times \dot{\xi} = \omega_0 + \nabla \times \Psi + \nabla(a_0^k + \Psi^k - \dot{\xi}^k) \times \nabla\xi^k, \quad \alpha \in \Omega, \quad t \in [0,T[, \tag{46}
$$

$$
\nabla \cdot \xi + \frac{1}{2}(\partial_i\xi^i\partial_j\xi^j - \partial_i\xi^j\partial_j\xi^i) + \frac{1}{6}\varepsilon_{i_1i_2i_3}\varepsilon_{j_1j_2j_3}\partial_{i_1}\xi^{j_1}\partial_{i_2}\xi^{j_2}\partial_{i_3}\xi^{j_3} = 0, \quad \alpha \in \Omega, \quad t \in [0,T[, \tag{47}
$$

$$
\Delta(a_0 + \Psi) + \nabla \cdot \big(\mathfrak{g}\,\nabla(a_0 + \nabla\Psi)\big) = \dot{\xi}, \quad \alpha \in \Omega, \quad t \in [0,T[, \tag{48}
$$

$$
(1 + \nabla\cdot\xi)\nabla\cdot\Psi - (\partial_j a_0^i + \partial_j\Psi^i)\Big(\partial_j\xi^i - \frac{1}{2}\varepsilon_{ii_1i_2}\varepsilon_{jj_1j_2}\partial_{i_1}\xi^{j_1}\partial_{i_2}\xi^{j_2}\Big) = 0, \quad \alpha \in \overline{\Omega}, \quad t \in [0,T[, \tag{49}
$$

$$
\begin{aligned}
B^i &= (\nabla \times a_0)^i + (\nabla \times \Psi)^i + \big(\nabla(a_0^k + \Psi^k) \times \nabla\xi^k\big)^i\\
&\quad + \big(\nabla\xi^i \times \nabla(a_0^k + \Psi^k)\big)^k + (\nabla\xi^k \times \nabla\xi^i)\cdot\nabla(a_0^k + \Psi^k), \quad \forall i, \quad \alpha \in \Omega, \quad t \in [0,T[,
\end{aligned} \tag{50}
$$

$$
E = -\ddot{\xi} - \dot{\xi} \times B, \quad \alpha \in \Omega, \quad t \in [0,T[, \tag{51}
$$

$$
U = \dot{\xi}, \quad P = \dot{\xi} - a_0 - \Psi, \quad \alpha \in \Omega, \quad t \in [0,T[, \tag{52}
$$

$$
\dot{\xi}\cdot\nu(\alpha + \xi) = 0, \quad a_0 \times \nu(\alpha + \xi) + \Psi \times \nu(\alpha + \xi) = 0, \quad \alpha \in \partial\Omega, \quad t \in [0,T[, \tag{53}
$$

*where the matrix $\mathfrak{g}$ is given by*

$$
\begin{aligned}
\mathfrak{g}_{ij} &= \delta_{ij}\nabla\cdot\xi + (1 + \nabla\cdot\xi)(\delta_{ij}\nabla\cdot\xi - \partial_i\xi^j - \partial_j\xi^i) + \partial_k\xi^i\partial_k\xi^j\\
&\quad + \frac{1}{2}(1 + \nabla\cdot\xi)(\varepsilon_{il_1l_2}\varepsilon_{jk_1k_2} + \varepsilon_{jl_1l_2}\varepsilon_{ik_1k_2})\partial_{l_1}\xi^{k_1}\partial_{l_2}\xi^{k_2}\\
&\quad - \frac{1}{2}\varepsilon_{kk_1k_2}(\varepsilon_{il_1l_2}\partial_k\xi^j + \varepsilon_{jl_1l_2}\partial_k\xi^i)\partial_{l_1}\xi^{k_1}\partial_{l_2}\xi^{k_2}\\
&\quad + \frac{1}{4}\varepsilon_{ii_1i_2}\varepsilon_{jj_1j_2}\partial_{i_1}\xi^{k_1}\partial_{i_2}\xi^{k_2}(\partial_{j_1}\xi^{k_1}\partial_{j_2}\xi^{k_2} - \partial_{j_1}\xi^{k_2}\partial_{j_2}\xi^{k_1}).
\end{aligned} \tag{54}
$$

**Proof.** Inserting the decomposition (45) into equations (35)-(42), using (31)-(33) and taking into account the initial condition $\nabla \cdot a_0 = 0$, after some algebra but without any difficulty, we obtain formula (46)-(54). Algebra involved in the derivation of equations (46)-(54) is standard tensorial calculus and the well-known formula for any $(3 \times 3)$-matrix $M$,

$$\det(I+M) = 1 + \mathrm{Tr}(M) + \frac{1}{2}\left(\mathrm{Tr}(M)^2 - \mathrm{Tr}(M^2)\right) + \det(M),$$

where $\mathrm{Tr}(M)$ denotes the trace of the matrix $M$. This last formula is only used for the derivation of (47) as it is done in [13]. Note that the order of equations in Proposition 2 is the same as the one of Proposition 1. □

*3.2. Construction of the solution from recursion relations*

In this section, using the Lagrangian formulation of Proposition 2, we design a recursive scheme to construct $\xi$ and $\Psi$. The convergence analysis of such a scheme is performed in Section 3.3. In order to design this scheme we introduce the following formal time-Taylor expansions of $\xi$ and $\Psi$,

$$\xi(t,\alpha) = \sum_{\sigma>0} \xi_\sigma(\alpha) t^\sigma, \quad \text{and} \quad \Psi(t,\alpha) = \sum_{\sigma>0} \Psi_\sigma(\alpha) t^\sigma. \tag{55}$$

Using these time-Taylor series and Proposition 2 we obtain a constructive scheme to determine recursively all the time-Taylor coefficients $\{\xi_\sigma\}_{\sigma>0}$ and $\{\Psi_\sigma\}_{\sigma>0}$. Schematically we obtain the following recursive procedure, for $\sigma > 1$,

$$\begin{aligned} \xi_\sigma &= \mathcal{F}_\xi\,[a_0]\big(\{\xi_{\sigma'}\}_{\sigma'<\sigma}, \{\Psi_{\sigma'}\}_{\sigma'<\sigma}\big), \\ \Psi_{\sigma-1} &= \mathcal{F}_\Psi[a_0]\big(\{\xi_{\sigma'}\}_{\sigma'\le\sigma}, \{\Psi_{\sigma'}\}_{\sigma'<\sigma-1}\big), \end{aligned} \tag{56}$$

where the functionals $\mathcal{F}_\xi[a_0](\cdot)$ and $\mathcal{F}_\Psi[a_0](\cdot)$, which depends on $a_0$, can be seen as some integro-differential or pseudo-differential operators of order zero. This recursive scheme is initialized with $\xi_1 = u_0$ and $a_0 = \mathcal{L}^{-1}\xi_1$, where $\mathcal{L}$ refers to the linear differential operator associated with a boundary value problem of elliptic type. The detailed algorithm is described in

**Proposition 3** *(Recursive scheme).*

1) *Initialization of the recursive algorithm.*
*The time-Taylor coefficient $\xi_1$ is given by*

$$\xi_1 = u_0. \tag{57}$$

*The initial magnetic vector potential $a_0$ is solution of the following non-homogeneous elliptic boundary value problem,*

$$\begin{cases} \Delta a_0 = \xi_1, & \alpha \in \Omega, \\ \nabla \cdot a_0 = 0, & \alpha \in \partial\Omega, \\ a_0 \times \nu = 0, & \alpha \in \partial\Omega. \end{cases} \tag{58}$$

2) *Determination of the time-Taylor coefficients $\xi_\sigma$ for $\sigma > 1$.*

*The Helmholtz–Hodge decomposition of the time-Taylor coefficient* $\xi_\sigma$ *reads*

$$\xi_\sigma = \nabla\varphi_\sigma + \nabla\times\Phi_\sigma, \quad \nabla\cdot\Phi_\sigma = 0, \quad \alpha\in\Omega, \tag{59}$$

*where the Helmholtz–Hodge potentials* $\varphi_\sigma$ *and* $\Phi_\sigma$ *are respectively a scalar and a three-dimensional vector.*

