Transient growth in MRI compressible in a magnetized accretion disk

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Outline

-General problem of matter infall on central star in astrophysical accretion disks

- +MRI short tutorial

-Specificity of shear flows L and NL (transient growth, non normal modes, …)

NL : Types of nonlinear cascades in spectrally stable shear flows: direct, inverse and ***transverse cascades*** (Origin of the nonlinear transverse cascade and its role in the self-sustenance of subcritical turbulence)

Realization of a transverse cascade on a specific example

see previous studies, GC +GM et al… for examples like :

– 2D HD and 2D MHD spectrally stable shear flow

– 3D MHD Keplerian flows with net azimuthal B field

HERE work in progress : 3D MHD Keplerian and compressible flows with vertical B field

- **Our Specific Results**:
- -Linear modes and mode coupling : possible generation of spiral density waves from magnetic modes (or converse)
- -NL some preliminary results in the analysis of computational NL data in the compressible case

-Summary and conclusions

Matter infall problem

Disk in rotation : centrifugal force +gravitational force +pressure gradient…

Differential rotation : rotation rate Ω +shear (locally plane) S <
Differential rotation :

Accretion rate : since the angular momentum L is conserved the matter should not fall ; falling yield losses of gravitational energy (+radiation then involved…) and of L inward : need for a dissipation mechanism to transport (and redistribute) partly L outward to enable radiation. The usual visocity is not enough : need to invoke an enhanced viscosity such as driven by turbulence…

***MRI (magneto-rotational instability) is a candidate to trigger turbulence in the case of magnetized disks**

***Here we "advocate" more the transient growth mecanism (TG) as a by-pass transition to turbulence than can applied also in dead zonse of disks (no B field) , and for spectrally otherwise stable flows.**

-TG is specific/typical for shear flows

There are various kind of disks (3 main kinds) with various possible instabilities for them (such as barocline instabilites , shear instability and so on …)

(very) short tutorial for MRI

BBBB field B field acts as magnetic spring
see on blackboard…

-Case of differential rotation $\Omega(r)$: rotation +shear S
epicyclic frequency $\varsigma^2=4\Omega^2+r\, d\Omega^2/dr$
Keplerian case : $\Omega(r)$ in $1/r^q$, $q=3/2$, $S=-q\Omega$, $2\Omega/S=-4/3$
$\varsigma^2=2\Omega_0(2-q)\Omega_0$, $\Omega_0=\Omega(r_0)$

-Instability for weak enough kB: $(kB)^2/\mu_0\rho<(-rd\Omega^2/dr)$
$(k_z v_{Az})^2<(2-q)\ \Omega^2$,$v_A{}^2=B_0{}^2/\rho\mu_0$ Alfven velocity

stabilization for larger (kB)
-Growth rate $\gamma=$…
Maximum growth rate $\gamma=S/2(=3/4\Omega)$ for kB=4/9…

-Rayleigh stability criterium ; extension if $B\neq0$

Specificity of shear flows (1)

list items
Non modal analysis :
Sheared waves decomposition k(t) , algebraic growth/exponential ones,
curve of time evolution (TG +saturated turbulent case)
+see on blackboard some illustrations

Specificity of shear flows (2)

(cf decomposition vortex-mode in HD case)
Transverse cascades (in angles in k space) : see next slides

The concept of transverse cascade and its basic scheme of activity first elaborated in

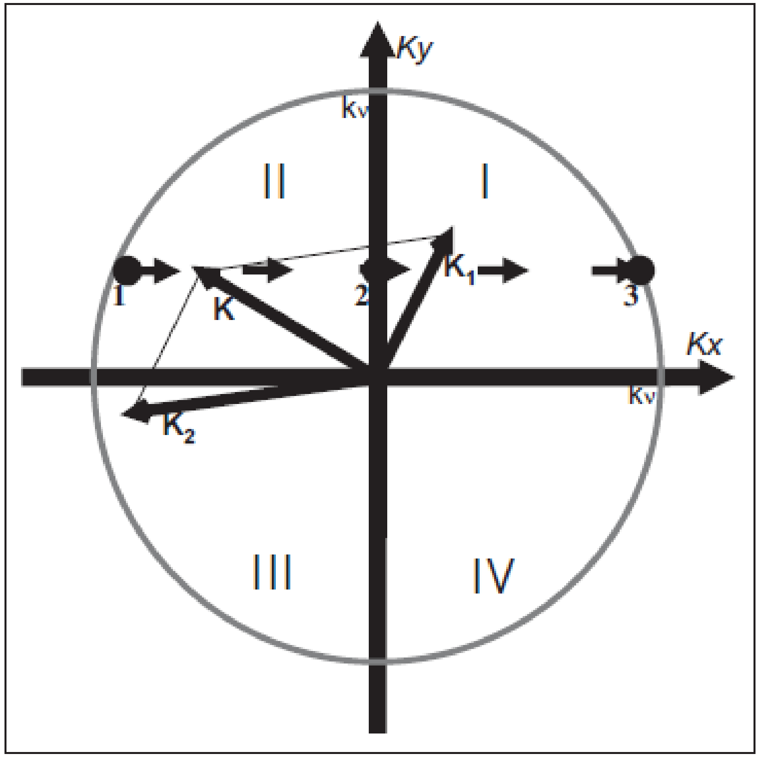
A&A 402, 401–407 (2003)
DOI: 10.1051/0004-6361:20030269

Astronomy & Astrophysics

On hydrodynamic shear turbulence in Keplerian disks: Via transient growth to bypass transition

G. D. Chagelishvili[1], J.-P. Zahn[2], A. G. Tevzadze[1], and J. G. Lominadze[1]

$k_x(t) = k_x(0) - Ak_y t, \qquad A \equiv (r\partial_r\Omega)|_{r=r_0}$ is the shear parameter



Nonlinear interaction between harmonics with wavenumbers $\mathbf{k}_1$ and $\mathbf{k}_2$ generates a new harmonic with

$$\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2$$

A simple sketch of: nonlinear transverse cascade and a scenario of the self-sustenance of (subcritical) turbulence in spectrally stable shear flows

Examples of previous results :

2D HD plane shear flow

Horton W., Kim J-H., Chagelishvili G.D., Bowman J. and Lominadze J., *Angular redistribution of nonlinear perturbations: a universal feature of nonuniform flows*, Phys. Rev. E., 2010, **81**, 066304

2D MHD plane shear flow

Mamatsashvili G., Gogichaishvili D. Chagelishvili G. and Horton W., *Nonlinear transverse cascade and 2D MHD subcritical turbulence in plane shear flows*, Phys. Rev. E., 2014, **89**, 043101

3D HD plane shear flow

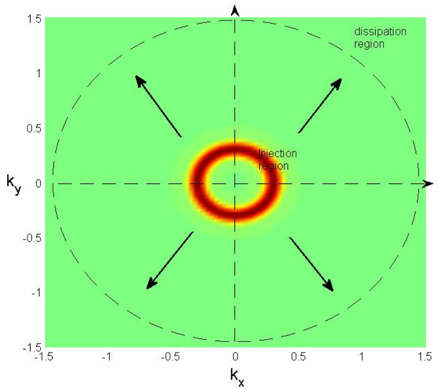
Mamatsashvili G., Khujadze G., Chagelishvili G., Dong S., Jimenez J., and Foysi H., *Dynamics of homogeneous shear turbulence – a key role of the nonlinear transverse cascade in the bypass concept,* Phys. Rev. E., 2016, 94, 023111

3D MHD Keplerian flow

Gogichaishvili D., Mamatsashvili G., Chagelishvili G. and Horton W., 2016, *Nonlinear* transverse cascade *and subcritical MHD turbulence in disks with azimuthal magnetic field, 2016,* in preparation….

Types of nonlinear cascades in Fourier k-space in spectrally stable shear flows

- Usually, linear processes (dynamics) in many flow systems depend on a certain combination of wavenumbers (e.g., $k^2 = k_x^2 + k_y^2 + k_z^2$ in isotropic case).
- As a result, nonlinear processes often appear to be also dependent on a similar combination of wavenumbers and change the latter, leading to ***direct*** or/and ***inverse*** cascades



- **Classical examples:** Kolmogorov's isotropic uniform HD turbulence or Iroshnikov-Kraichnan theory of MHD turbulence, where nonlinear cascades change only wavenumber magnitude k of harmonics

- **Non-normal nature of linear dynamics of spectrally/modally stable shear flows and its consequences were well-understood and extensively analyzed by the HD community in the 1990s (e.g., *Reddy et al. 93, Trefethen et al. 93, Reddy & Henningson 93, Schmid & Henningson 01, Schmid 07*)**

Implications of non-normality

- Eigenfunctions of linearized equations in modal analysis of shear flows are non-orthogonal and strongly interfere

- Due to the non-normality perturbations display important and diverse finite-time (***transient***) phenomena, which are typically missed out in the classical modal/spectral analysis

- Non-normality induced phenomena can be grasped by appealing to a more general ***non-modal analysis***

- **In spectrally stable shear flows, the subtle interplay of linear transient, or nonmodal amplification phenomena and nonliner feedback determines subcritical transition and maintenance of turbulence**

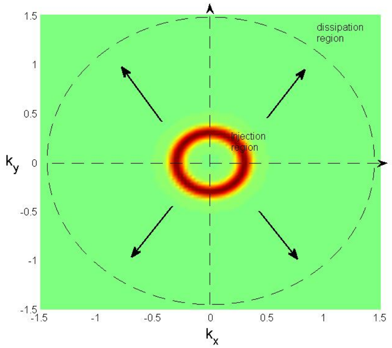
- Generally, linear nonmodal processes are ***anisotropic*** in wavenumber/Fourier space due to flow shear. (In the case of specific flows considered below, it depends on the combination $k_x k_y / k^2$)

- **This strong anisotropy of linear processes in shear flows, in turn, leads to anisotropy of nonlinear processes (cascades) in k-space**

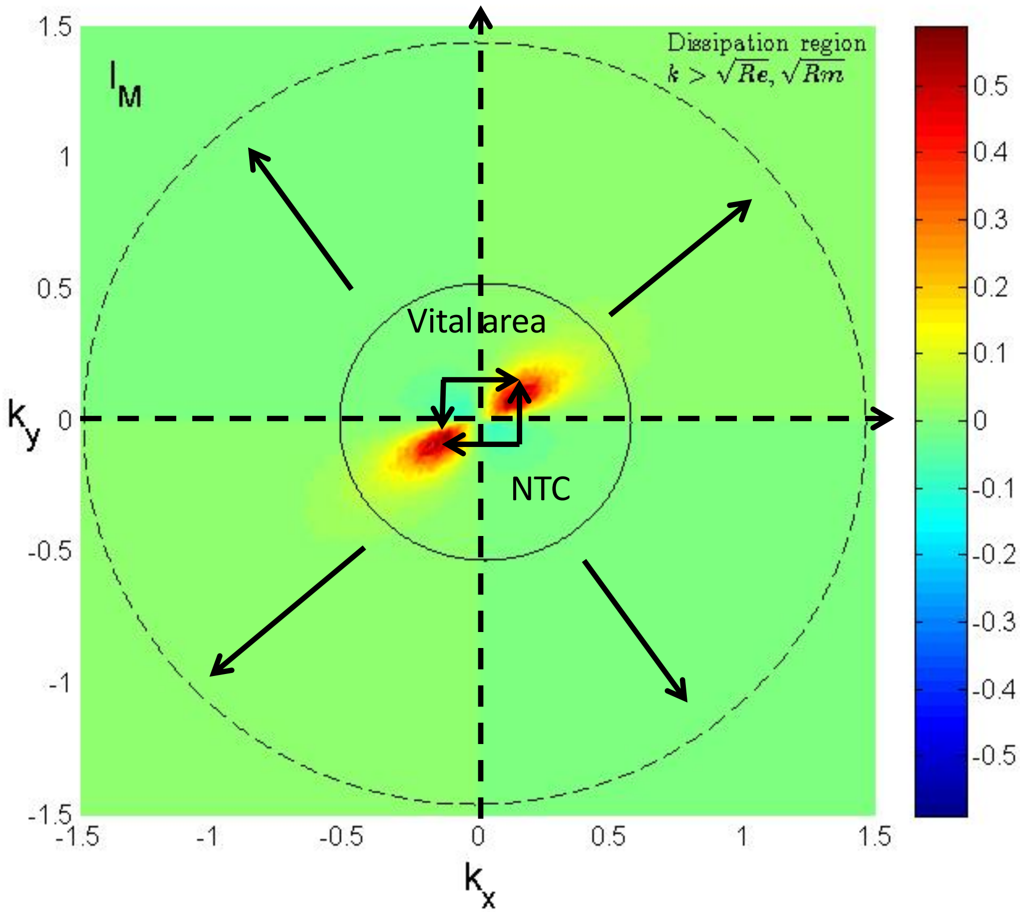
  (e.g., Chagelishvili et al 2002, Horton et al. 2010, Mamatsashvili et al. 2014)

- We refer to this shear-induced anisotropic redistribution of Fourier harmonics over wavevector angles as a ***nonlinear transverse cascade (NTC)*** – a new type of nonlinear cascade emerging in shear flows

Sketch of energy injection area (red and blue) and nonlinear (direct) cascades (long arrows) in the presence of shear



Classical case without shear flow. Forcing is at narrow wavenumber range



Showcase 1 – essence of NTC and its role in self-sustenance of perturbations (turbulence) in some real flows

2D incompressible HD plane Couette flow (*Horton et al. 2010, Phys. Rev. E*)

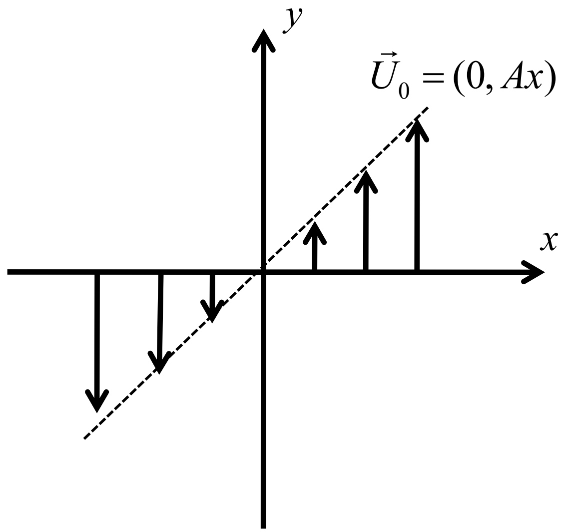
$$\mathbf{U_0}(x) = (0, Ax)$$

$$\mathbf{U}(x,y) = \mathbf{U_0}(x) + \mathbf{v}$$

$$v_x = -\frac{\partial \psi}{\partial y}; \quad v_y = \frac{\partial \psi}{\partial x}$$

$$\left[\frac{\partial}{\partial t} + U_0(x)\frac{\partial}{\partial y}\right]\Delta\psi + J(\psi, \Delta\psi) - \nu\Delta^2\psi = 0$$

$$e(x,y,t) = \frac{1}{2}\rho\left[\left(\frac{\partial \psi}{\partial y}\right)^2 + \left(\frac{\partial \psi}{\partial x}\right)^2\right]$$



$\vec{U}$ - is the velocity, ν - is the kinematic viscosity, J, Δ -2D Jacobian and Laplacian