*The scalar potential* $\varphi_\sigma$ *satisfies the following non-homogeneous elliptic boundary value problem,*

$$\begin{cases} \Delta\varphi_\sigma = \nabla\cdot\xi_\sigma, & \alpha\in\Omega,\\ \partial_\nu\varphi_\sigma = \xi_\sigma\cdot\nu, & \alpha\in\partial\Omega, \end{cases} \tag{60}$$

*where*

$$\begin{aligned} \nabla\cdot\xi_\sigma = &-\frac{1}{2}\sum_{\substack{\sigma_1+\sigma_2=\sigma\\ \sigma_1,\,\sigma_2>0}} (\partial_i\xi^i_{\sigma_1}\partial_j\xi^j_{\sigma_2} - \partial_i\xi^j_{\sigma_1}\partial_j\xi^i_{\sigma_2})\\ &-\frac{1}{6}\varepsilon_{i_1i_2i_3}\varepsilon_{j_1j_2j_3}\sum_{\substack{\sigma_1+\sigma_2+\sigma_3=\sigma\\ \sigma_1,\,\sigma_2,\,\sigma_3>0}} \partial_{i_1}\xi^{j_1}_{\sigma_1}\partial_{i_2}\xi^{j_2}_{\sigma_2}\partial_{i_3}\xi^{j_3}_{\sigma_3}, \end{aligned} \tag{61}$$

$$\xi_\sigma\cdot\nu = -\sum_{\substack{\sigma_1+\sigma_2=\sigma\\ \sigma_1,\,\sigma_2>0}} \xi_{\sigma_1}\cdot\nu_{\sigma_2}, \tag{62}$$

*with*

$$\nu_\sigma(\alpha) := \sum_{1\le|\beta|\le\sigma} \partial^\beta\nu(\alpha)\sum_{i=1}^{\sigma}\sum_{P_i(\sigma,\beta)}\prod_{j=1}^{i}\frac{(\xi^1_{\ell_j})^{k^1_j}}{k^1_j!}\cdots\frac{(\xi^3_{\ell_j})^{k^3_j}}{k^3_j!}. \tag{63}$$

*In* (63) *the set* $P_i(\sigma,\beta)$ *is defined by*

$$P_i(\sigma,\beta) := \Big\{(\ell_1,\dots,\ell_i),\ (k_1,\dots,k_i);\quad 0<\ell_1<\dots<\ell_i;\\ |k_j|>0,\ j\in[1,i];\quad \sum_{j=1}^{i}k_j=\beta,\quad \sum_{j=1}^{i}|k_j|\ell_j=\sigma\Big\}. \tag{64}$$

*The vector potential* $\Phi_\sigma$ *satisfies the following non-homogeneous elliptic boundary value problem,*

$$\begin{cases} \Delta\Phi_\sigma = -\nabla\times\xi_\sigma, & \alpha\in\Omega,\\ \nabla\cdot\Phi_\sigma = 0, & \alpha\in\partial\Omega,\\ \Phi_\sigma\times\nu = 0, & \alpha\in\partial\Omega, \end{cases} \tag{65}$$

*where*

$$\begin{aligned} \nabla\times\xi_\sigma = &\frac{1}{\sigma}\nabla\times\Psi_{\sigma-1} + \frac{1}{\sigma}\nabla a^k_0\times\nabla\xi^k_{\sigma-1}\\ &+\sum_{\substack{\sigma_1+\sigma_2+1=\sigma\\ \sigma_1,\,\sigma_2>0}}\frac{1}{\sigma}\nabla\Psi^k_{\sigma_1}\times\nabla\xi^k_{\sigma_2} - \sum_{\substack{\sigma_1+\sigma_2=\sigma\\ \sigma_1,\,\sigma_2>0}}\frac{\sigma_1}{\sigma}\nabla\xi^k_{\sigma_1}\times\nabla\xi^k_{\sigma_2}. \end{aligned} \tag{66}$$

3) *Determination of the time-Taylor coefficients $\Psi_\sigma$ for $\sigma > 0$.*
*The time-Taylor coefficient $\Psi_\sigma$ satisfies the following non-homogeneous elliptic boundary value problem,*

$$\begin{cases} \Delta\Psi_\sigma = f_\sigma, & \alpha \in \Omega, \\ \nabla\cdot\Psi_\sigma = h_\sigma, & \alpha \in \partial\Omega, \\ \Psi_\sigma \times \nu = g_\sigma, & \alpha \in \partial\Omega, \end{cases} \tag{67}$$

*where*

$$f_\sigma := (\sigma+1)\xi_{\sigma+1} - \nabla\cdot(\mathfrak{g}_\sigma\nabla a_0) - \sum_{\substack{\sigma_1+\sigma_2=\sigma \\ \sigma_1,\,\sigma_2>0}} \nabla\cdot(\mathfrak{g}_{\sigma_1}\nabla\Psi_{\sigma_2}), \tag{68}$$

$$\begin{aligned} h_\sigma := \partial_j a_0^i \partial_j \xi_\sigma^i + &\sum_{\substack{\sigma_1+\sigma_2=\sigma \\ \sigma_1,\,\sigma_2>0}} \Big\{\partial_j\xi^i_{\sigma_1}\partial_j\Psi^i_{\sigma_2} - \nabla\cdot\xi_{\sigma_1}\,\nabla\cdot\Psi_{\sigma_2} - \frac{1}{2}\varepsilon_{ii_1i_2}\varepsilon_{jj_1j_2}\partial_{i_1}\xi^{j_1}_{\sigma_1}\partial_{i_2}\xi^{j_2}_{\sigma_2}\partial_j a_0^i\Big\} \\ &-\frac{1}{2}\sum_{\substack{\sigma_1+\sigma_2+\sigma_3=\sigma \\ \sigma_1,\,\sigma_2,\,\sigma_3>0}} \varepsilon_{ii_1i_2}\varepsilon_{jj_1j_2}\partial_{i_1}\xi^{j_1}_{\sigma_1}\partial_{i_2}\xi^{j_2}_{\sigma_2}\partial_j\Psi^i_{\sigma_3}, \end{aligned} \tag{69}$$

$$g_\sigma := -a_0 \times \nu_\sigma - \sum_{\substack{\sigma_1+\sigma_2=\sigma \\ \sigma_1,\,\sigma_2>0}} \Psi_{\sigma_1}\times\nu_{\sigma_2}, \tag{70}$$

*with*

$$\begin{aligned} (\mathfrak{g}_\sigma)_{ij} := 2\delta_{ij}\nabla\cdot\xi_\sigma - \partial_i\xi^j_\sigma - \partial_j\xi^i_\sigma \;+ &\sum_{\substack{\sigma_1+\sigma_2=\sigma \\ \sigma_1,\,\sigma_2>0}} \Big\{\nabla\cdot\xi_{\sigma_1}\,(\delta_{ij}\nabla\cdot\xi_{\sigma_2} - \partial_i\xi^j_{\sigma_2} - \partial_j\xi^i_{\sigma_2}) \\ &+ \partial_k\xi^i_{\sigma_1}\partial_k\xi^j_{\sigma_2} \;+\; \frac{1}{2}(\varepsilon_{il_1l_2}\varepsilon_{jk_1k_2} + \varepsilon_{jl_1l_2}\varepsilon_{ik_1k_2})\partial_{l_1}\xi^{k_1}_{\sigma_1}\partial_{l_2}\xi^{k_2}_{\sigma_2}\Big\} \\ &+ \frac{1}{2}\sum_{\substack{\sigma_1+\sigma_2+\sigma_3=\sigma \\ \sigma_1,\,\sigma_2,\,\sigma_3>0}} \Big\{\nabla\cdot\xi_{\sigma_3}\,(\varepsilon_{il_1l_2}\varepsilon_{jk_1k_2} + \varepsilon_{jl_1l_2}\varepsilon_{ik_1k_2})\partial_{l_1}\xi^{k_1}_{\sigma_1}\partial_{l_2}\xi^{k_2}_{\sigma_2} \\ &- \varepsilon_{kk_1k_2}(\varepsilon_{il_1l_2}\partial_k\xi^j_{\sigma_3} + \varepsilon_{jl_1l_2}\partial_k\xi^i_{\sigma_3})\partial_{l_1}\xi^{k_1}_{\sigma_1}\partial_{l_2}\xi^{k_2}_{\sigma_2}\Big\} \\ &+ \frac{1}{4}\sum_{\substack{\sigma_1+\sigma_2+\sigma_3+\sigma_4=\sigma \\ \sigma_1,\,\sigma_2,\,\sigma_3,\,\sigma_4>0}} \varepsilon_{ii_1i_2}\varepsilon_{jj_1j_2}\partial_{i_1}\xi^{k_1}_{\sigma_1}\partial_{i_2}\xi^{k_2}_{\sigma_2}(\partial_{j_1}\xi^{k_1}_{\sigma_3}\partial_{j_2}\xi^{k_2}_{\sigma_4} - \partial_{j_1}\xi^{k_2}_{\sigma_3}\partial_{j_2}\xi^{k_1}_{\sigma_4}). \end{aligned} \tag{71}$$

Before giving the proof of Proposition 3, we make the important

**Remark 5** *(A nonlinear recursive scheme).* Observe that the recursive scheme (56) is nonlinear. Indeed, fixing $\sigma > 1$, we assume that we know all the following time-Taylor coefficients $\{\xi_{\sigma'}\}_{\sigma'<\sigma}$ and $\{\Psi_{\sigma'}\}_{\sigma'<\sigma-1}$. From these known time-Taylor coefficients, the aim is to obtain the next unknown time-Taylor coefficients $\xi_\sigma$ and $\Psi_{\sigma-1}$, that we call the current time-Taylor coefficients at the rank $\sigma$. Introducing the notation $\mathcal{X} := \xi_\sigma$ and $\mathcal{Y} := \Psi_{\sigma-1}$ for the current time-Taylor coefficients, the scheme (56) rewrites as

$$\begin{aligned} \mathcal{X} &= \mathcal{F}_\xi\,[a_0]\big(\{\xi_{\sigma'}\}_{\sigma'<\sigma},\, \{\Psi_{\sigma'}\}_{\sigma'<\sigma-1},\, \mathcal{Y}\big), \\ \mathcal{Y} &= \mathcal{F}_\Psi[a_0]\big(\{\xi_{\sigma'}\}_{\sigma'<\sigma},\, \mathcal{X},\, \{\Psi_{\sigma'}\}_{\sigma'<\sigma-1}\big), \end{aligned} \tag{72}$$

or

$$\begin{aligned} \mathcal{X} &= \mathcal{F}_\xi\,[a_0]\Big(\{\xi_{\sigma'}\}_{\sigma'<\sigma},\, \{\Psi_{\sigma'}\}_{\sigma'<\sigma-1},\, \mathcal{F}_\Psi[a_0]\big(\{\xi_{\sigma'}\}_{\sigma'<\sigma},\, \mathcal{X},\, \{\Psi_{\sigma'}\}_{\sigma'<\sigma-1}\big)\Big), \\ \mathcal{Y} &= \mathcal{F}_\Psi[a_0]\big(\{\xi_{\sigma'}\}_{\sigma'<\sigma},\, \mathcal{X},\, \{\Psi_{\sigma'}\}_{\sigma'<\sigma-1}\big). \end{aligned} \tag{73}$$

In other words the boundary value problems (60), (65) and (67) are coupled together and thus constitute a closed nonlinear system (with Helmholtz–Hodge decomposition (59)) in terms of the current time-Taylor coefficients $(\xi_\sigma, \Psi_{\sigma-1})$ or $(\mathcal{X}, \mathcal{Y})$. This nonlinear coupling is just a resurgence of the nonlinearity existing between $\xi$ and $\Psi$ in equations (46)-(49) of Proposition 2, or between $X$ and $A$ in equations (35)-(38) of Proposition 1. This situation is very different from the incompressible Euler equations [13,11], which would correspond in the scheme (56) to set $a_0 = 0$, and $\{\Psi_\sigma = 0\}_{\sigma>0}$, i.e. $\mathcal{F}_\Psi \equiv 0$ and

$$\mathcal{X} = \xi_\sigma = \mathcal{F}_\xi\,[0]\big(\{\xi_{\sigma'}\}_{\sigma'<\sigma},\, \{\Psi_{\sigma'} = 0\}_{\sigma'<\sigma}\big) = \mathcal{CZ}\big(\{\xi_{\sigma'}\}_{\sigma'<\sigma}\big),$$

where $\mathcal{CZ}(\cdot)$ stands for a Calderón–Zygmund integro-differential operator of order zero. In this case we clearly observe that for any $\sigma > 1$, the current time-Taylor coefficient $\mathcal{X} = \xi_\sigma$ is obtained only from coefficients $\{\xi_{\sigma'}\}_{\sigma'<\sigma}$ by solving linear boundary value problems in terms of the current time-Taylor coefficient $\mathcal{X} = \xi_\sigma$ or in terms of the current time-Taylor coefficients for the Helmholtz–Hodge potentials $(\varphi_\sigma, \Phi_\sigma)$ via the Helmholtz–Hodge decomposition (59).

For the proof of the regularity result of Theorem 2, we do not need to solve explicitly the nonlinearity (72) or (73), as we will see in Section 3.3, because the final a priori estimate holds on a generating function, an object which groups together all the time-Taylor coefficients $\xi_\sigma$ and $\Psi_\sigma$. By contrast, from a numerical perspective, this nonlinearity must be solved explicitly, at least in an approximative way. As often, this can be performed by applying a Picard iteration method (first-order approximation) or a Newton iteration method (second-order approximation) to the nonlinear equations (72) or (73). Usually, for a given precision, the Picard or Newton iterative procedure converges quite fastly, with very few iterations.

**Proof of Proposition 3.** We start the proof by setting an Helmholtz–Hodge decomposition for the time-Taylor coefficients $\xi_\sigma$ of the displacement vector $\xi$. This Helmholtz–Hodge decomposition must incorporate suitable boundary conditions, which must be consistent with the natural boundary conditions of our problem. Using the Helmholtz–Hodge decomposition for vectors on a bounded, simply-connected and regular domain $\Omega$ (with $\mathscr{C}^\infty$ boundary $\partial\Omega$) of Euclidean spaces (see, e.g., [45,101,4,71,5]), there exist a scalar-valued function $\varphi_\sigma$, and a vector-valued function $\Phi_\sigma$ such that the coefficient $\xi_\sigma$ can be rewritten as equation (59). Since the domain $\Omega$ is simply connected and regular, the Helmholtz decomposition (59) does not contain any harmonic fields. Moreover for non-homogeneous boundary value problems involving vector potentials, kernels (i.e. solutions of the corresponding homogeneous boundary value problems) are empty, and there is no integrability or solvability conditions. As in [13], the divergence (resp. the curl) of (59) gives the first equation of (60) (resp. (65)). Taking the scalar product of (59) with the normal vector $\nu$ and assuming

$$\nu \cdot \nabla \times \Phi_\sigma = 0 \quad \text{on } \partial\Omega, \tag{74}$$

we obtain the boundary condition of the boundary value problem (60). At this point, we have the choice between two boundary conditions for the Laplace equation,

$$\Delta\Phi_\sigma = -\nabla \times \xi_\sigma \quad \text{on } \Omega, \tag{75}$$

going hand in hand with the gauge or the constraint condition,

$$\nabla \cdot \Phi_\sigma = 0 \quad \text{on } \overline{\Omega}. \tag{76}$$

Keeping in mind the divergence theorem, from (76) a first natural choice for the boundary condition of (75) is $\Phi_\sigma \cdot \nu = 0$ on $\partial\Omega$, plus boundary condition (74). With these boundary conditions the Laplace equation (75) plus the gauge condition (76) are well-posed in Sobolev spaces $H^s$ (see, e.g., Section 3 and especially Theorem 3.5 of Chapter 1 in [45]). The second choice is the boundary condition $\Phi_\sigma \times \nu = 0$ on $\partial\Omega$, which is a consequence of (74) and the vector analysis formula (17). This last solution is the choice that we have done in the boundary value problem (65), which is also well-posed in Sobolev spaces $H^s$ (see, e.g., Section 3 and especially Theorem 3.6 of Chapter 1 in [45], or [101,4,71,5]). Since these two boundary conditions are different, the associated Helmholtz–Hodge potentials $\Phi_\sigma$ are also different, but their curl is the same, which finally gives the same value for $\xi_\sigma$.