ψ - is the stream function, e - is the kinetic energy density

Fourier transform of the stream function: $\psi(x,y,t)=\int \psi_{\mathbf{k}}(t)\exp(ik_x x+ik_y y)d^2\mathbf{k}$

Spectral density of energy: $E(\mathbf{k},t)=k^2\,|\psi(\mathbf{k},t)|^2\,/2$

Equation for spectral energy density (in non-dimensional form):

$$\left(\frac{\partial}{\partial t}-Ak_y\frac{\partial}{\partial k_x}\right)E_{\mathbf{k}}=2A\frac{k_x k_y}{k^2}E_{\mathbf{k}}-\frac{2}{\mathrm{Re}}k^2E_{\mathbf{k}}+N_{\mathbf{k}}$$

where Re is the Reynolds number

$2A\frac{k_x k_y}{k^2}$ is of linear origin and describes energy exchange between the flow and perturbation harmonics

Nonlinear term $N_{\mathbf{k}}=\frac{1}{2}\int(k'_x k''_y-k'_y k''_x)k'^2(\psi_{\mathbf{k}}{}^*\psi_{\mathbf{k}'}\psi_{\mathbf{k}''}+\psi_{\mathbf{k}}\psi_{\mathbf{k}'}{}^*\psi_{\mathbf{k}''}{}^*)\delta(\mathbf{k}-\mathbf{k}'-\mathbf{k}'')d\mathbf{k}'d\mathbf{k}''$ describes energy redistribution by nonlinearity in the **k**-plane, while leaving the total energy unchanged, $\int N_{\mathbf{k}}d^2\mathbf{k}=0$

Direct numerical simulations of the evolution of localized (vortical) perturbations and dynamical picture in k-plane

The DNS are done using the dealiased Fourier pseudo-spectral code

2D domain size: $25l \times 25l$, resolution 256×256 , minimal and maximal wavenumbers in the domain $k_{\min} = 2\pi / 25l = 0.25 l^{-1}, k_{\max} = 85 l^{-1}$

Initial form of the localized vortical perturbation

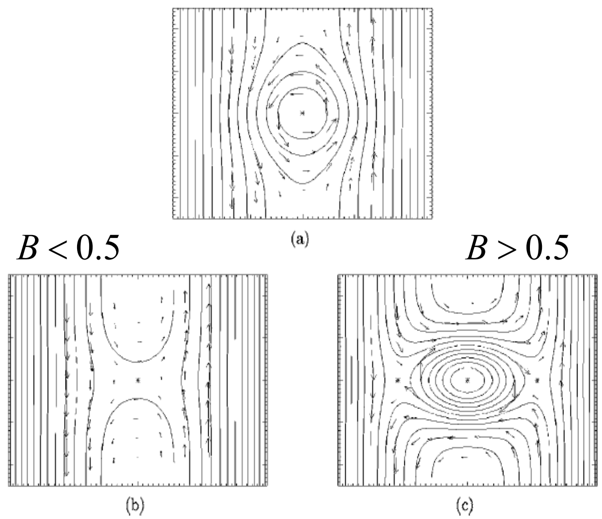
$$\psi(x, y, 0) = Bn(x, y)\exp(-x^2 - y^2)$$

n=-1 - coherent cyclonic vortex (a)

n=1 - coherent anticyclonic vortex (b,c)

B is the amplitude, we set B=3

Re=1000



Evolution of total energy of cyclonic and anticyclonic vortices

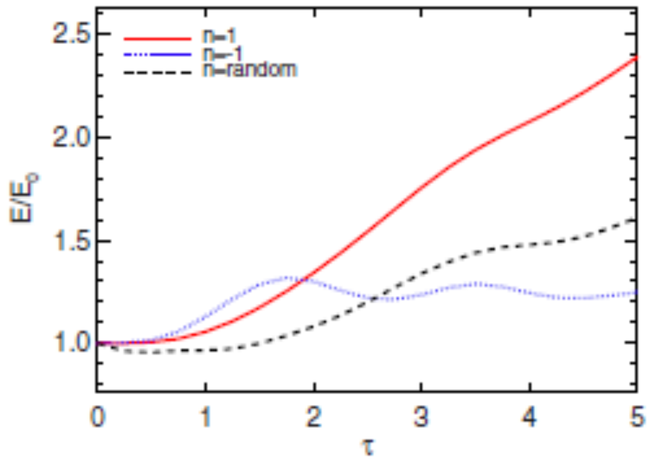
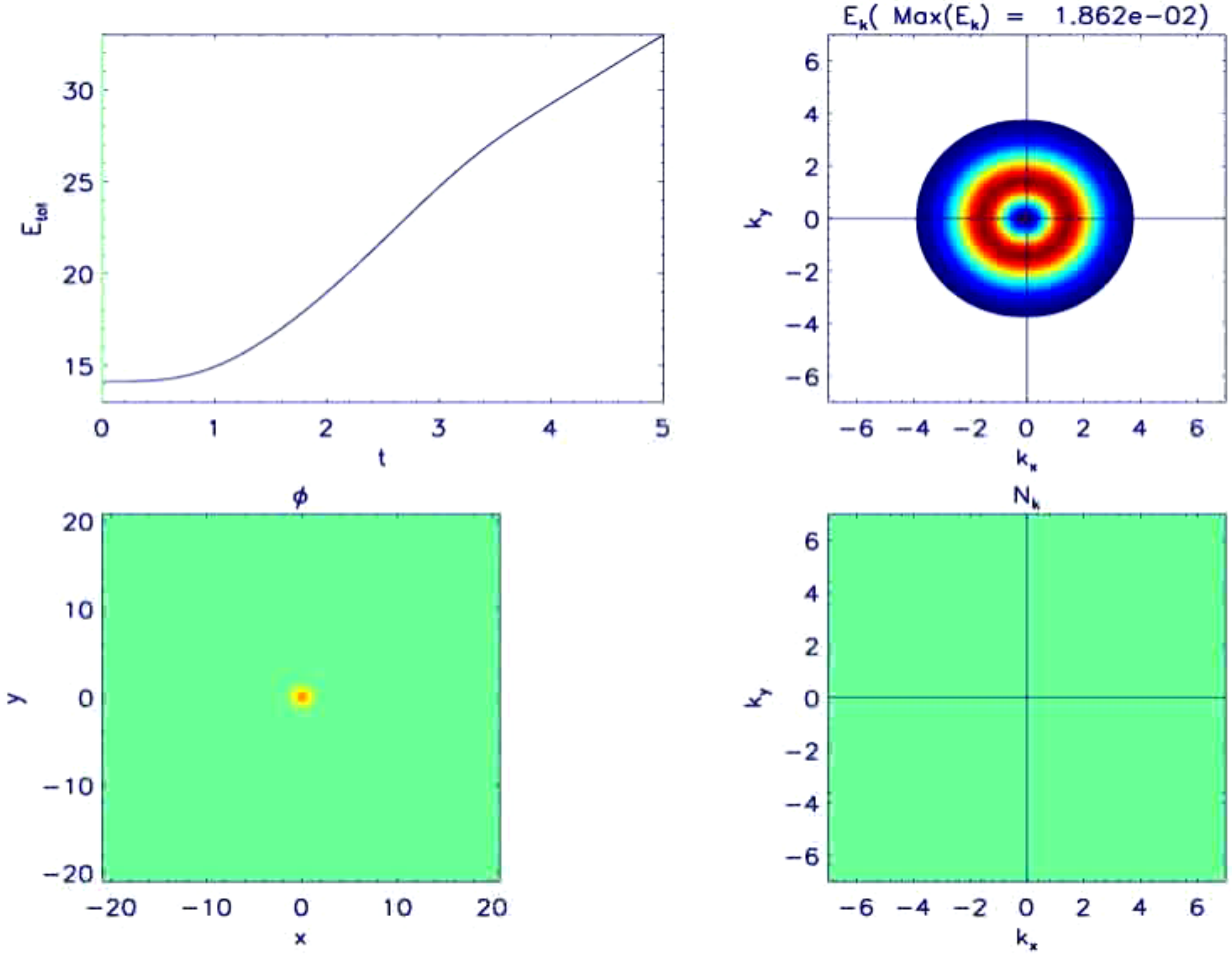
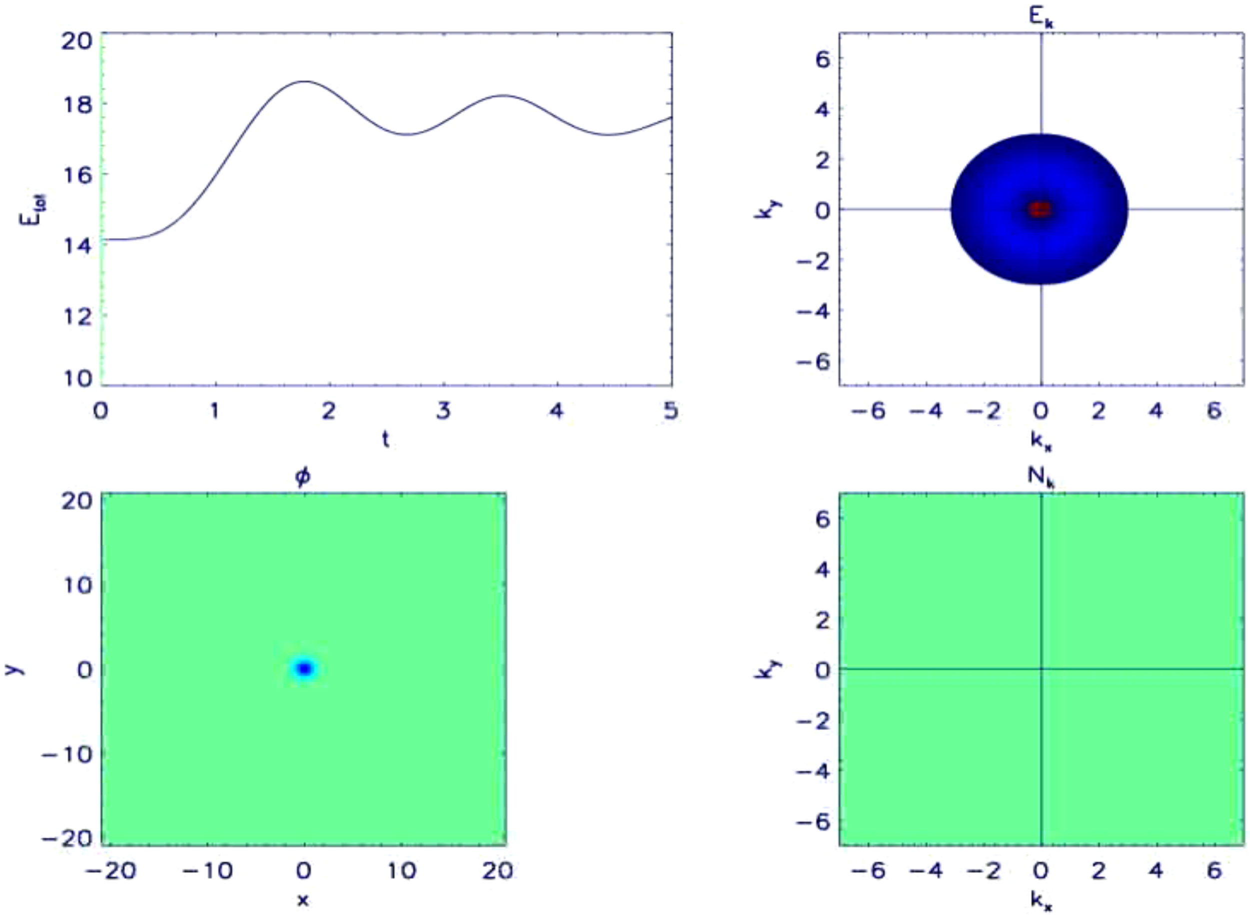


FIG. 1. (Color online) Normalized total energy vs time for anticyclonic (solid red), cyclonic (dotted blue) and stochastic (dashed black) perturbations for $R_e=1000$ and $B=3$

Anticyclonic vortex



Cyclonic vortex



See Showcase 2 and 3…

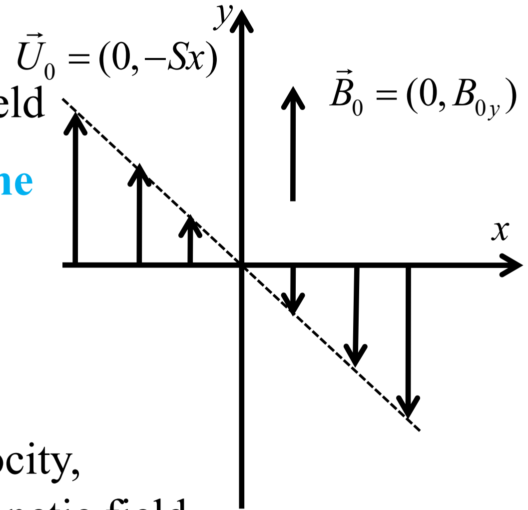
2D incompressible MHD constant shear flow (*Mamatsashvili et al. 2014)*

Equilibrium: flow with constant shear S

threaded by parallel (azimuthal) uniform magnetic field

This equilibrium is spectrally stable in linear regime

Equations of non-ideal MHD



$$\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U} \cdot \nabla)\mathbf{U} = -\frac{\nabla P}{\rho} + \frac{(\mathbf{B} \cdot \nabla)\mathbf{B}}{4\pi\rho} + \nu \nabla^2 \mathbf{U},$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{U} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B},$$

$$\nabla \cdot \mathbf{U} = 0,$$

$$\nabla \cdot \mathbf{B} = 0,$$

$\mathbf{U}$ - velocity,
$\mathbf{B}$ - magnetic field
P - total (thermal +magnetic) pressure
ρ -density
ν -viscosity
η -resistivity

Nonlinear evolution – self-sustained 2D MHD turbulence

We solve nonlinear 2D (*x,y*-plane) MHD equations for perturbed velocity, ***u*** and magnetic field ***b*** in a rectangular domain with sizes $L_x \times L_y$ divided into $N_x \times N_y$ cells using a spectral code **SNOOPY** (by Dr. G. Lesur, http://ipag.obs.ujf-grenoble.fr/~lesurg/snoopy.html, see also Lesur & Longaretti 2007).

The SNOOPY is based on a spectral (Fourier) method (with FFTW),

- Nonlinear terms are computed with pseudo-spectral algorithm
- de-aliasing is done using the "3/2" rule,
- Time integration is done by an explicit 3rd order Runge-Kutta scheme and by implicit exact scheme for dissipative terms

Spectral representation of the main equations

Units and characteristic parameters

$$St \to t, \quad \left(\frac{x}{\ell}, \frac{y}{\ell}\right) \to (x, y), \quad \frac{\mathbf{u}}{u_A} \to \mathbf{u}, \qquad k\frac{u_A}{S} \to k$$

$$\frac{p}{\rho_0 u_A^2} \to p, \quad \frac{\mathbf{b}}{B_{0y}} \to \mathbf{b}, \quad \frac{E_{K,M}}{\rho_0 u_A^2} \to E_{K,M}.$$

where, $u_A = \dfrac{B_{0y}}{\sqrt{4\pi\rho_0}}$ is the Alfven speed and $\ell = \dfrac{u_A}{S}$ is the Alfven length.