We then continue the proof by establishing the central non-homogeneous elliptic boundary value problems involved in the recursive scheme, namely (58), (60), (65) and (67). Substituting formal time series (55) into the Cauchy invariants equation (46), and collecting terms of the same power $\sigma > 0$, we obtain, after some algebra, equation (57) for $\sigma = 1$ and equation (66) for $\sigma > 1$. Similarly, substituting formal time series (55) into equation (48), and collecting terms of the same power $\sigma \geq 0$, we obtain, for $\sigma = 0$, the first equation of the boundary value problem (58), and for $\sigma > 0$, the first equation of the boundary value problem (67) with definitions (68) and (71). We note that the boundary value problem (58) is equivalent to the boundary value problem (19) for strong solutions (see, e.g., [71,5]). Using the Faà di Bruno formula [29,54,38], we obtain the time series expansion of the composed function $\nu(\alpha + \xi(t, \alpha))$ namely,

$$\nu(\alpha + \xi(t, \alpha)) = \sum_{\sigma \geq 0} \nu_\sigma(\alpha) t^\sigma, \tag{77}$$

with $\nu_0 = \nu$ and $\nu_\sigma$ given by definition (63) for $\sigma > 0$. Using (77) in the first equation of (53) we obtain, for $\sigma = 1$, the boundary condition $\xi_1 \cdot \nu = 0$ on $\partial\Omega$, and for $\sigma > 1$, the boundary condition of the boundary value problem (60) with definition (62). Similarly, using (77) in the second equation of (53) we obtain, for $\sigma = 0$, the boundary condition of the boundary value problem (58), and for $\sigma > 0$, the boundary condition of the boundary value problem (67) with definition (70). Finally substituting formal time series (55) into equations (47), (49), and (54), after collecting terms of the same power $\sigma > 0$, we obtain respectively equation (61), the second equation of the boundary value problem (67) with definition (69), and definition (71). Let us note that in the time series expansion of (49) the term of power $\sigma = 0$ does not exist because we have used the initial condition $\nabla \cdot a_0 = 0$ when deriving (49) from (38). If we did not use the initial condition $\nabla \cdot a_0 = 0$ in deriving (49) from (38), new terms (of degree zero and one in $\sigma$-power) involving $\nabla \cdot a_0$ would appear in (49). But the time series expansion of this modified version of (49) or directly the time series expansion of (38), by using (45) and (55), would lead to the equation $\nabla \cdot a_0 = 0$, for the term of power $\sigma = 0$ in the corresponding time series expansion. Therefore, if we do not assume initially the condition $\nabla \cdot a_0 = 0$, we retrieve it from the time series expansion of the Lagrangian formulation of the e-MHD equations. This means that some constraints on the initial condition are already encoded in the Lagrangian formulation of the equations. This ends the proof of Proposition 3. □

### *3.3. Convergence analysis of the recursive scheme*

Here, we prove Theorem 2. For this we need to prove first that the time series expansion (45), for $\xi$ and $\Psi$, converge and are time-ultradifferentiable in the log-superlinear Faà di Bruno class $\mathcal{C}\{\mathcal{M}\}(]0, T[; H^s(\Omega))$ (see Definition 1). Using $(\xi,\, \Psi) \in \mathcal{C}\{\mathcal{M}\}(]0, T[; H^s(\Omega))$, $u_0 \in H^s(\Omega)$, and $a_0 \in H^{s+2}(\Omega)$, we deduce from (45) that $(X,\, A) \in \mathcal{C}\{\mathcal{M}\}(]0, T[; H^s(\Omega))$. Next, using $(\xi,\, \Psi) \in \mathcal{C}\{\mathcal{M}\}(]0, T[; H^s(\Omega))$, the Lagrangian formulation (50)-(51) for $B$ and $E$, and the algebra property (81) for Sobolev spaces $H^s(\Omega)$, we obtain that $(B,\, E) \in \mathcal{C}\{\mathcal{M}\}(]0, T[; H^{s-1}(\Omega))$. As already observed in Remark 4, from equation (51) the regularity result $(X,\, A) \in \mathcal{C}\{\mathcal{M}\}(]0, T[; H^s(\Omega))$ implies $(U,\, P) \in \mathcal{C}\{\mathcal{M}\}(]0, T[; H^s(\Omega))$. Then the rest of the proof is devoted to show $(\xi,\, \Psi) \in \mathcal{C}\{\mathcal{M}\}(]0, T[; H^s(\Omega))$.

To simplify the notation, the norm $\|\cdot\|_{H^s(\Omega)}$ will be sometimes denoted by $\|\cdot\|_{H^s}$. Any space-time dependent vector-valued function $\psi : ]0,T[\times\Omega \mapsto \mathbb{R}^3$, belongs to the space $\mathcal{C}\{\mathcal{M}\}(]0,T[;H^s(\Omega))$ if, and only if, there exists a real positive number $\varrho$ such that the set

$$\left\{\frac{\|\partial_t^\sigma \psi\|_{H^s(\Omega)}}{\varrho^\sigma\, \sigma!\, M_\sigma}, \quad \sigma\in\mathbb{N}, \quad t\in ]0,T[\right\}, \tag{78}$$

is bounded. A sufficient condition to obtain $(\xi,\, \Psi)\in\mathcal{C}\{\mathcal{M}\}(]0,T[;H^s(\Omega))$ is that the generating function $t\mapsto\zeta(t)$, defined by

$$\zeta(t) = \sum_{\sigma>0}\left(\|\xi_\sigma\|_{H^s(\Omega)} + \|\Psi_\sigma\|_{H^s(\Omega)}\right)\varrho^{-\sigma} M_\sigma^{-1} t^\sigma, \tag{79}$$

is uniformly bounded on $]0,T[$.

To derive a priori estimates we need three tools. The first one is the Lemma 1 of [11] that, for the sake of completeness, we here give as

**Lemma 1.** *Let $\psi : \partial\Omega \mapsto \mathbb{R}$ be a LSL–FdB ultradifferentiable function defined on $\partial\Omega$, which is also LSL–FdB ultradifferentiable. Then, there exist positive constants $C$ and $R$, which depend on $\psi$, $\partial\Omega$, $s$, $M_0$, and $C_{\mathrm{D}}$, such that*

$$\|\partial^\beta\psi\|_{H^s(\partial\Omega)} \le CR^{-|\beta|}|\beta|!M_{|\beta|}, \quad |\beta|\ge 0. \tag{80}$$

**Proof.** For the proof of Lemma 1, we refer to the proof of Lemma 1 of [11]. □

We also use repeatedly the property that the Sobolev space $H^{\mathfrak{s}}(\Omega)$, with $\mathfrak{s} > d/2$ ($\Omega$ being here a bounded domain of $\mathbb{R}^d$, $d\ge 1$), is an algebra with respect to the pointwise multiplication, i.e. there exists a constant $C_a = C_a(\mathfrak{s})$, which depends on $\mathfrak{s}$, such that

$$\|\psi\varphi\|_{H^s(\Omega)} \le C_a\|\psi\|_{H^s(\Omega)}\|\varphi\|_{H^s(\Omega)}, \quad \forall\psi,\ \varphi\in H^{\mathfrak{s}}(\Omega), \quad \mathfrak{s}>d/2. \tag{81}$$

The last tool is the continuous surjection of the trace operator $\psi\mapsto\psi_{|\partial\Omega}$ from $H^{\mathfrak{s}}(\Omega)$ to $H^{\mathfrak{s}-1/2}(\partial\Omega)$, for $\mathfrak{s}\ge 1$, with the continuity constant $C_\partial$ (see, e.g., [79]), i.e.

$$\|\psi_{|\partial\Omega}\|_{H^{\mathfrak{s}-1/2}(\partial\Omega)} \le C_\partial\|\psi\|_{H^s(\Omega)}, \quad \forall\psi\in H^{\mathfrak{s}}(\Omega), \quad \mathfrak{s}\ge 1. \tag{82}$$

We now derive some a priori estimates. For this, we use elliptic regularity estimates in Sobolev spaces for non-homogeneous boundary value problems, which are recalled in Appendix A. Using Theorem 3 of Appendix A, for the solution $\varphi_\sigma$ of the non-homogeneous Neumann boundary value problem (60), we have the following elliptic regularity estimates

$$\|\varphi_\sigma\|_{H^{s+1}(\Omega)} \le C_1\left(\|\nabla\cdot\xi_\sigma\|_{H^{s-1}(\Omega)} + \|\xi_\sigma\cdot\nu\|_{H^{s-1/2}(\partial\Omega)}\right). \tag{83}$$

Using Theorem 4 of Appendix A, we have the following elliptic regularity estimates

$$\|\Phi_\sigma\|_{H^{s+1}(\Omega)} \le C_2\|\nabla\times\xi_\sigma\|_{H^{s-1}(\Omega)}, \tag{84}$$

for the solution $\Phi_\sigma$ of the non-homogeneous boundary value problem (65), and

$$\|\Psi_\sigma\|_{H^s(\Omega)} \le C_3\left(\|f_\sigma\|_{H^{s-2}(\Omega)} + \|g_\sigma\|_{H^{s-1/2}(\partial\Omega)} + \|h_\sigma\|_{H^{s-3/2}(\partial\Omega)}\right), \tag{85}$$

for the solution $\Psi_\sigma$ of the non-homogeneous boundary value problem (67). Using the Helmholtz–Hodge decomposition (59) for the coefficient $\xi_\sigma$, we obtain for $\sigma > 0$,

$$\begin{aligned}\|\xi_\sigma\|_{H^s(\Omega)} &\leq \|\nabla\cdot\varphi_\sigma\|_{H^s(\Omega)} + \|\nabla\times\Phi_\sigma\|_{H^s(\Omega)}\\ &\leq \|\varphi_\sigma\|_{H^{s+1}(\Omega)} + \|\Phi_\sigma\|_{H^{s+1}(\Omega)}\\ &\leq C_{12}\left(\|\nabla\times\xi_\sigma\|_{H^{s-1}(\Omega)} + \|\nabla\cdot\xi_\sigma\|_{H^{s-1}(\Omega)} + \|\xi_\sigma\cdot\nu\|_{H^{s-1/2}(\partial\Omega)}\right),\end{aligned} \tag{86}$$

with $C_{12} = \max\{C_1, C_2\}$. Using (85)-(86), we obtain from the definition of the generating function (79),

$$\begin{aligned}\zeta(t) \leq C_{123}\sum_{\sigma>0}\Big(&\|\nabla\cdot\xi_\sigma\|_{H^{s-1}(\Omega)} + \|\nabla\times\xi_\sigma\|_{H^{s-1}(\Omega)} + \|\xi_\sigma\cdot\nu\|_{H^{s-1/2}(\partial\Omega)}\\ &+ \|f_\sigma\|_{H^{s-2}(\Omega)} + \|g_\sigma\|_{H^{s-1/2}(\partial\Omega)} + \|h_\sigma\|_{H^{s-3/2}(\partial\Omega)}\Big)\varrho^{-\sigma}M_\sigma^{-1}t^\sigma,\end{aligned} \tag{87}$$

with $C_{123} = \max\{C_{12}, C_3\}$. We must estimate the right hand side of (87). This is here that the assumption of Theorem 2 on the regularity of the boundary $\partial\Omega$ plays a crucial role. Since the boundary $\partial\Omega$ is a LSL–FdB ultradifferentiable manifold, then the normal vector $\nu : \partial\Omega \mapsto \mathbb{R}^3$ is LSL–FdB ultradifferentiable and using Lemma 1, there exist positive real constants $C_\nu$, and $R_\nu$ such that, for $0 \leq s < \infty$, and $|\beta| \geq 0$,

$$\|\partial^\beta\nu\|_{H^s(\partial\Omega)} \leq C_\nu R_\nu^{-|\beta|}|\beta|!M_{|\beta|}, \tag{88}$$

where the sequence $\{M_\sigma\}_{\sigma\geq 0}$ satisfies Definition 1. An estimate of the right-hand side of (87) is given by

**Proposition 4.** *Let $s > 3/2$. Then there exist positive constants*

$$\begin{aligned}C_d &= C_d(C_a, M_0),\\ C_r &= C_r(C_a, M_0, M_1, \varrho, \|a_0\|_{H^s}),\\ C_n &= C_n(C_a, M_0, C_\nu, C_\partial),\\ C_f &= C_f(C_a, M_0, \|a_0\|_{H^s}),\\ C_g &= C_g(C_a, M_0, C_\nu, C_\partial, \|a_0\|_{H^s}),\\ C_h &= C_h(C_a, M_0, C_\partial),\end{aligned}$$

*such that*

$$\sum_{\sigma>0}\|\nabla\cdot\xi_\sigma\|_{H^{s-1}(\Omega)}\varrho^{-\sigma}M_\sigma^{-1}t^\sigma \leq C_d\zeta^2(t)(1+\zeta(t)), \tag{89}$$

$$\sum_{\sigma>0}\|\nabla\times\xi_\sigma\|_{H^{s-1}(\Omega)}\varrho^{-\sigma}M_\sigma^{-1}t^\sigma \leq \|u_0\|_{H^s}\varrho^{-1}M_1^{-1}t + C_r\zeta(t)\big(t + (1+t)\zeta(t)\big), \tag{90}$$

$$\sum_{\sigma>0}\|\xi_\sigma\cdot\nu\|_{H^{s-1/2}(\partial\Omega)}\varrho^{-\sigma}M_\sigma^{-1}t^\sigma \leq C_n\zeta(t)\big(1 - K_\nu^{-1}\zeta(t)\big)^{-1}, \tag{91}$$

$$\sum_{\sigma>0}\|f_\sigma\|_{H^{s-2}(\Omega)}\varrho^{-\sigma}M_\sigma^{-1}t^\sigma \leq \ddot{\zeta}(t) + C_f\zeta(t)\big(1+\zeta(t)+\zeta^2(t)+\zeta^3(t)+\zeta^4(t)\big), \tag{92}$$

$$\sum_{\sigma>0}\|g_\sigma\|_{H^{s-1/2}(\partial\Omega)}\varrho^{-\sigma}M_\sigma^{-1}t^\sigma \leq C_g(1+\zeta(t))\big(1 - K_\nu^{-1}\zeta(t)\big)^{-1}, \tag{93}$$

$$\sum_{\sigma>0}\|h_\sigma\|_{H^{s-3/2}(\partial\Omega)}\varrho^{-\sigma}M_\sigma^{-1}t^\sigma \leq C_h\zeta(t)\big(1+\zeta(t)+\zeta^2(t)\big), \tag{94}$$

*with* $K_\nu^{-1} = C_a C_\partial / R_\nu$, *and* $\tilde{\zeta}(t) = (\varrho/C_{\mathrm{D}}) \sum_{\sigma>0} \left(\|\xi_\sigma\|_{H^s(\Omega)} + \|\Psi_\sigma\|_{H^s(\Omega)}\right) (\varrho/C_{\mathrm{D}})^{-\sigma} M_\sigma^{-1} t^\sigma$.

**Proof.** In order not to need to determine the purely numerical constants, which arise in a priori estimates and are not relevant, we introduce the notation $A \lesssim B$ and $A \simeq B$ defined as follows. The notation $A \lesssim B$ (resp. $A \simeq B$) means that there exists a purely numerical constant $C_{\mathrm{num}}$ such that $A \leq C_{\mathrm{num}} B$ (resp. $A = C_{\mathrm{num}} B$). We start with estimate (89). Using the algebra property (81), and the superlinearity property (27), we obtain from (61),

$$\frac{\|\nabla \cdot \xi_\sigma\|_{H^{s-1}}}{\varrho^\sigma M_\sigma} \lesssim C_a M_0 \sum_{\substack{\sigma_1+\sigma_2=\sigma \\ \sigma_1,\,\sigma_2>0}} \frac{\|\xi_{\sigma_1}\|_{H^s}}{\varrho^{\sigma_1} M_{\sigma_1}} \frac{\|\xi_{\sigma_2}\|_{H^s}}{\varrho^{\sigma_2} M_{\sigma_2}} + C_a^2 M_0^2 \sum_{\substack{\sigma_1+\sigma_2+\sigma_3=\sigma \\ \sigma_1,\,\sigma_2,\,\sigma_3>0}} \frac{\|\xi_{\sigma_1}\|_{H^s}}{\varrho^{\sigma_1} M_{\sigma_1}} \frac{\|\xi_{\sigma_2}\|_{H^s}}{\varrho^{\sigma_2} M_{\sigma_2}} \frac{\|\xi_{\sigma_3}\|_{H^s}}{\varrho^{\sigma_3} M_{\sigma_3}}.$$