Dynamics is characterized by the hydrodynamic Re and magnetic Rm Reynolds numbers in terms of Alfven speed and length

$$\mathrm{Re} = \frac{u_A \ell}{\nu} = \frac{u_A^2}{\nu S}, \quad \mathrm{Rm} = \frac{u_A \ell}{\eta} = \frac{u_A^2}{\eta S}.$$

and by the nondimensional domain size: L_x, L_y

The strength of the imposed magnetic field is measured by the ratio of the mean flow kinetic energy to the magnetic energy

$$\beta = \frac{\pi \rho_0 S^2 L_x^2}{3B_{0y}^2} = \frac{S^2 L_x^2}{12u_A^2} = \frac{L_x^2}{12\ell^2}.$$

Fourier transform of variables: $f(\mathbf{r},t) = \int \bar{f}(\mathbf{k},t)\exp(i\mathbf{k}\cdot\mathbf{r})dk_x dk_y$

Dynamical equations in spectral plane

$$\left(\frac{\partial}{\partial t} + k_y \frac{\partial}{\partial k_x}\right)\bar{u}_x = -\mathrm{i}k_x\bar{p} + \mathrm{i}k_y\bar{b}_x - \frac{k^2}{\mathrm{Re}}\bar{u}_x + \mathrm{i}k_y N_1 + \mathrm{i}k_x N_2,$$

$$\left(\frac{\partial}{\partial t} + k_y \frac{\partial}{\partial k_x}\right)\bar{u}_y = \bar{u}_x - \mathrm{i}k_y\bar{p} + \mathrm{i}k_y\bar{b}_y - \frac{k^2}{\mathrm{Re}}\bar{u}_y + \mathrm{i}k_x N_1 + \mathrm{i}k_y N_3,$$

$$\left(\frac{\partial}{\partial t} + k_y \frac{\partial}{\partial k_x}\right)\bar{b}_x = \mathrm{i}k_y\bar{u}_x - \frac{k^2}{\mathrm{Rm}}\bar{b}_x + \mathrm{i}k_y N_4,$$

$$\left(\frac{\partial}{\partial t} + k_y \frac{\partial}{\partial k_x}\right)\bar{b}_y = -\bar{b}_x + \mathrm{i}k_y\bar{u}_y - \frac{k^2}{\mathrm{Rm}}\bar{b}_y - \mathrm{i}k_x N_4,$$

$$k_x\bar{u}_x + k_y\bar{u}_y = 0, \quad k_x\bar{b}_x + k_y\bar{b}_y = 0,$$

where the terms $N_1(\mathbf{k},t), N_2(\mathbf{k},t)$, $N_3(\mathbf{k},t)$ and $N_4(\mathbf{k},t)$ describe nonlinear transfers of spatial harmonics over wavenumbers

$$N_1(\mathbf{k},t) = \int d^2\mathbf{k}' \left[\bar{b}_x(\mathbf{k}',t)\bar{b}_y(\mathbf{k}-\mathbf{k}',t) - \bar{u}_x(\mathbf{k}',t)\bar{u}_y(\mathbf{k}-\mathbf{k}',t)\right]$$

$$N_2(\mathbf{k},t) = \int d^2\mathbf{k}' \left[\bar{b}_x(\mathbf{k}',t)\bar{b}_x(\mathbf{k}-\mathbf{k}',t) - \bar{u}_x(\mathbf{k}',t)\bar{u}_x(\mathbf{k}-\mathbf{k}',t)\right]$$

$$N_3(\mathbf{k},t) = \int d^2\mathbf{k}' \left[\bar{b}_y(\mathbf{k}',t)\bar{b}_y(\mathbf{k}-\mathbf{k}',t) - \bar{u}_y(\mathbf{k}',t)\bar{u}_y(\mathbf{k}-\mathbf{k}',t)\right]$$

$$N_4(\mathbf{k},t) = \int d^2\mathbf{k}' \left[\bar{u}_x(\mathbf{k}',t)\bar{b}_y(\mathbf{k}-\mathbf{k}',t) - \bar{u}_y(\mathbf{k}',t)\bar{b}_x(\mathbf{k}-\mathbf{k}',t)\right]$$

We derive evolution equations for the kinetic $\bar{E}_M = |\bar{b}_x|^2 + |\bar{b}_y|^2$, and magnetic $\bar{E}_K = |\bar{u}_x|^2 + |\bar{u}_y|^2$ spectral energies

$$\frac{\partial \bar{E}_K}{\partial t} + \frac{\partial}{\partial k_x}(k_y \bar{E}_K) = I_K + I_{K-M} + D_K + N_K,$$

Drift due to flow | Injection | cross terms | Dissipation | nonlinear transfers

$$\frac{\partial \bar{E}_M}{\partial t} + \frac{\partial}{\partial k_x}(k_y \bar{E}_M) = I_M + I_{M-K} + D_M + N_M,$$

Energy injection: $I_K = \bar{u}_x \bar{u}_y^* + \bar{u}_x^* \bar{u}_y = -\frac{2k_x k_y}{k^2}\bar{E}_K, \quad I_M = -\bar{b}_x \bar{b}_y^* - \bar{b}_x^* \bar{b}_y = \frac{2k_x k_y}{k^2}\bar{E}_M,$

Cross terms: $I_{K-M} = \mathrm{i}k_y\left(\bar{u}_x^* \bar{b}_x + \bar{u}_y^* \bar{b}_y - \bar{u}_x \bar{b}_x^* - \bar{u}_y \bar{b}_y^*\right), \quad I_{M-K} = -I_{K-M},$

Dissipation: $D_M = -\frac{2k^2}{\mathrm{Re}_m}\bar{E}_M$, $D_M = -\frac{2k^2}{\mathrm{Re}_m}\bar{E}_M$

Nonlinear transfers $N_K(\mathbf{k}, t) = \mathrm{i}(k_y \bar{u}_x^* + k_x \bar{u}_y^*)N_1(\mathbf{k}, t) + \mathrm{i}k_x \bar{u}_x^*[N_2(\mathbf{k}, t) - N_3(\mathbf{k}, t)] + \text{c.c.}$

$$N_M(\mathbf{k}, t) = \mathrm{i}(k_y \bar{b}_x^* - k_x \bar{b}_y^*)N_4(\mathbf{k}, t) + \text{c.c.}\ .$$

Stresses are related to injection terms

$$\langle u_x u_y \rangle = \frac{1}{2} \int I_K(\mathbf{k}, t) d^2\mathbf{k}, \quad \langle -b_x b_y \rangle = \frac{1}{2} \int I_M(\mathbf{k}, t) d^2\mathbf{k}$$

The net effect of nonlinear transfers is zero,

$$\int [N_K(\mathbf{k}, t) + N_M(\mathbf{k}, t)] d^2\mathbf{k} = 0,$$

Fiducial run

We start simulations by imposing random noise perturbations of velocity and magnetic field with rms amplitude $\langle \mathbf{u}^2 \rangle^{1/2} = \langle \mathbf{b}^2 \rangle^{1/2} = 0.84$

Other parameters are:

Domain size: $L_x = L_y = 400 u_A / S$

Resolution: 512×512

Reynolds numbers as defined above: $\mathrm{Re} = \mathrm{Rm} = 5$

For usually defined Reynolds numbers in terms of domain size L and mean flow velocity U_0, these correspond to $\mathrm{Re}^* = SL^2 / \nu = \mathrm{Rm}^* = SL^2 / \eta = 2 \times 10^5$

Energy evolution

The perturbation kinetic E_K and magnetic E_M energy densities are defined

$$E_K = \frac{\rho_0 \mathbf{u}^2}{2}, \quad E_M = \frac{\mathbf{b}^2}{8\pi}.$$

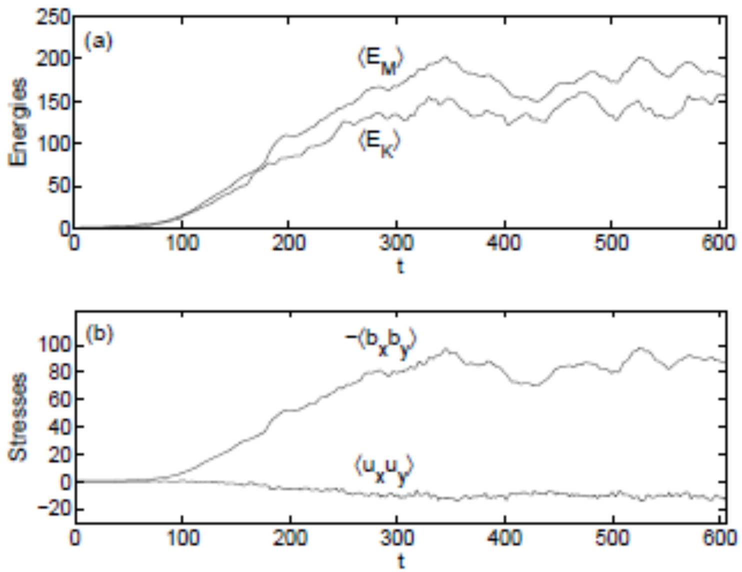
From nonlinear equations for perturbations one can derive evolution equation for the domain averaged total energy $E = E_K + E_M$

$$\frac{d\langle E \rangle}{dt} = S \left\langle \rho_0 u_x u_y - \frac{b_x b_y}{4\pi} \right\rangle - \rho_0 \nu \langle (\nabla u_x)^2 + (\nabla u_y)^2 \rangle - \frac{\eta}{4\pi} \langle (\nabla b_x)^2 + (\nabla b_y)^2 \rangle.$$

Reynolds stress = $\langle \rho_0 u_x u_y \rangle$

Maxwell stress = $-\langle b_x b_y \rangle / 4\pi$

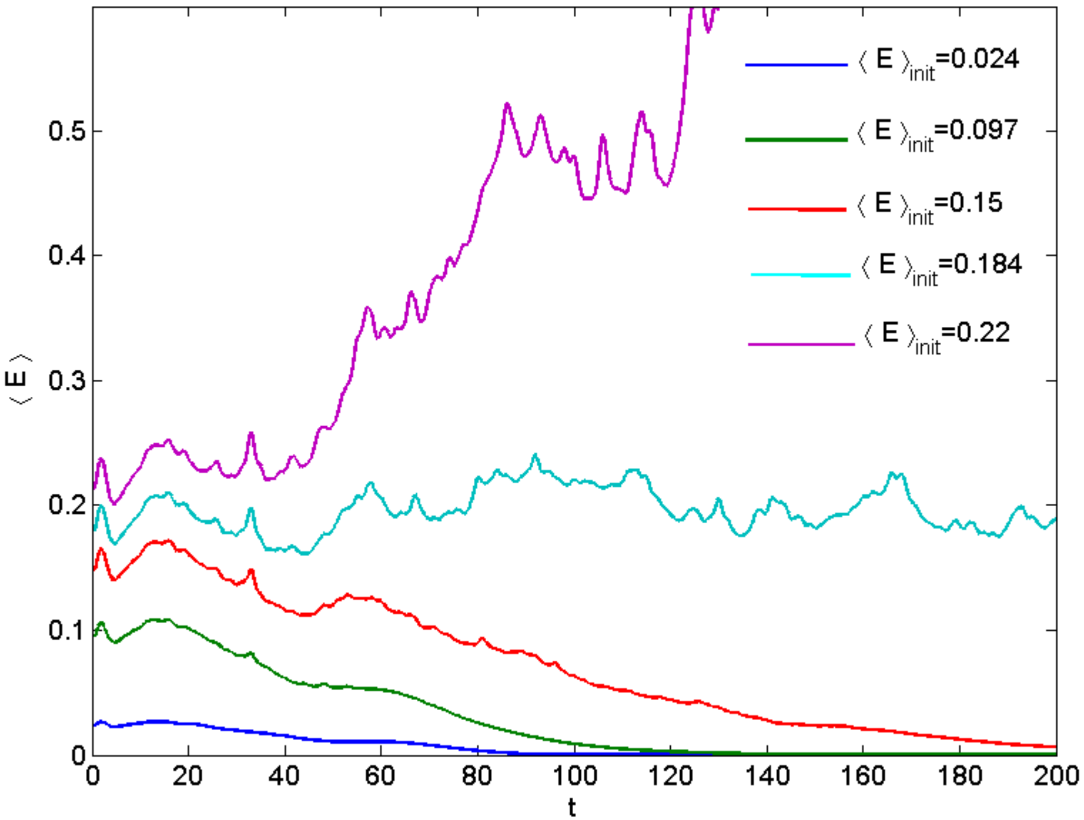
Time evolution of the domain-averaged energies and stresses



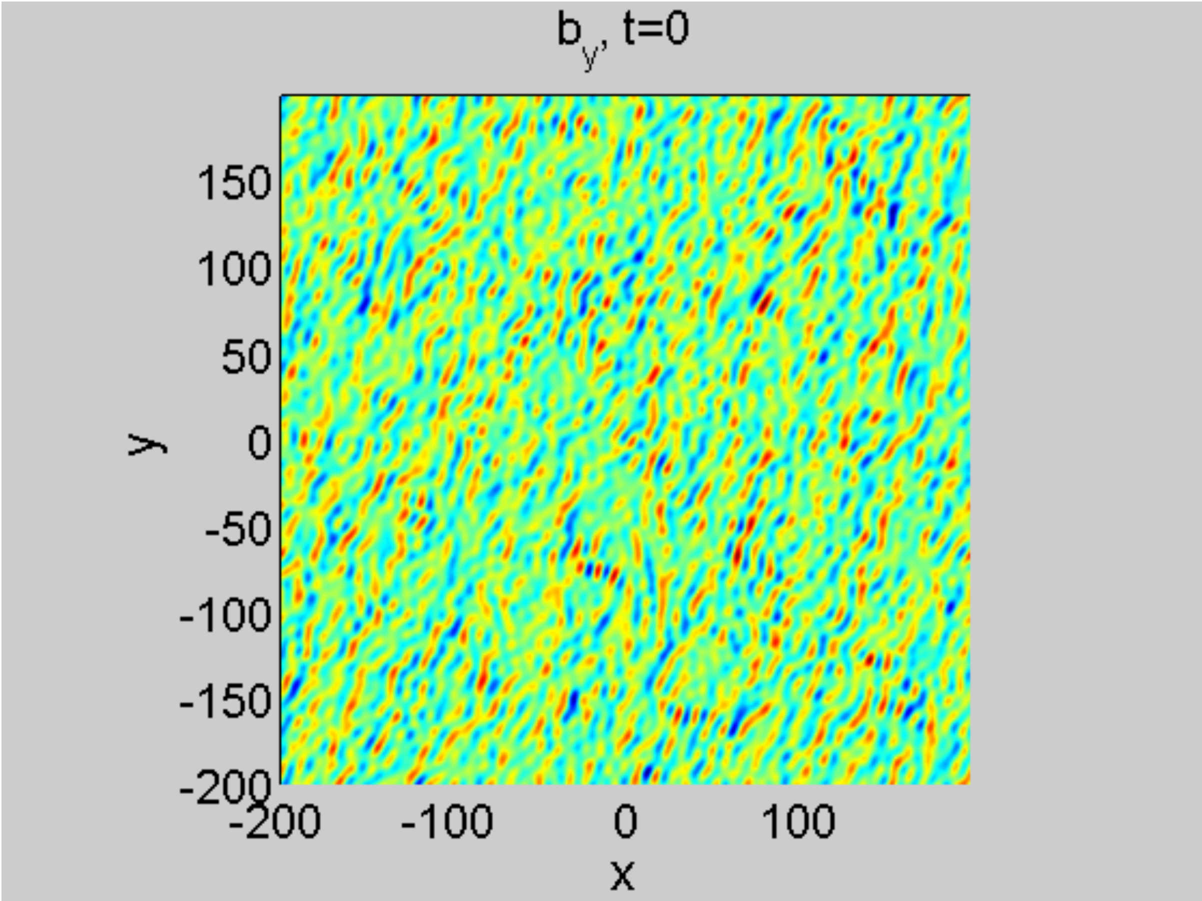
At the beginning, these quantities grow as a result of transient amplification of separate harmonics. Then, at about t=250, the amplification saturates to a quasi-steady turbulent state. **Positive Maxwell stress dominates negative Reynolds one and maintains turbulence. Thus, the self-sustenance occurs owing to magnetic field perturbations**