Multiplying the above estimate by $t^\sigma$ and summing the result over the index $\sigma$, we obtain (89) with

$$C_d \simeq C_a M_0 \max\{1,\, C_a M_0\}.$$

We continue with the proof of (90). Using the algebra property (81), and the superlinearity property (27), we obtain from (57) and (66),

$$\begin{aligned} \frac{\|\nabla \times \xi_\sigma\|_{H^{s-1}}}{\varrho^\sigma M_\sigma} \lesssim{}& \frac{\|u_0\|_{H^s}}{\varrho M_1} \delta_{1\sigma} + \frac{M_0}{\varrho M_1} \frac{\|\Psi_{\sigma-1}\|_{H^s}}{\varrho^{\sigma-1} M_{\sigma-1}} + \frac{C_a M_0}{\varrho M_1} \|a_0\|_{H^s} \frac{\|\xi_{\sigma-1}\|_{H^s}}{\varrho^{\sigma-1} M_{\sigma-1}} \\ &+ \frac{C_a M_0^2}{\varrho M_1} \sum_{\substack{\sigma_1+\sigma_2+1=\sigma \\ \sigma_1,\,\sigma_2>0}} \frac{\|\Psi_{\sigma_1}\|_{H^s}}{\varrho^{\sigma_1} M_{\sigma_1}} \frac{\|\xi_{\sigma_2}\|_{H^s}}{\varrho^{\sigma_2} M_{\sigma_2}} + C_a M_0 \sum_{\substack{\sigma_1+\sigma_2=\sigma \\ \sigma_1,\,\sigma_2>0}} \frac{\|\xi_{\sigma_1}\|_{H^s}}{\varrho^{\sigma_1} M_{\sigma_1}} \frac{\|\xi_{\sigma_2}\|_{H^s}}{\varrho^{\sigma_2} M_{\sigma_2}} \end{aligned}$$

Multiplying the above estimate by $t^\sigma$ and summing the result over the index $\sigma$, we obtain (89) with

$$C_r \simeq M_0 \max\{(1 + C_a \|a_0\|_{H^s}) \varrho^{-1} M_1^{-1},\, C_a M_0 \varrho^{-1} M_1^{-1},\, C_a\}.$$

We continue with the proof of (91). Using the algebra property (81), the superlinearity property (27), and the continuous surjection of the trace operator (82), we obtain from (62),

$$\frac{\|\xi_\sigma \cdot \nu\|_{H^{s-1/2}(\partial\Omega)}}{\varrho^\sigma M_\sigma} \leq C_\partial C_a M_0 \sum_{\substack{\sigma_1+\sigma_2=\sigma \\ \sigma_1,\,\sigma_2>0}} \frac{\|\xi_{\sigma_1}\|_{H^s(\Omega)}}{\varrho^{\sigma_1} M_{\sigma_1}} \frac{\|\nu_{\sigma_2}\|_{H^{s-1/2}(\partial\Omega)}}{\varrho^{\sigma_2} M_{\sigma_2}}. \tag{95}$$

We then have to control $\|\nu_\sigma\|_{H^{s-1/2}(\partial\Omega)}$ in (95). Using the algebra property (81), the continuous surjection of the trace operator (82), and the Cauchy-like estimate (88), we obtain from (63)-(64),

$$\begin{aligned} \frac{\|\nu_\sigma\|_{H^{s-1/2}(\partial\Omega)}}{\varrho^\sigma M_\sigma} &\leq C_a \sum_{1\leq|\beta|\leq\sigma} \frac{\|\partial^\beta \nu\|_{H^{s-1/2}(\partial\Omega)}}{\varrho^\sigma M_\sigma} (C_a C_\partial)^{|\beta|} \sum_{i=1}^{\sigma} \sum_{P_i(\sigma,\beta)} \prod_{j=1}^{i} \frac{\|\xi^1_{\ell_j}\|^{k^1_j}_{H^s(\Omega)}}{k^1_j!} \cdots \frac{\|\xi^3_{\ell_j}\|^{k^3_j}_{H^s(\Omega)}}{k^3_j!} \\ &\leq C_a C_\nu \sum_{1\leq|\beta|\leq\sigma} \left(\frac{C_a C_\partial}{R_\nu}\right)^{|\beta|} \frac{|\beta|! M_{|\beta|}}{\varrho^\sigma M_\sigma} \sum_{i=1}^{\sigma} \sum_{P_i(\sigma,\beta)} \prod_{j=1}^{i} \frac{\|\xi^1_{\ell_j}\|^{k^1_j}_{H^s}}{k^1_j!} \cdots \frac{\|\xi^3_{\ell_j}\|^{k^3_j}_{H^s}}{k^3_j!} \\ &\leq C_a C_\nu \sum_{1\leq|\beta|\leq\sigma} \left(\frac{C_a C_\partial}{R_\nu}\right)^{|\beta|} \frac{|\beta|!}{\varrho^\sigma} \end{aligned}$$

$$\sum_{i=1}^{\sigma} \sum_{P_i(\sigma,\beta)} \frac{M_{|\beta|} M_{\ell_1}^{|k_1|} \dots M_{\ell_i}^{|k_i|}}{M_\sigma} \prod_{j=1}^{i} \left( \frac{\|\xi_{\ell_j}\|_{H^s}}{M_{\ell_j}} \right)^{|k_j|} \frac{1}{k_j!}. \tag{96}$$

It is now convenient to introduce the following notation,

$$\mu_1 := \ell_1, \ \dots, \ \mu_{|k_1|} := \ell_1, \ \mu_{|k_1|+1} := \ell_2, \ \dots, \ \mu_{|k_1|+|k_2|} := \ell_2, \ \dots, \ \mu_{|k_1|+\dots+|k_i|} := \ell_i,$$

in terms of which we have

$$\begin{aligned} M_{\ell_1}^{|k_1|} \dots M_{\ell_i}^{|k_i|} &= M_{\mu_1} \dots M_{\mu_{|k_1|}} M_{\mu_{|k_1|+1}} \dots M_{\mu_{|k_1|+|k_2|}} \dots M_{\mu_{|k_1|+\dots+|k_i|}} \\ &= M_{\mu_1} \dots M_{\mu_{|\beta|}}. \end{aligned}$$

Using the FdB-stability property (28) and definition (64) for the set $P_i(\sigma, \beta)$, we obtain

$$\frac{M_{|\beta|} M_{\ell_1}^{|k_1|} \dots M_{\ell_i}^{|k_i|}}{M_\sigma} = \frac{M_{|\beta|} M_{\mu_1} \dots M_{\mu_{|\beta|}}}{M_\sigma} \le \frac{M_{\mu_1+\dots+\mu_{|\beta|}}}{M_\sigma} = \frac{M_{|k_1|\ell_1+\dots+|k_i|\ell_i}}{M_\sigma} = 1. \tag{97}$$

Using (97) and the definition of the generating function (79), we obtain from (96),

$$\begin{aligned} \frac{\|\nu_\sigma\|_{H^{s-1/2}(\partial\Omega)}}{\varrho^\sigma M_\sigma} &\le C_a C_\nu \sum_{1\le|\beta|\le\sigma} \left( \frac{C_a C_\partial}{R_\nu} \right)^{|\beta|} |\beta|! \sum_{i=1}^{\sigma} \sum_{P_i(\sigma,\beta)} \prod_{j=1}^{i} \left( \frac{\|\xi_{\ell_j}\|_{H^s}}{\varrho^{\ell_j} M_{\ell_j}} \right)^{|k_j|} \frac{1}{k_j!} \\ &\le \frac{C_a C_\nu}{\sigma!} \sigma! \sum_{1\le|\beta|\le\sigma} \left( \frac{C_a C_\partial}{R_\nu} \right)^{|\beta|} |\beta|! \sum_{i=1}^{\sigma} \sum_{P_i(\sigma,\beta)} \prod_{j=1}^{i} \frac{\left(\partial_t^{\ell_j} \zeta(0)\right)^{|k_j|}}{k_j! (\ell_j)^{|k_j|}}. \end{aligned} \tag{98}$$

We introduce the function $\mathcal{K} : \mathbb{R}^3 \mapsto \mathbb{R}$, defined by

$$\mathcal{K}(x_1, x_2, x_3) = \prod_{i=1}^{3} \left( 1 - K_\nu^{-1} x_i \right)^{-1/3}, \quad K_\nu^{-1} := C_a C_\partial / R_\nu, \tag{99}$$

and such that

$$(\partial^\beta \mathcal{K})(0,0,0) = \partial_z^{|\beta|} \mathcal{K}(z,z,z)_{|z=0} = |\beta|! K_\nu^{-|\beta|}. \tag{100}$$

Using (99)-(100), and setting

$$\Theta^{(\sigma)}(0) = \sigma! \sum_{1\le|\beta|\le\sigma} (\partial^\beta \mathcal{K})(0,0,0) \sum_{i=1}^{\sigma} \sum_{P_i(\sigma,\beta)} \prod_{j=1}^{i} \frac{\left(\partial_t^{\ell_j} \zeta(0)\right)^{|k_j|}}{k_j! (\ell_j)^{|k_j|}},$$

we obtain from (98),

$$\frac{\|\nu_\sigma\|_{H^{s-1/2}(\partial\Omega)}}{\varrho^\sigma M_\sigma} \le C_a C_\nu \frac{\Theta^{(\sigma)}(0)}{\sigma!}. \tag{101}$$

Multiplying (101) by $t^\sigma$, summing the result over the index $\sigma$, and using the Faà di Bruno formula, we obtain

$$\sum_{\sigma>0} \frac{\|\nu_\sigma\|_{H^{s-1/2}(\partial\Omega)}}{\varrho^\sigma M_\sigma} t^\sigma \leq C_a C_\nu \sum_{\sigma>0} \Theta^{(\sigma)}(0) \frac{t^\sigma}{\sigma!}$$
$$\leq C_a C_\nu \Theta(t)$$
$$\leq C_a C_\nu \mathcal{K}(\zeta(t), \zeta(t), \zeta(t))$$
$$\leq C_a C_\nu \big(1 - K_\nu^{-1}\zeta(t)\big)^{-1}. \quad (102)$$

Multiplying (95) by $t^\sigma$, summing the result over the index $\sigma$, and using (102), we obtain (91) with

$$C_n = C_a^2 C_\nu C_\partial M_0.$$

We now deal with estimate (92). Using the algebra property (81), the differentiation property (26), and the superlinearity property (27), we obtain from (68),

$$\frac{\|f_\sigma\|_{H^{s-2}}}{\varrho^\sigma M_\sigma} \leq \frac{\varrho}{C_\mathrm{D}} \frac{(\sigma+1)\|\xi_{\sigma+1}\|_{H^{s-2}}}{(\varrho/C_\mathrm{D})^{\sigma+1} M_{\sigma+1}} + \frac{C_a}{\varrho^\sigma M_\sigma} \|\mathfrak{g}_\sigma\|_{H^{s-1}} \|a_0\|_{H^s} + C_a M_0 \sum_{\substack{\sigma_1+\sigma_2=\sigma \\ \sigma_1,\, \sigma_2>0}} \frac{\|\mathfrak{g}_{\sigma_1}\|_{H^{s-1}}}{\varrho^{\sigma_1} M_{\sigma_1}} \frac{\|\Psi_{\sigma_2}\|_{H^s}}{\varrho^{\sigma_2} M_{\sigma_2}}. \quad (103)$$

We then have to control $\|\mathfrak{g}_\sigma\|_{H^{s-1}}$ in (103). Using the algebra property (81), and the superlinearity property (27), we obtain from (71),

$$\frac{\|(\mathfrak{g}_\sigma)_{ij}\|_{H^{s-1}}}{\varrho^\sigma M_\sigma} \lesssim \frac{\|\xi_\sigma\|_{H^s}}{\varrho^\sigma M_\sigma} + C_a M_0 \sum_{\substack{\sigma_1+\sigma_2=\sigma \\ \sigma_1,\, \sigma_2>0}} \frac{\|\xi_{\sigma_1}\|_{H^s}}{\varrho^{\sigma_1} M_{\sigma_1}} \frac{\|\xi_{\sigma_2}\|_{H^s}}{\varrho^{\sigma_2} M_{\sigma_2}}$$
$$+ C_a^2 M_0^2 \sum_{\substack{\sigma_1+\sigma_2+\sigma_3=\sigma \\ \sigma_1,\, \sigma_2,\, \sigma_3>0}} \frac{\|\xi_{\sigma_1}\|_{H^s}}{\varrho^{\sigma_3} M_{\sigma_3}} \frac{\|\xi_{\sigma_2}\|_{H^s}}{\varrho^{\sigma_2} M_{\sigma_2}} \frac{\|\xi_{\sigma_3}\|_{H^s}}{\varrho^{\sigma_3} M_{\sigma_3}}$$
$$+ C_a^3 M_0^3 \sum_{\substack{\sigma_1+\sigma_2+\sigma_3+\sigma_4=\sigma \\ \sigma_1,\, \sigma_2,\, \sigma_3,\, \sigma_4>0}} \frac{\|\xi_{\sigma_1}\|_{H^s}}{\varrho^{\sigma_3} M_{\sigma_3}} \frac{\|\xi_{\sigma_2}\|_{H^s}}{\varrho^{\sigma_2} M_{\sigma_2}} \frac{\|\xi_{\sigma_3}\|_{H^s}}{\varrho^{\sigma_3} M_{\sigma_3}} \frac{\|\xi_{\sigma_4}\|_{H^s}}{\varrho^{\sigma_4} M_{\sigma_4}}.$$

Multiplying the above expression by $t^\sigma$ and summing the result over the index $\sigma$, we obtain

$$\sum_{\sigma>0} \frac{\|(\mathfrak{g}_\sigma)_{ij}\|_{H^{s-1}}}{\varrho^\sigma M_\sigma} t^\sigma \lesssim \zeta(t)\big(1 + C_a M_0 \zeta(t) + (C_a M_0 \zeta(t))^2 + (C_a M_0 \zeta(t))^3\big). \quad (104)$$

Multiplying (103) by $t^\sigma$, summing the result over the index $\sigma$, and using (104), we obtain (92) with

$$C_f \simeq \max\big\{C_a\|a_0\|_{H^s},\, C_a M_0(1 + C_a\|a_0\|_{H^s}),\, C_a^2 M_0^2(1 + C_a\|a_0\|_{H^s}),\, C_a^3 M_0^3(1 + C_a\|a_0\|_{H^s}),\, C_a^4 M_0^4\big\}.$$

We now deal with estimate (93). Using the algebra property (81), the superlinearity property (27), and the continuous surjection of the trace operator (82), we obtain from (70),

$$\frac{\|g_\sigma\|_{H^{s-1/2}(\partial\Omega)}}{\varrho^\sigma M_\sigma} \leq C_a C_\partial \|a_0\|_{H^s} \frac{\|\nu_\sigma\|_{H^{s-1/2}(\partial\Omega)}}{\varrho^\sigma M_\sigma} + C_a C_\partial M_0 \sum_{\substack{\sigma_1+\sigma_2=\sigma \\ \sigma_1,\, \sigma_2>0}} \frac{\|\Psi_{\sigma_1}\|_{H^s(\Omega)}}{\varrho^{\sigma_1} M_{\sigma_1}} \frac{\|\nu_{\sigma_2}\|_{H^{s-1/2}(\partial\Omega)}}{\varrho^{\sigma_2} M_{\sigma_2}}. \quad (105)$$