Subcritical transition

Total energy evolution at different initial amplitudes



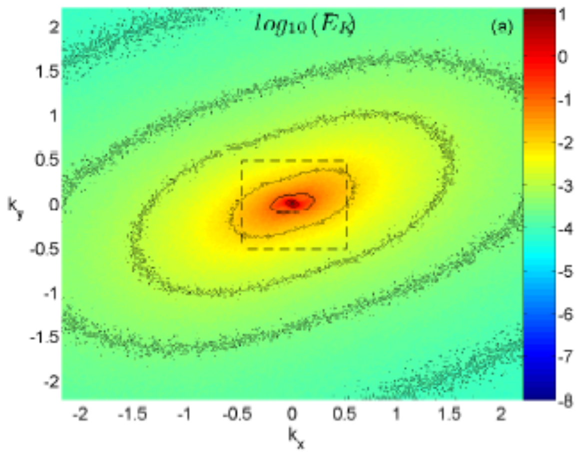
Turbulence's map in the coordinate plane

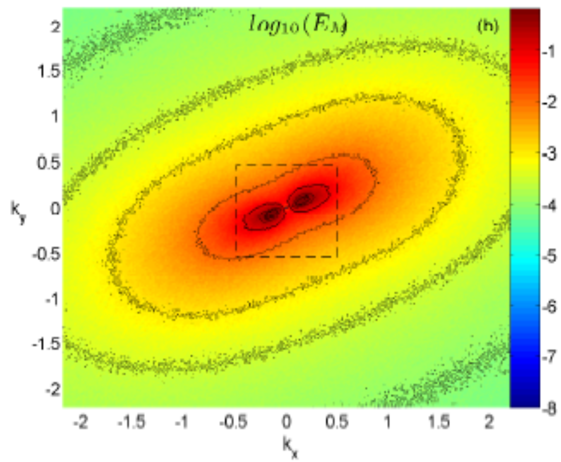


From the simulation data, we calculate both spectra as well as individual energy injection I_K, I_M and nonlinear transfer N_K, N_M terms entering spectral equations above and therefore get valuable/essential information and understanding about the roles of different linear and nonlinear processes (and their interplay) in the self-sustenance of the turbulence in the flow

2D spectra in the turbulent state

kinetic energy **magnetic energy**

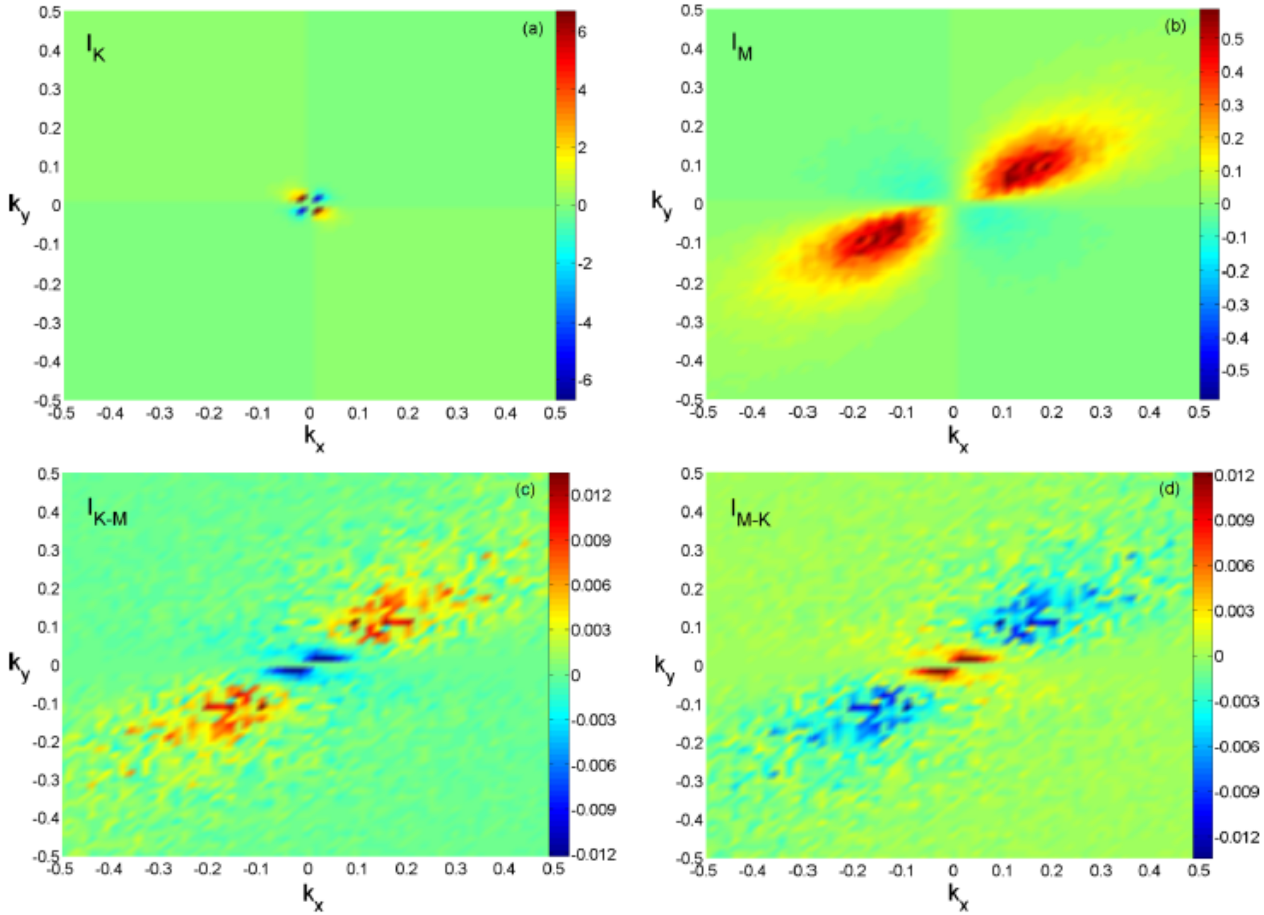




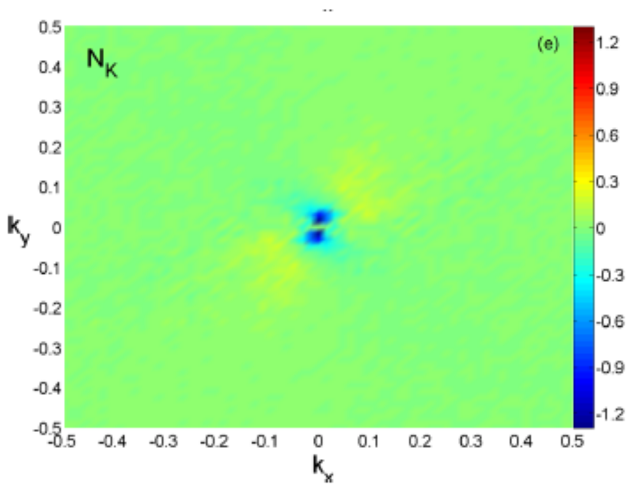
Both spectra are clearly anisotropic, having more power at the $k_x / k_y > 0$ side

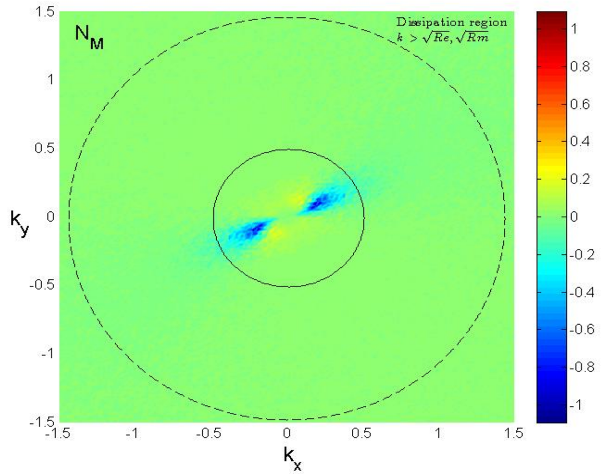
This anisotropy is due to shear

Action of injection I_K, I_M and cross terms I_{K-M}, I_{M-K} as a function of wavenumbers in the turbulent state (averaged over quasi-steady state)

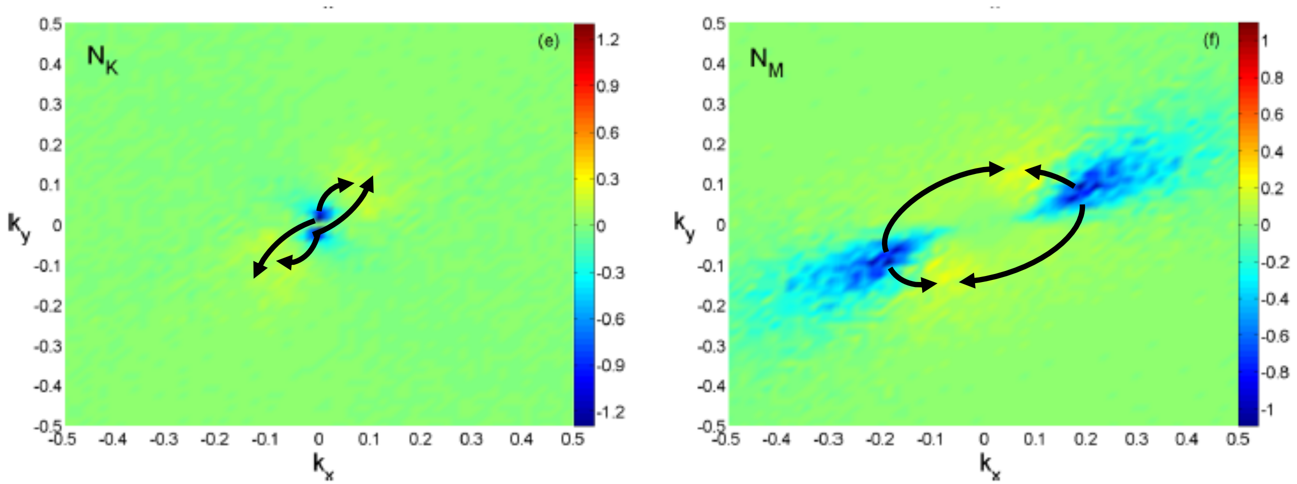


Action of nonlinear transfer terms N_K, N_M in k-plane in the turbulent state and the essence of transverse cascade





Action of nonlinear transfer terms N_K, N_M in k-plane in the turbulent state and the essence of transverse cascade

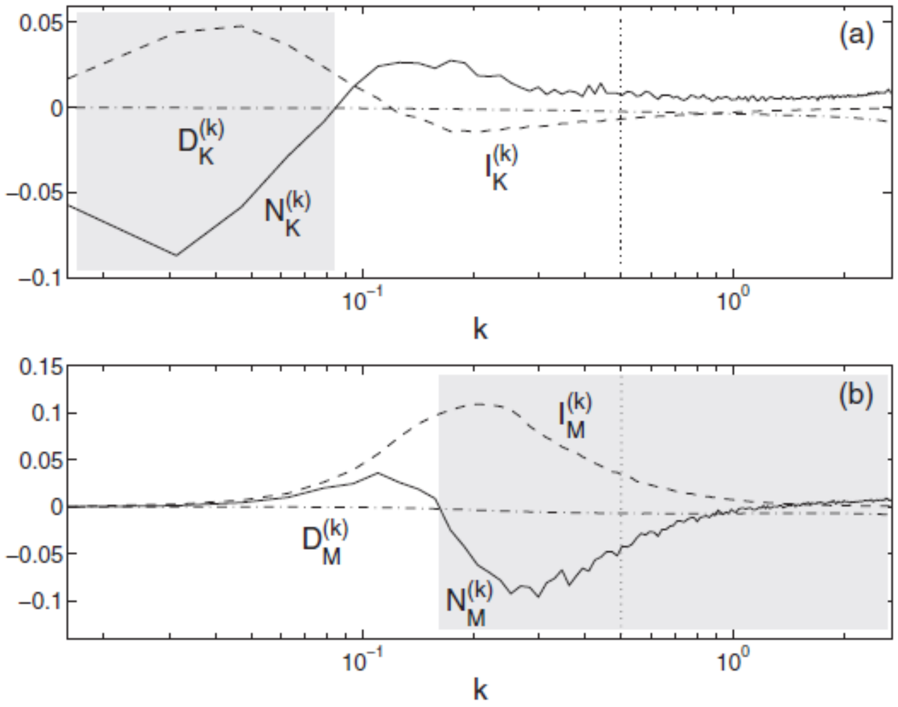


These terms transfer respectively kinetic and magnetic energies from blue regions where they are negative to yellow regions where they are positive.

Both energy transfers are anisotropic and depend on the azimuthal angle.

This picture of energy redistribution is different from that seen in classical cases without shear

Angle averaged injection and transfer terms as a function of *k*

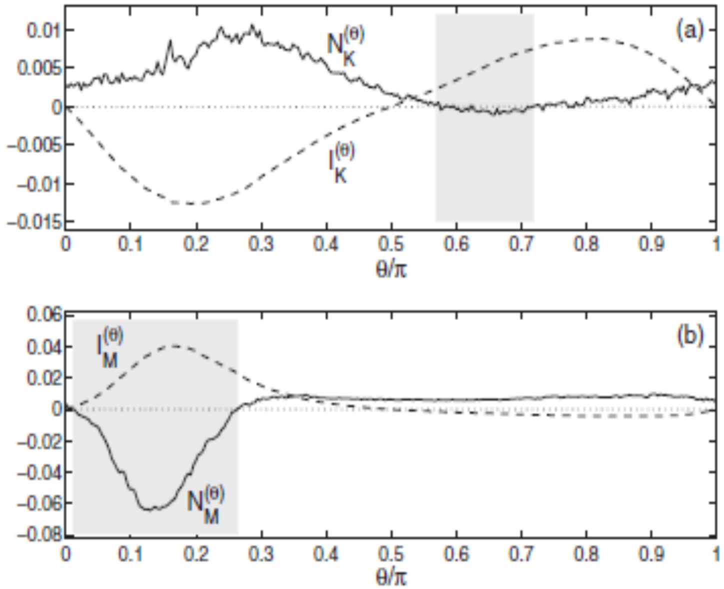


Injection and transfer nonlinear transfer terms overlap over a broad range of wavenumbers

Angular dependence of injection and transfer terms – interplay of transient growth and nonlinear transverse cascades

k-integrated injection and transfer terms represented as a function of polar angle θ

$$I_{K,M}^{(\theta)} = \int_{k_{\min}}^{k_{\max}} I_{K,M} k dk, \quad N_{K,M}^{(\theta)} = \int_{k_{\min}}^{k_{\max}} N_{K,M} k dk$$



Note the opposite relative trends between $I_K^{(\theta)}, N_K^{(\theta)}$ and $I_M^{(\theta)}, N_M^{(\theta)}$

$$I_K^{(\theta)} \leqslant 0 \quad \text{and} \quad N_K^{(\theta)} > 0 \quad \text{at} \quad 0 \leqslant \theta \leqslant \pi/2,$$
$$I_K^{(\theta)} \geqslant 0 \quad \text{and} \quad N_K^{(\theta)} \approx 0 \quad \text{at} \quad \pi/2 < \theta \leqslant \pi.$$

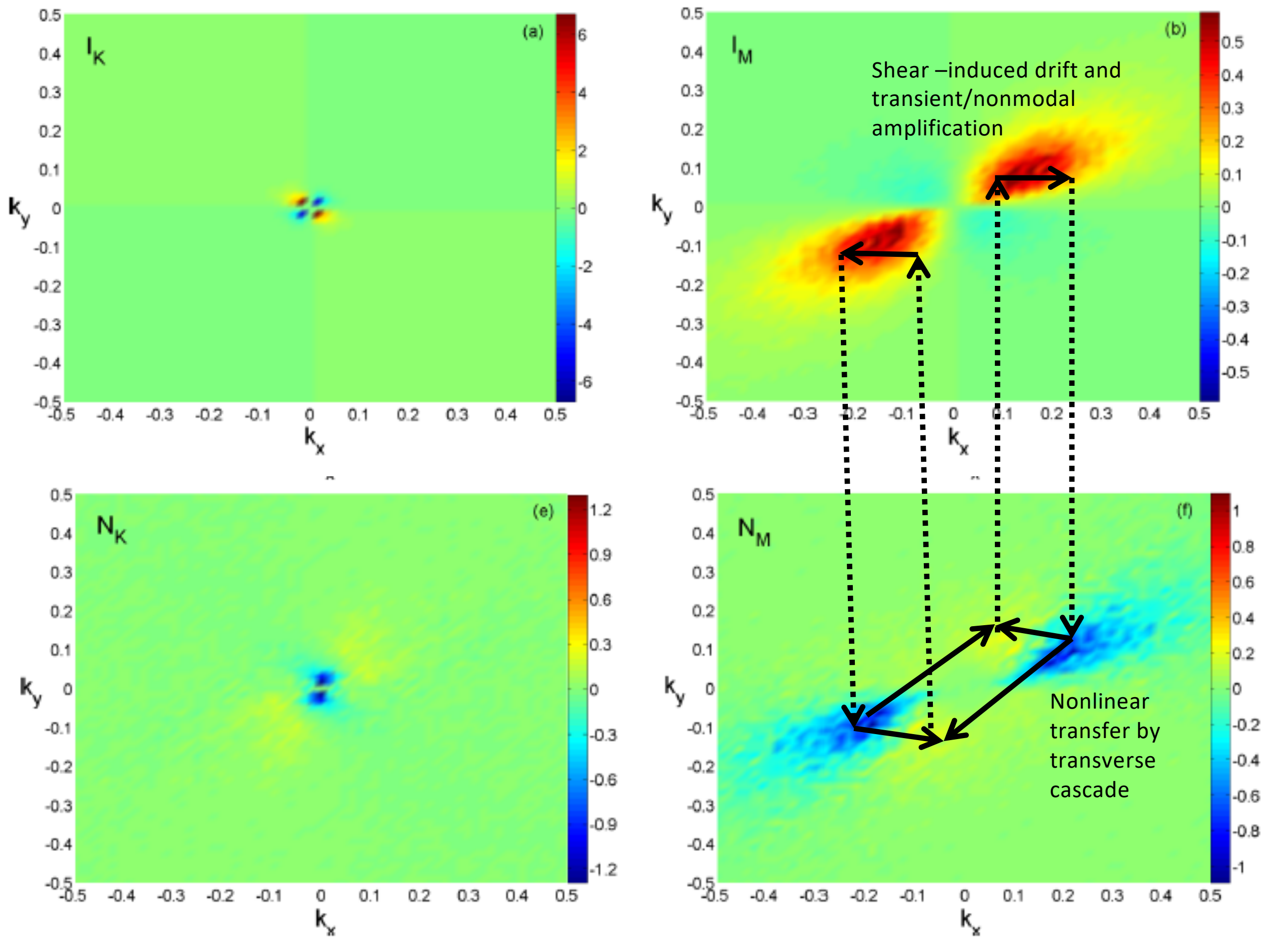
On the other hand,

$$I_M^{(\theta)} \geqslant 0 \quad \text{and} \quad N_M^{(\theta)} \leqslant 0 \quad \text{at} \quad 0 \leqslant \theta \lesssim 0.3\pi,$$
$$I_M^{(\theta)} \approx 0 \quad \text{and} \quad N_M^{(\theta)} > 0 \quad \text{at} \quad 0.3\pi \lesssim \theta \leqslant \pi.$$

Interplay of transient growth and nonlinear transverse cascade

$$\frac{\partial \bar{E}_M}{\partial t} + \frac{\partial}{\partial k_x}(k_y \bar{E}_M) = I_M + I_{M-K} + D_M + N_M,$$

This mechanism works at $k < S / u_A$



Showcase 3 – 3D MHD Keplerian shear flow

Linear MRI

– exponential growth of axisymmetric perturbations
(e.g., Balbus & Hawley 1991, Goodman & Xu 1994)

– (transient) growth of non-axisymmetric perturbations
(e.g., Balbus & Hawley 1992, Brandenburg & Dintrans 2006, Mamatsashvili et al 2013, Squire & Bhattacharjee 2014)

- **Magnetic field configuration**

– Vertical field MRI (e.g., Balbus & Hawley 1991, Ryu & Goodman 1994, Bodo et al. 2008)

– Azimuthal field MRI (e.g.,Ogilvie & Pringle 1996, Papaloizou &Terquem 1997, Simon & Hawley 2009)

– Zero-net flux MRI (e.g., Fromang et al. 2007, Lesur & Ogilvie 2008, Bodo et al. 2014, Shi et al. 2016)

("Helical MRI", Kirillov et al…)

- **Effects of dissipation** – different dynamics depending on the **Re** and **Pm** numbers (Fromang et al. 2007, Lesur & Longaretti 2007, 2011, Pessah & Chan 2008)

Effects of nonlinearity – essence of sustenance of MRI-driven turbulence and saturation

- Direct/inverse cascades of kinetic and magnetic energies in Fourier/spectral space

- Nonlocality of transfers and dependence of saturation amplitude on Pm (Simon et al 2009, Lesur & Longaretti 2011)

- a new – ***transverse*** – type of turbulent cascades in shear flows (Horton et al 2010, Mamatsashvili et al 2014)

Need to put all these results into a consistent framework - systematization

Transverse cascade and self-sustenance of MRI-turbulence with net azimuthal field

Incompressible conducting fluid with constant viscosity, thermal diffusivity and resistivity in the shearing box with vertically constant stratification N^2 (Boussinesq approximation) (Lesur & Ogilvie 2010)

$$\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U}\cdot\nabla)\mathbf{U} = -\frac{1}{\rho}\nabla P + \frac{(\mathbf{B}\cdot\nabla)\mathbf{B}}{4\pi\rho} - 2\mathbf{\Omega}\times\mathbf{U} + 2q\Omega^2 x\,\mathbf{e_x} - \Lambda N^2\theta\,\mathbf{e_z} + \nu\nabla^2\mathbf{U},$$

$$\frac{\partial\theta}{\partial t} + \mathbf{U}\cdot\nabla\theta = \frac{u_z}{\Lambda} + \chi\nabla^2\theta, \qquad \nabla\cdot\mathbf{U} = 0,$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{U}\times\mathbf{B}) + \eta\nabla^2\mathbf{B}, \qquad \nabla\cdot\mathbf{B} = 0,$$

where $\Lambda = -g/N^2$ with gravitation acceleration ***g*** and Brunt-Väisälä frequency N

Ω - is the angular velocity of the local rotating reference frame

time is normalized by Ω^{-1}, distances by scale height H, velocity by ΩH and magnetic field by $\Omega H\sqrt{4\pi\rho}$

Equilibrium:

Keplerian shear flow $\vec{U}_0 = (0, -q\Omega x, 0)$ with q=1.5 threaded by constant azimuthal/toroidal magnetic field $\vec{B}_0 = (0, B_{0y}, 0)$, $B_{0y} = \sqrt{2/\beta} = 0.1$, $\beta = 8\pi\rho\Omega^2 H^2 / B_{0y}^2 = 200$

Vertically constant Brunt-Väisälä frequency $N^2 = 0.25\Omega^2$

Reynolds number Re, magnetic Reynolds number Rm, and thermal Reynolds number Re_{th}:

$$\mathrm{Re} = \frac{\Omega H^2}{\nu}, \quad \mathrm{R}m = \frac{\Omega H^2}{\eta}, \quad \mathrm{R}_{th} = \frac{\Omega H^2}{\chi}, \qquad \mathrm{Re} = Rm = \mathrm{R}_{th} = 5000$$

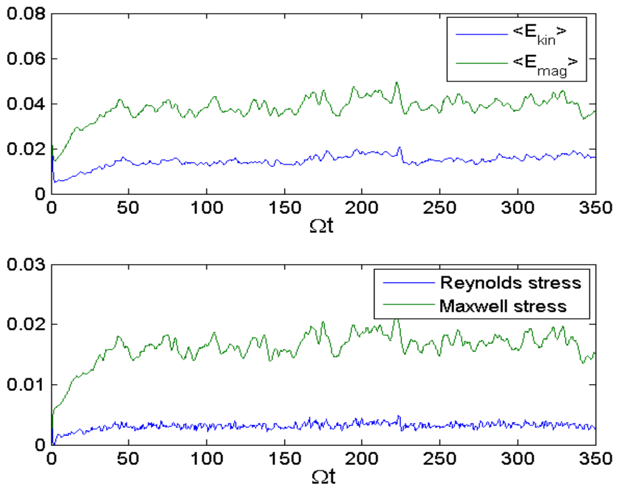
Box size and resolution: $(L_x, L_y, L_z) = (4,4,1)H, \quad (N_x, N_y, N_z) = (256, 256, 64)$ (i.e., 64 grid points per H)

Simulations are performed with **SNOOPY** code (Lesur & Longaretti 2007,G. Lesur's homepage)

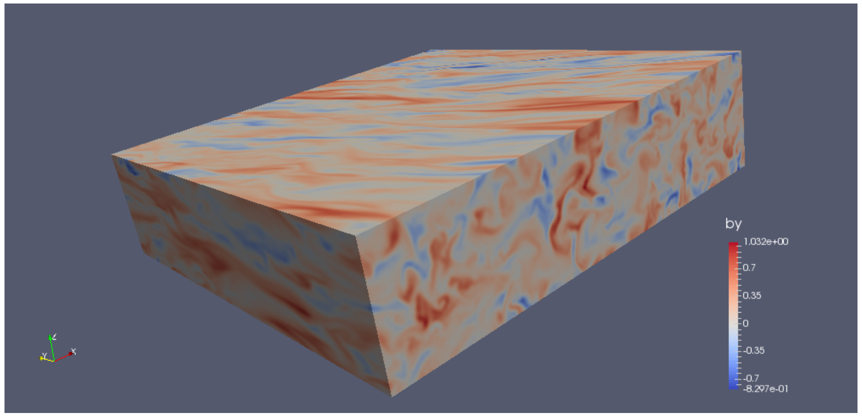
Main goals:

Numerically investigate nonlinear transfers sustaining MRI-turbulence and the role of the transverse cascade

Initial amplification and settling into self-sustaining MHD turbulent state



structure of turbulence in physical space (*by* at t=250)



Main equations in spectral/Fourier space

$$f(\mathbf{r},t) = \int \bar{f}(\mathbf{k},t)\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r})\,d^3\mathbf{k}$$

$$f \equiv (\mathbf{u}, p, \theta, \mathbf{b})$$

Momentum equation

$$\left(\frac{\partial}{\partial t} + qk_y\frac{\partial}{\partial k_x}\right)\frac{|\bar{u}_x|^2}{2} = 2\left(1-\frac{k_x^2}{k^2}\right)\mathcal{H} + 2(1-q)\frac{k_xk_y}{k^2}|\bar{u}_x|^2 + N^2\frac{k_xk_z}{2k^2}(\bar{u}_x\bar{\theta}^* + \bar{u}_x^*\bar{\theta}) - \mathcal{I}_x^{(ub)} + \mathcal{D}_x^{(u)} + \mathcal{N}_x^{(u)}$$

$$\left(\frac{\partial}{\partial t} + qk_y\frac{\partial}{\partial k_x}\right)\frac{|\bar{u}_y|^2}{2} = \left[q-2-2(q-1)\frac{k_y^2}{k^2}\right]\mathcal{H} - 2\frac{k_xk_y}{k^2}|\bar{u}_y|^2 + N^2\frac{k_yk_z}{2k^2}(\bar{u}_y\bar{\theta}^* + \bar{u}_y^*\bar{\theta}) - \mathcal{I}_y^{(ub)} + \mathcal{D}_y^{(u)} + \mathcal{N}_y^{(u)},$$

$$\left(\frac{\partial}{\partial t} + qk_y\frac{\partial}{\partial k_x}\right)\frac{|\bar{u}_z|^2}{2} = (1-q)\frac{k_yk_z}{2k^2}(\bar{u}_x\bar{u}_z^* + \bar{u}_x^*\bar{u}_z) - \frac{k_xk_z}{2k^2}(\bar{u}_y\bar{u}_z^* + \bar{u}_y^*\bar{u}_z) - \frac{N^2}{2}\left(1-\frac{k_z^2}{k^2}\right)(\bar{\theta}\bar{u}_z^* + \bar{\theta}^*\bar{u}_z) - \mathcal{I}_z^{(ub)} + \mathcal{D}_z^{(u)} + \mathcal{N}_z^{(u)},$$

Where the Reynolds stress: $\mathcal{H} = (\bar{u}_x\bar{u}_y^* + \bar{u}_x^*\bar{u}_y)/2$

Kinetic and magnetic exchange terms: $\mathcal{I}_i^{(ub)} = \mathrm{i}k_yB_{0y}(\bar{u}_i\bar{b}_i^* - \bar{u}_i^*\bar{b}_i)/2\,,$ $\quad i=x,y,z$