Multiplying (105) by $t^\sigma$, summing the result over the index $\sigma$, and using (102), we obtain (93) with

$$C_g = C_a^2 C_\nu C_\partial \max\{M_0,\, \|a_0\|_{H^s}\}.$$

Finally we deal with estimate (94). Using the algebra property (81), the superlinearity property (27), and the continuous surjection of the trace operator (82), we obtain from (69),

$$\begin{aligned}\frac{\|h_\sigma\|_{H^{s-3/2}(\partial\Omega)}}{\varrho^\sigma M_\sigma} &\leq C_\partial \frac{\|h_\sigma\|_{H^{s-1}(\Omega)}}{\varrho^\sigma M_\sigma} \\ &\lesssim C_\partial C_a \|a_0\|_{H^s} \frac{\|\xi_\sigma\|_{H^s}}{\varrho^\sigma M_\sigma} \\ &\quad + C_\partial \sum_{\substack{\sigma_1+\sigma_2=\sigma \\ \sigma_1,\,\sigma_2>0}} \left\{ C_a M_0 \frac{\|\xi_{\sigma_1}\|_{H^s}}{\varrho^{\sigma_1} M_{\sigma_1}} \frac{\|\Psi_{\sigma_2}\|_{H^s}}{\varrho^{\sigma_2} M_{\sigma_2}} + C_a^2 M_0 \|a_0\|_{H^s} \frac{\|\xi_{\sigma_1}\|_{H^s}}{\varrho^{\sigma_1} M_{\sigma_1}} \frac{\|\xi_{\sigma_2}\|_{H^s}}{\varrho^{\sigma_2} M_{\sigma_2}} \right\} \\ &\quad + C_\partial C_a^2 M_0^2 \sum_{\substack{\sigma_1+\sigma_2+\sigma_3=\sigma \\ \sigma_1,\,\sigma_2,\,\sigma_3>0}} \frac{\|\xi_{\sigma_1}\|_{H^s}}{\varrho^{\sigma_1} M_{\sigma_1}} \frac{\|\xi_{\sigma_2}\|_{H^s}}{\varrho^{\sigma_2} M_{\sigma_2}} \frac{\|\Psi_{\sigma_3}\|_{H^s}}{\varrho^{\sigma_3} M_{\sigma_3}}. \end{aligned} \tag{106}$$

Multiplying (106) by $t^\sigma$ and summing the result over the index $\sigma$, we obtain (94) with

$$C_h \simeq C_a C_\partial \max\left\{\|a_0\|_{H^s},\, M_0(1 + C_a\|a_0\|_{H^s}),\, C_a M_0^2\right\}.$$

Therefore the proof of Proposition 4 is complete. □

We now complete the proof of Theorem 2. Combining (87) and estimates of Proposition 4 we obtain the following differential inequality,

$$\begin{aligned}\zeta(t) \leq C_{123}\Big\{ &\|u_0\|_{H^s} \varrho^{-1} M_1^{-1} t + \dot{\tilde{\zeta}}(t) + C_{drfh}(1+t)\zeta(t)\big(1+\zeta(t)\big) \\ &+ C_{dfh}\zeta^3(t) + C_f\zeta^4(t) + C_f\zeta^5(t) + C_{ng}\big(1+\zeta(t)\big)\big(1 - K_\nu^{-1}\zeta(t)\big)^{-1}\Big\},\end{aligned} \tag{107}$$

where

$$C_{drfh} = C_d + C_r + C_f + C_h, \quad C_{dfh} = C_d + C_f + C_h, \quad \text{and} \quad C_{ng} = C_n + C_g.$$

Setting

$$\begin{aligned}\lambda(t) &:= \|u_0\|_{H^s} \varrho^{-1} M_1^{-1} t, \\ Q(t) &:= \lambda(t) - C_{123}^{-1}\zeta(t) + C_{drfh}(1+t)\zeta(t)\big(1+\zeta(t)\big) + C_{dfh}\zeta^3(t) + C_f\zeta^4(t) + C_f\zeta^5(t), \\ Z(t) &:= Q(t) + C_{ng}\big(1+\zeta(t)\big)\big(1 - K_\nu^{-1}\zeta(t)\big)^{-1},\end{aligned}$$

inequality (107) can be recast as

$$-\dot{\tilde{\zeta}}(t) \leq Z(t),$$

which gives, after time integration, the following final inequality

$$\tilde{\zeta}(t) + \int_0^t Z(\tau)d\tau \geq 0. \tag{108}$$

A sufficient condition for inequality (108) to hold is to have both

$$Q(t) \geq 0, \quad \text{and} \quad \zeta(t) \leq K_\nu. \tag{109}$$

Following [13,11], we can show that there exists a time $T > 0$, with

$$T = T(\|u_0\|_{H^s},\, \|a_0\|_{H^s},\, M_0,\, M_1,\, C_a,\, C_\partial,\, C_\nu,\, K_\nu,\, \varrho)$$

such that for all $t \in ]0, T[$, the sufficient condition (109) is satisfied, which ends the proof Theorem 2.

## Appendix A. Regularity estimates for some non-homogeneous elliptic boundary value problems

The literature concerning non-homogeneous elliptic boundary value problems is so huge that we can not cite all of it. We only cite a few, which are relevant for our problem, such as [2,3,87,88,79,85,50,45,101,44,4,71,5]. The following theorems are extracted from [79,16,45,101,71].

**Theorem 3.** *Let $\Omega$ be a bounded and simply-connected domain of $\mathbb{R}^3$ with $\mathscr{C}^\infty$ boundary. Let $\nu$ be the outward pointing unit normal to the boundary $\partial\Omega$, whose the elementary measure is denoted by $d\Gamma$. Let $p$ and $s$ be respectively an integer and a real such that $1 < p < \infty$ and $s \geq 0$. Let $f : \Omega \mapsto \mathbb{R}$, and $g : \partial\Omega \mapsto \mathbb{R}$ be such that $f \in W^{s,p}(\Omega)$ and $g \in W^{s+1-1/p,p}(\partial\Omega)$. We consider the following non-homogeneous boundary value problem,*

$$\begin{cases} \Delta\varphi = f & \text{on } \Omega, \\ \partial_\nu\varphi = g & \text{on } \partial\Omega. \end{cases} \tag{110}$$

*The non-homogeneous boundary value problem (110) is solvable if and only if the data obey the integrability or solvability condition,*

$$\int_\Omega f d\Omega = \int_{\partial\Omega} g d\Gamma. \tag{111}$$

*The non-homogeneous boundary value problem (110) has a unique solution up to a constant. Moreover there exists a constant $C = C(s, p, \Omega)$ such that this solution satisfies the following regularity estimate*

$$\|\varphi\|_{W^{s+2,p}(\Omega)} \leq C \left( \|f\|_{W^{s,p}(\Omega)} \; + \; \|g\|_{W^{s+1-1/p,p}(\partial\Omega)} \right). \tag{112}$$

**Proof.** A proof of Theorem 3 can be found for example in [45] (Theorem 1.10) or in [16] (Lemma 1, see also [79]). □

**Theorem 4.** *Let $\Omega$ be a bounded and simply-connected domain of $\mathbb{R}^3$ with $\mathscr{C}^\infty$ boundary. Let $\nu$ be the outward pointing unit normal to the boundary $\partial\Omega$. Let $p$ and $s$ be respectively an integer and a real such that $1 < p < \infty$ and $s \geq 0$. Let $f : \Omega \mapsto \mathbb{R}^3$, $g : \partial\Omega \mapsto \mathbb{R}^3$, and $h : \partial\Omega \mapsto \mathbb{R}$ be such that $f \in W^{s,p}(\Omega)$, $g \in W^{s+2-1/p,p}(\partial\Omega)$, and $h \in W^{s+1-1/p,p}(\partial\Omega)$. We consider the following non-homogeneous boundary value problem,*

$$\begin{cases} \Delta\Phi = f & \text{on } \Omega, \\ \nabla \cdot \Phi = h & \text{on } \partial\Omega, \\ \Phi \times \nu = g & \text{on } \partial\Omega. \end{cases} \tag{113}$$

*The non-homogeneous boundary value problem (113) has a unique solution. Moreover there exists a constant $C = C(s, p, \Omega)$ such that this solution satisfies the following regularity estimate*

$$\|\Phi\|_{W^{s+2,p}(\Omega)} \leq C\left(\|f\|_{W^{s,p}(\Omega)} + \|g\|_{W^{s+2-1/p,p}(\partial\Omega)} + \|h\|_{W^{s+1-1/p,p}(\partial\Omega)}\right). \tag{114}$$

**Proof.** A proof of Theorem 4 can be found for example in [101] (Corollary 3.4.8 and Lemma 3.4.7) or in [71] (Lemma 4.4). □

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