Nonlinear terms: $\mathcal{N}_i^{(u)} = (\bar{u}_iQ_i^* + \bar{u}_i^*Q_i)/2, \quad Q_i = \mathrm{i}\sum_j k_jN_{ij}^{(u)} - \mathrm{i}k_i\sum_{m,n}\frac{k_mk_n}{k^2}N_{mn}^{(u)}, \quad i,j,m,n = x,y,z.$

$$N_{ij}^{(u)}(\mathbf{k},t) = \int d^3\mathbf{k}'\left[\bar{b}_i(\mathbf{k}',t)\bar{b}_j(\mathbf{k}-\mathbf{k}',t) - \bar{u}_i(\mathbf{k}',t)\bar{u}_j(\mathbf{k}-\mathbf{k}',t)\right],$$

Viscous dissipation terms: $\mathcal{D}_i^{(u)} = -k^2|\bar{u}_i|^2/\mathrm{Re}$

Thermal equation

$$\left(\frac{\partial}{\partial t}+qk_y\frac{\partial}{\partial k_x}\right)\frac{|\bar{\theta}|^2}{2}=\frac{1}{2}(\bar{u}_z\bar{\theta}^*+\bar{u}_z^*\bar{\theta})-\frac{k^2}{\mathrm{Re}_{th}}|\bar{\theta}|^2+\mathcal{N}^{(\theta)},$$

$$\mathcal{N}^{(\theta)}=\frac{\mathrm{i}}{2}\bar{\theta}^*[k_xN_x^{(\theta)}+k_yN_y^{(\theta)}+k_zN_z^{(\theta)}]+c.c.,\quad N_i^{(\theta)}(\mathbf{k},t)=-\int d^3\mathbf{k}'\bar{u}_i(\mathbf{k}',t)\bar{\theta}(\mathbf{k}-\mathbf{k}',t)\qquad i=x,y,z$$

Induction equation

$$\left(\frac{\partial}{\partial t}+qk_y\frac{\partial}{\partial k_x}\right)\frac{|\bar{b}_x|^2}{2}=\mathcal{I}_x^{(ub)}+\mathcal{D}_x^{(b)}+\mathcal{N}_x^{(b)}$$

$$\left(\frac{\partial}{\partial t}+qk_y\frac{\partial}{\partial k_x}\right)\frac{|\bar{b}_y|^2}{2}=q\mathcal{M}+\mathcal{I}_y^{(ub)}+\mathcal{D}_y^{(b)}+\mathcal{N}_y^{(b)}$$

$$\left(\frac{\partial}{\partial t}+qk_y\frac{\partial}{\partial k_x}\right)\frac{|\bar{b}_z|^2}{2}=\mathcal{I}_z^{(ub)}+\mathcal{D}_z^{(b)}+\mathcal{N}_z^{(b)}$$

Maxwell stress: $\mathcal{M}=-(\bar{b}_x\bar{b}_y^*+\bar{b}_x^*\bar{b}_y)/2$

Nonlinear terms: $\mathcal{N}_x^{(b)}=\frac{\mathrm{i}}{2}\bar{b}_x^*[k_y\bar{\mathcal{E}}_z-k_z\bar{\mathcal{E}}_y]+c.c.,\ \mathcal{N}_y^{(b)}=\frac{\mathrm{i}}{2}\bar{b}_y^*[k_z\bar{\mathcal{E}}_x-k_x\bar{\mathcal{E}}_z]+c.c.,\ \mathcal{N}_z^{(b)}=\frac{\mathrm{i}}{2}\bar{b}_z^*[k_x\bar{\mathcal{E}}_y-k_y\bar{\mathcal{E}}_x]+c.c.$

where the Fourier transforms of electromotive force are:

$$\bar{\mathcal{E}}_x(\mathbf{k},t)=\int d^3\mathbf{k}'\left[\bar{u}_y(\mathbf{k}',t)\bar{b}_z(\mathbf{k}-\mathbf{k}',t)-\bar{u}_z(\mathbf{k}',t)\bar{b}_y(\mathbf{k}-\mathbf{k}',t)\right]\qquad\bar{\mathcal{E}}_z(\mathbf{k},t)=\int d^3\mathbf{k}'\left[\bar{u}_x(\mathbf{k}',t)\bar{b}_y(\mathbf{k}-\mathbf{k}',t)-\bar{u}_y(\mathbf{k}',t)\bar{b}_x(\mathbf{k}-\mathbf{k}',t)\right]$$

$$\bar{\mathcal{E}}_y(\mathbf{k},t)=\int d^3\mathbf{k}'\left[\bar{u}_z(\mathbf{k}',t)\bar{b}_x(\mathbf{k}-\mathbf{k}',t)-\bar{u}_x(\mathbf{k}',t)\bar{b}_z(\mathbf{k}-\mathbf{k}',t)\right]$$

Resistive dissipation terms: $\mathcal{D}_i^{(b)}=-k^2|\bar{b}_i|^2/\mathrm{Rm}$, $i=x,y,z$

Spectral equations

for the kinetic energy density: $\bar{E}_K = (|\bar{u}_x|^2 + |\bar{u}_y|^2 + |\bar{u}_z|^2)/2$

$$\frac{\partial \bar{E}_K}{\partial t} + \frac{\partial}{\partial k_x}(qk_y\bar{E}_K) = q\mathcal{H} - \mathcal{I}^{(u\theta)} - \mathcal{I}^{(ub)} + \mathcal{D}^{(u)} + \mathcal{N}^{(u)},$$

$$\mathcal{I}^{(u\theta)} = \frac{N^2}{2}(\bar{u}_z\bar{\theta}^* + \bar{u}_z^*\bar{\theta}), \quad \mathcal{I}^{(ub)} = \sum_i \mathcal{I}_i^{(ub)} \quad \mathcal{D}^{(u)} = \sum_i \mathcal{D}_i^{(u)} = -\frac{2k^2}{\mathrm{Re}}\bar{E}_K, \quad \mathcal{N}^{(u)} = \sum_i \mathcal{N}_i^{(u)}.$$

for the magnetic energy density: $\bar{E}_M = (|\bar{b}_x|^2 + |\bar{b}_y|^2 + |\bar{b}_z|^2)/2$

$$\frac{\partial \bar{E}_M}{\partial t} + \frac{\partial}{\partial k_x}(qk_y\bar{E}_M) = q\mathcal{M} + \mathcal{I}^{(ub)} + \mathcal{D}^{(b)} + \mathcal{N}^{(b)}, \quad \mathcal{N}^{(b)} = \sum_i \mathcal{N}_i^{(b)}, \quad \mathcal{D}^{(b)} = \sum_i \mathcal{D}_i^{(b)} = -\frac{2k^2}{\mathrm{Rm}}\bar{E}_M$$

Energy drift due to shear In **k**-space

Energy Injection by Maxwell stress

Kinetic-magnetic energy exchange

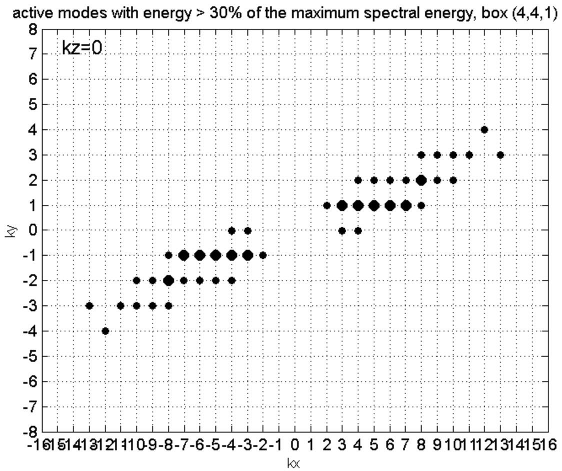
Dissipation due to resistivity

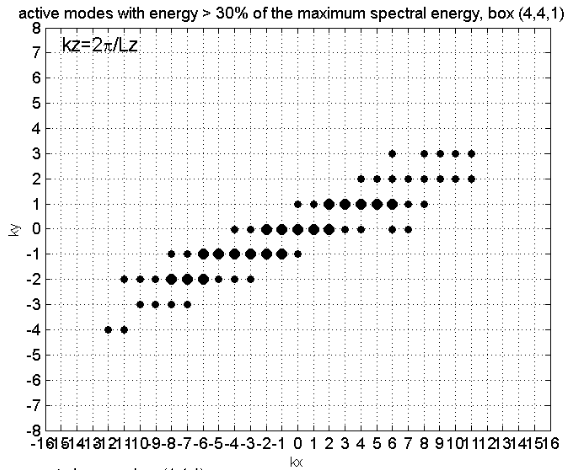
Nonlinear transfers

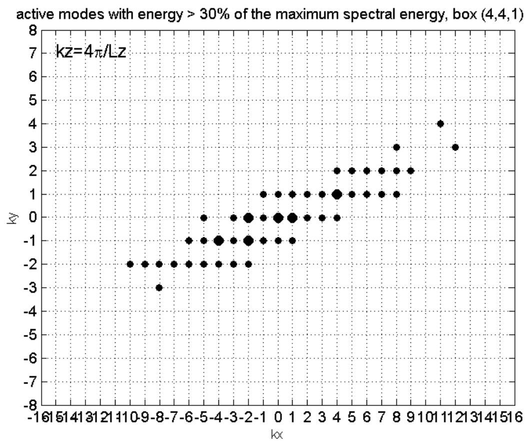
Magnetic energy is dominant (here) among kinetic and thermal energies

Maxwell stress is larger than Reynolds stress and primarily determines turbulence's energy supply and self-sustenance

Active modes with magnetic energy larger than 30% of the maximum at $k_z = (0,1,2)2\pi / L_z$
These modes play a main role in the self-sustaining process

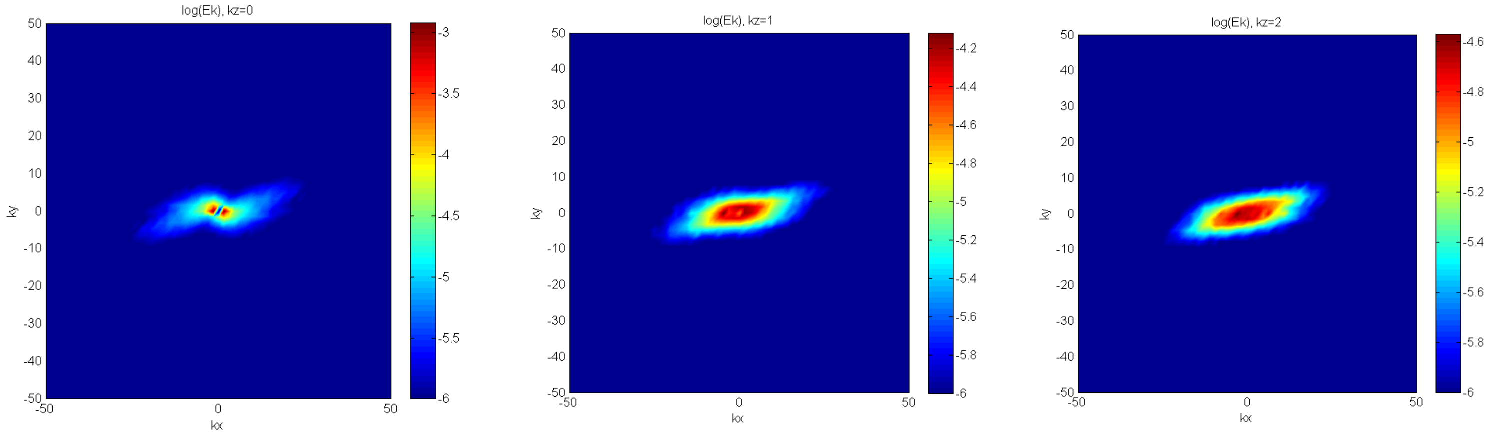




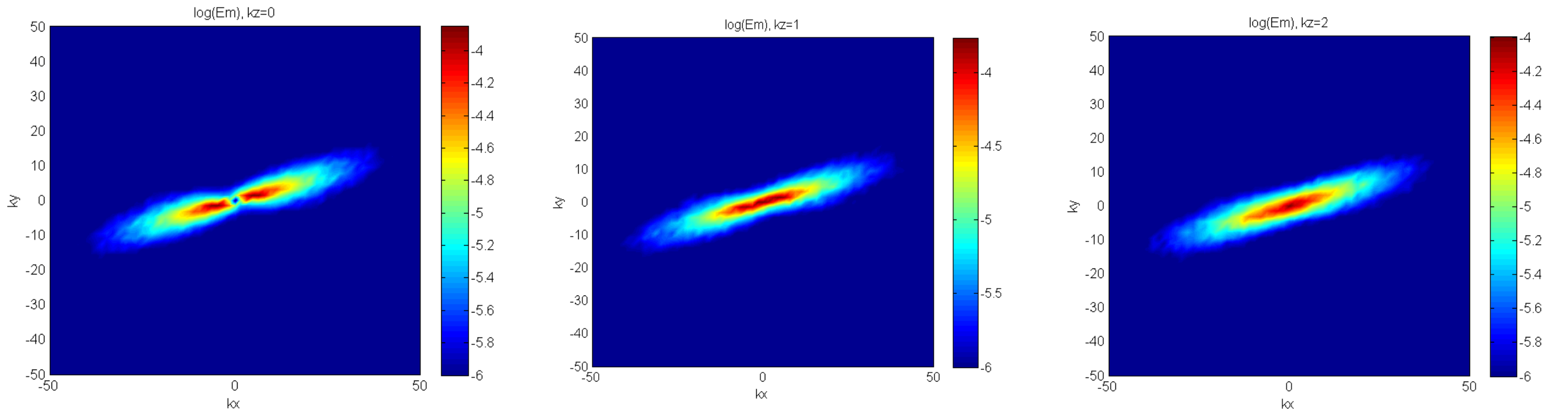


Kinetic and magnetic energy spectra are strongly anisotropic in Fourier space

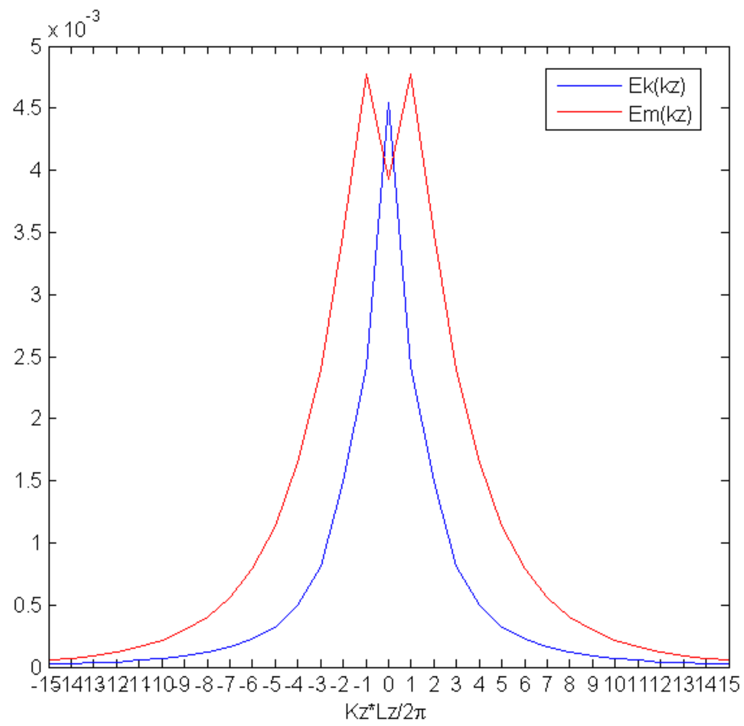
Kinetic energy spectra in (k_x, k_y)-plane at different $k_z = (0,1,2)2\pi / L_z$



Magnetic energy spectra in (k_x, k_y)-plane at different k_z

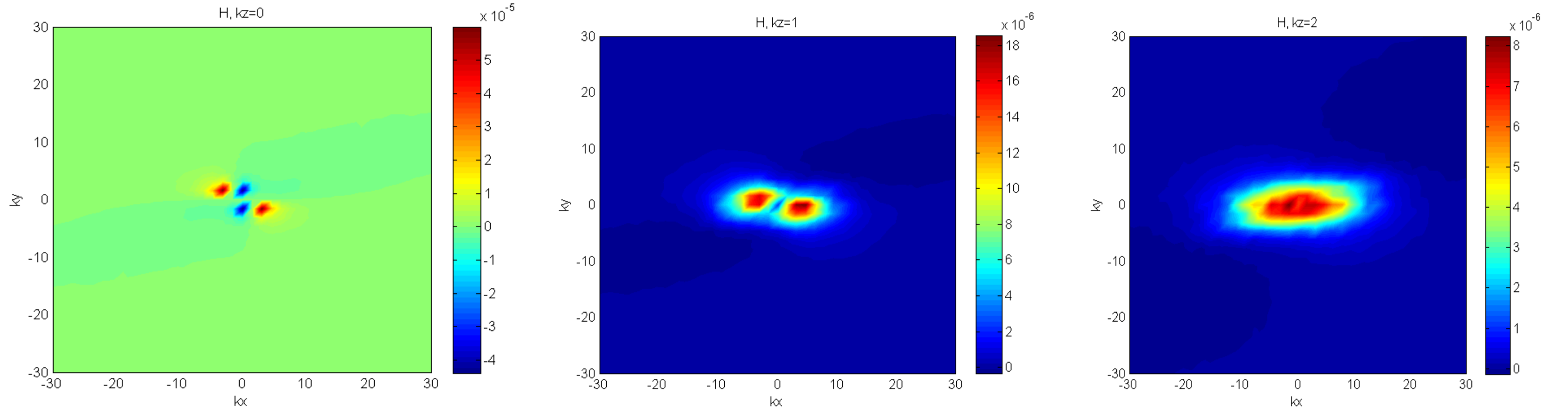


Kinetic and magnetic energy spectra integrated in (k_x, k_y) −plane and represented as a function of k_z

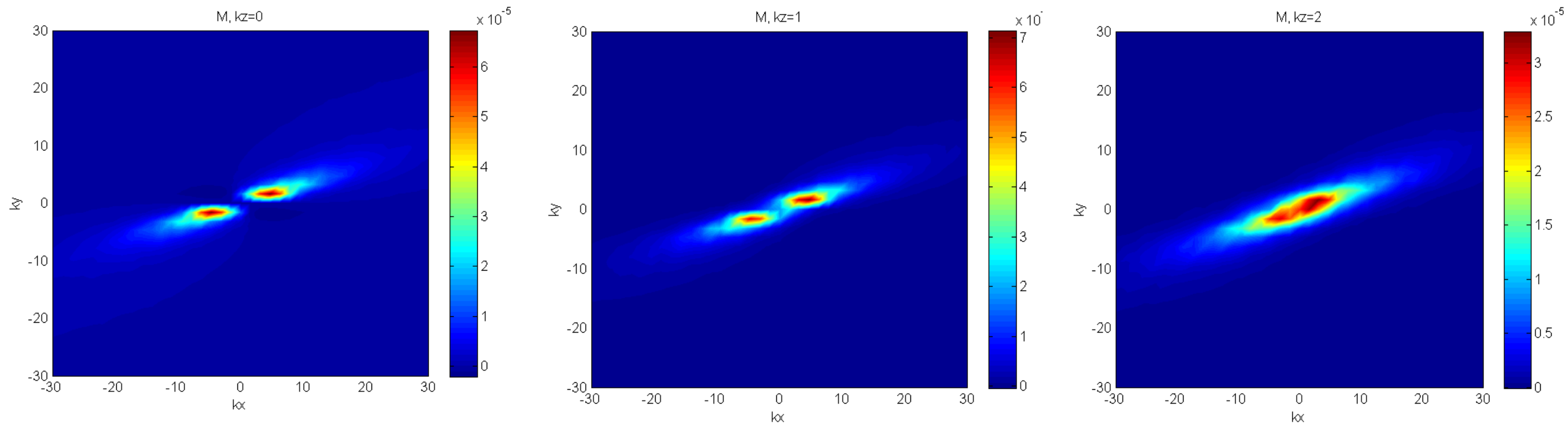


Action of the energy-injecting Reynold and Maxwell stress terms H, M in the turbulent state (averaged over quasi-steady state)

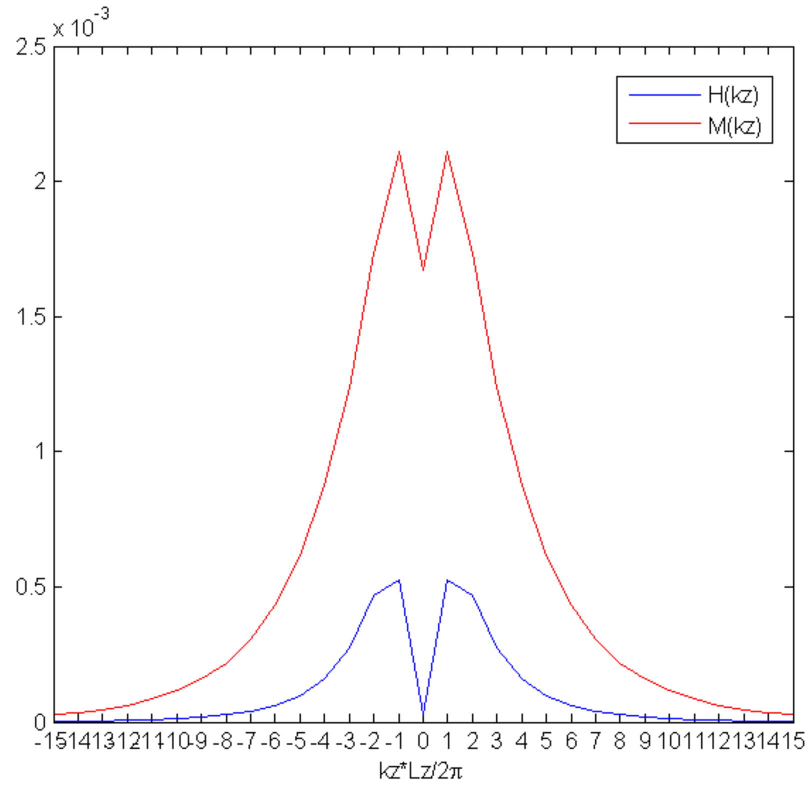
Reynolds stress H in (k_x, k_y)-plane at different $k_z = (0,1,2)2\pi / L_z$



Maxwell stress M in (k_x, k_y)-plane at different k_z. **Energy supply for turbulence is mainly due to Maxwell stress**

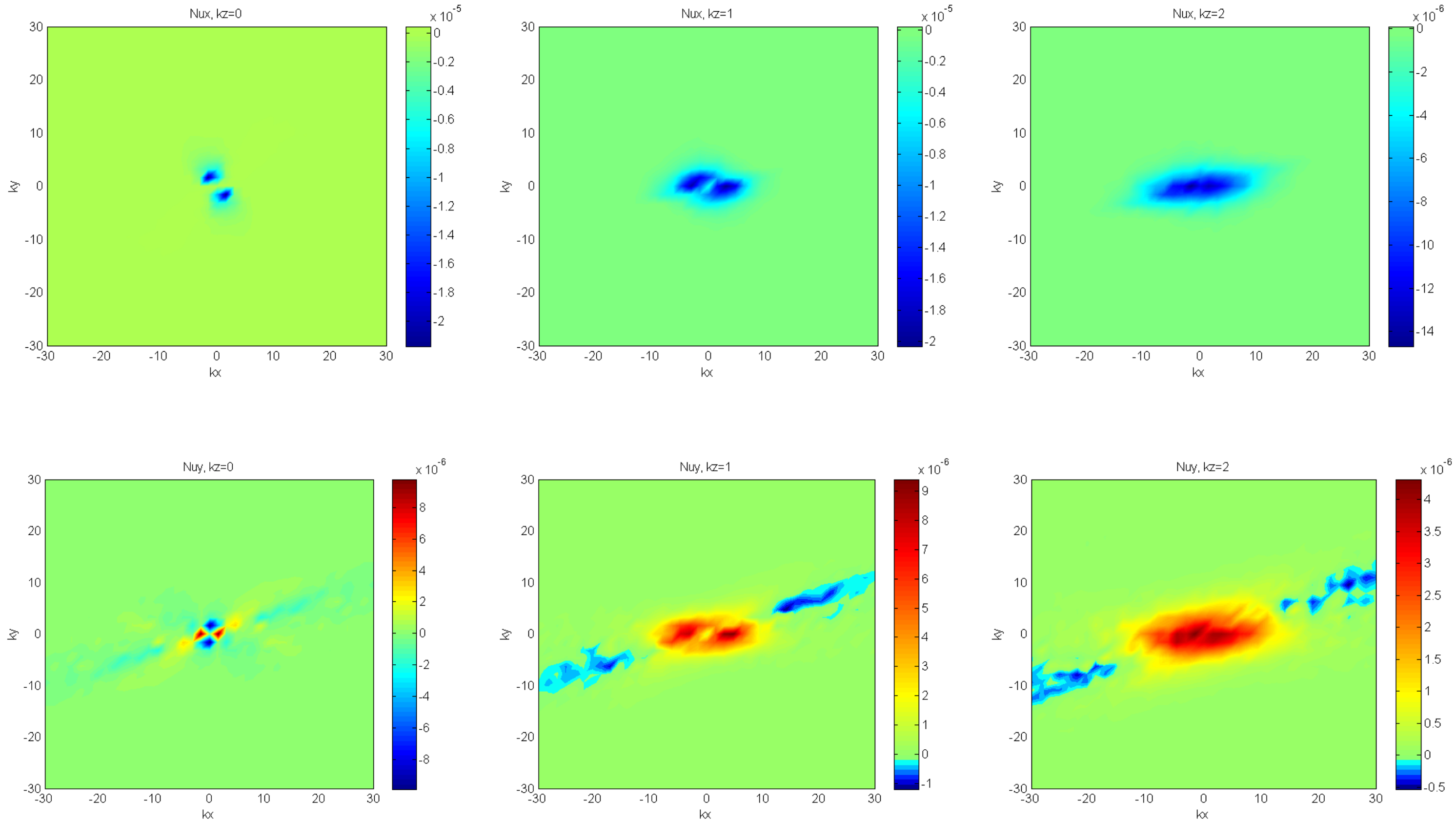


Spectra of the Reynolds and Maxwell stresses integrated in (k_x, k_y) −plane vs. k_z

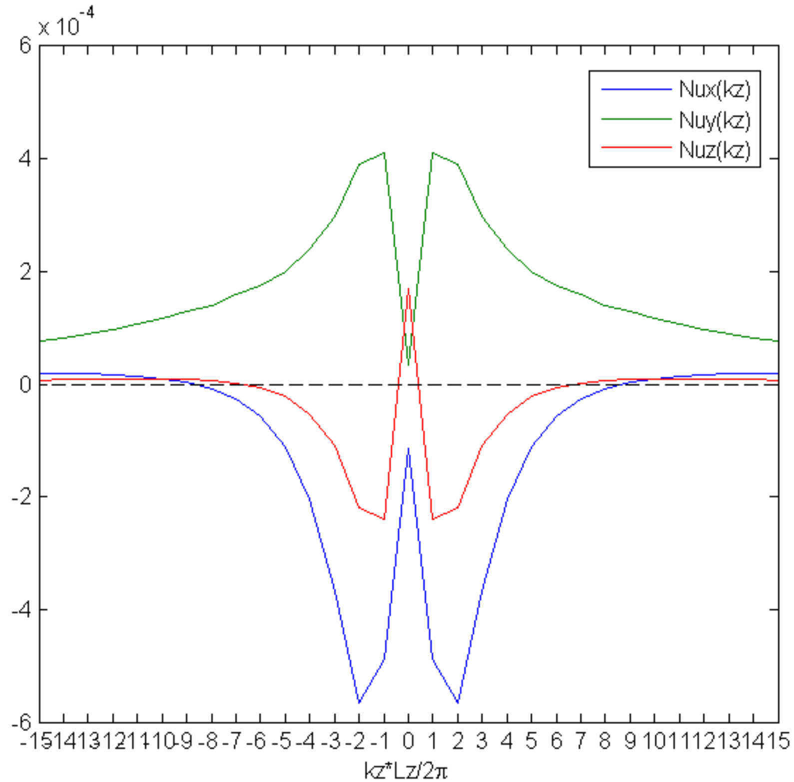


Action of nonlinear transfer terms in k-plane in the turbulent state

Hydrodynamic nonlinear terms $N_x^{(u)}, N_y^{(u)}$ at different $k_z = (0,1,2)2\pi / L_z$

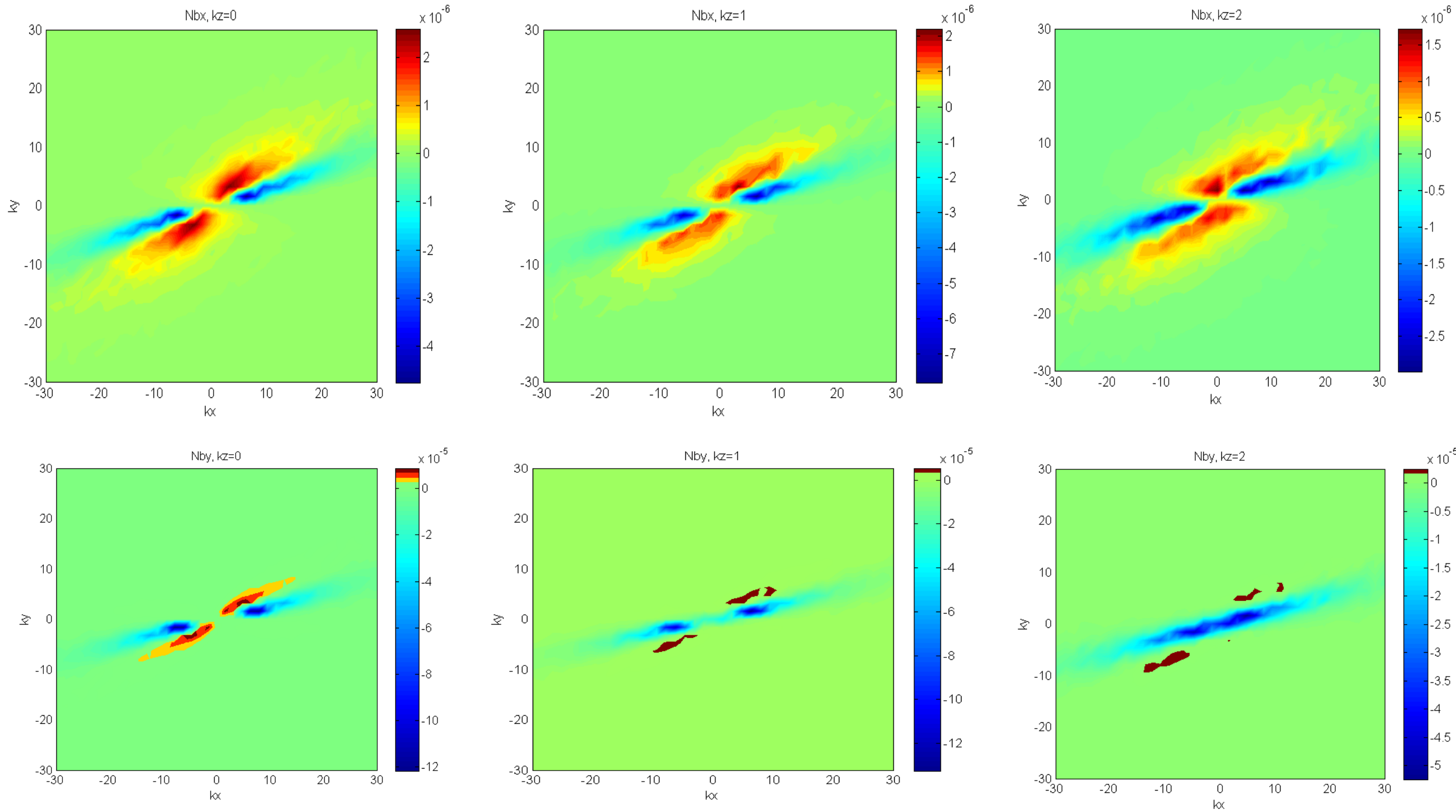


Hydrodynamic nonlinear terms integrated in (k_x, k_y) −plane vs. k_z

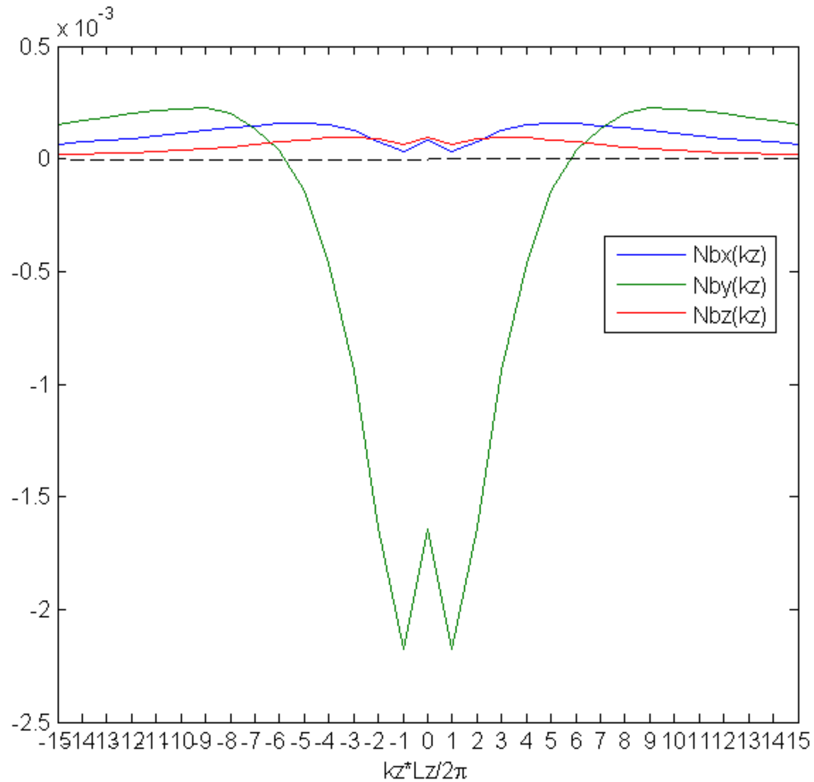


Action of nonlinear transfer terms in k-plane in the turbulent state

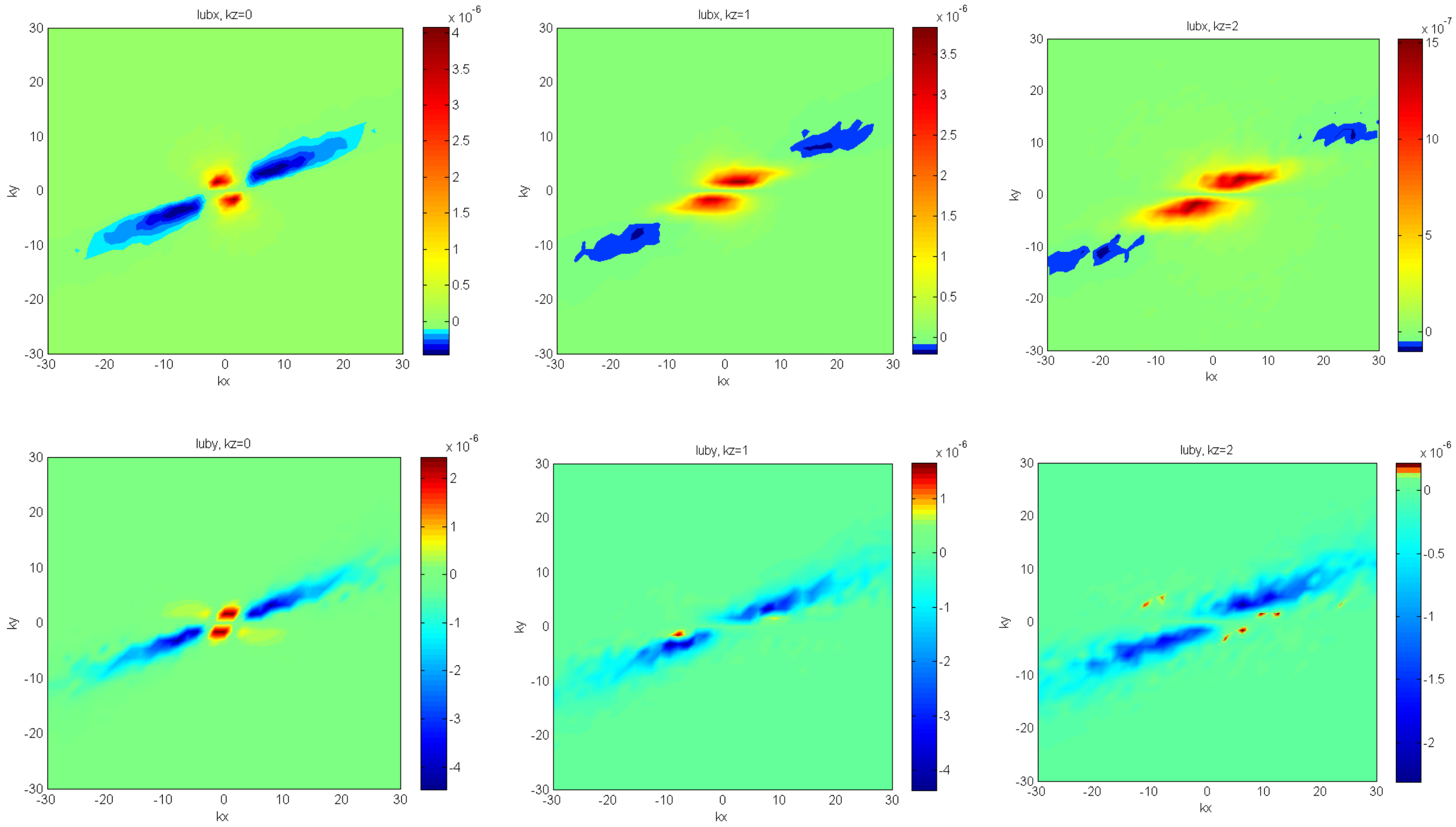
Magnetic nonlinear terms $N_x^{(b)}, N_y^{(b)}$ at different $k_z = (0,1,2)2\pi / L_z$



Magnetic nonlinear terms integrated in (k_x, k_y) −plane vs. k_z



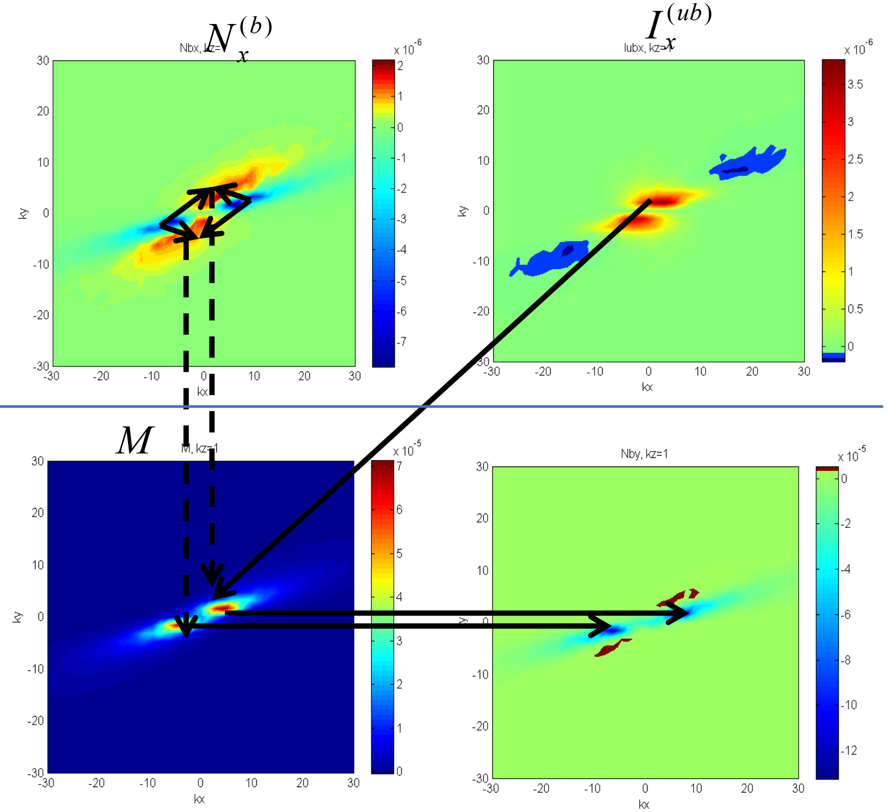
Magnetic and kinetic exchange terms $I_x^{(ub)}, I_y^{(ub)}$



Self-sustaining scheme for 3D MHD turbulennce

$$\left(\frac{\partial}{\partial t}+qk_y\frac{\partial}{\partial k_x}\right)\frac{|\bar{b}_x|^2}{2} = \mathcal{I}_x^{(ub)}+\mathcal{D}_x^{(b)}+\mathcal{N}_x^{(b)}$$

$$\left(\frac{\partial}{\partial t}+qk_y\frac{\partial}{\partial k_x}\right)\frac{|\bar{b}_y|^2}{2} = q\mathcal{M}+\mathcal{I}_y^{(ub)}+\mathcal{D}_y^{(b)}+\mathcal{N}_y^{(b)}$$



Conclusion /show cases 1,2,3

- **We find via numerical simulations self-sustained subcritical 2D and 3D turbulence in a spectrally stable MHD shear flow with constant shear and parallel (azimuthal) mean magnetic field**

- **We investigated turbulence dynamics and underlying self-sustaining mechanism in Fourier space by analysing individual linear and nonlinear terms in spectral equations using the simulation data**

- **We identified a new type of nonlinear cascade due to shear – anisotropic (i.e., along wavevector polar angle) transverse cascade**

- **Turbulence is sustained by interplay of linear nonmodal growth, which injects energy (via Maxwell stress), and nonlinear transverse cascade which ensures regeneration of transiently growing harmonics and in this way is crucial to self-sustenance**

- **Energy spectra is formed as a result of combined action of transient growth and transverse cascade and since both are anisotropic in spectral plane, the resulting spectra is also anisotropic and differs from classical one in the shearless limit**

Future extensions (in course)

- **Here we showed importance of nonlinear transverse cascade in the sustenance of subcritical turbulence in 2D and 3D plane and rotating smooth (spectrally stable) MHD constant shear flow with parallel (azimuthal) mean magnetic field**

- **This approach is easily extendable to more general cases**

  - MRI turbulence with compressibility (OUR CASE OF STUDY HERE)

  - zero-net flux MRI-turbulence and dynamo action

All these are quite relevant to the understanding of subcritical turbulence in sheared and magnetized complex environments, such as MRI-turbulence in magnetized astrophysical discs, MHD winds, geophysical magnetic fields

our specific case (work in progress) :

3D MHD KEPLERIAN AND COMPRESSIBLE FLOWS WITH VERTICAL MAGNETIC FIELD

Set –up
MHD equations
Linear case/modes adiabatic and non adiabatic
Linear mode coupling SDW/MRI
NL case : data analysis of simulations

Set-up: Geometry of the system (drawing)
Simulations with the PLUTO code (Bodo et al,) in shearing box approximation
Simulations parameters
Possible initial MRI …
See next slides

Plan for this part

- Contexte et conditions de l'étude
- Equations de base et leur linéarisation
- Modes propres : cas adiabatique
- Croissance transitoire (notions)
- Couplage linéaire de modes par le cisaillement : cas non adiabatique :ex génération d'ondes de densité (SDW) Extensions prévues (en cours) :

 - Comparaison : cas de MRI avec des modes non axi en croissance transitoire/ mode axi et non-axi en croissance exponentielle

(-Effet d'une stratification verticale du disque)

-Prédictions théorie spectrale linéaire/ DNS compressible

Contexte (1)

- Contexte :
- Interprétation de data de DNS du code astrophysique PLUTO (Bodo et al, 2007 et 2012)
- Introduction aux DNS par analyse linaire (dont croissance transitoire) et couplage linéaire de modes par le cisaillement dans le cas d'un disque , compressible, en rotation, cisaillé radialement et magnétisé (+plus tard stratifié vertical…)

Contexte (2)

- Conditions physiques de l'étude et des simulations.
- Approximation de boite cisaillée avec V_{Kepler} et cisaillement S=cte=$-q\Omega_0$//x(er), B_0 // Ω_0 //Oz (=ctes), $\beta=2\,(cs/Va)^2$=300-400,
- $Va^2=B_0{}^2/2\mu_0\rho$ Alfven velocity (p_m/vol=$B_0{}^2/2\mu_0$)
- q=3/2, autogravitation négligée (d.a /étoile)
- DNS : code (global) astro PLUTO dans le domaine réel 3D (b,u, ρ) , forme conservative (Godunov +RK en t), ici boites cisaillées périodiques de taille 128(256) (z). 512^2 (x,y)
- ->analyse des data par FFT cisaillée en k

-Motivations : succès passés de nos analyses linéaires spectrales dont asymptotiques en temps en croissance transitoire (TG)/ résultats des DNS associées.

General equations (NL)

MHD-3D NL compressible eqs (1-3) for u, ρ,b (7 components) :

$$\rho(\partial_t + u.\nabla)u = -\nabla(p + p_m + p_c) - 2\Omega_0 \times u + (B.\nabla)B / \mu_0 - g\rho\hat{e}_3 + \eta_e \Delta u$$

$$(\partial_t \rho + \nabla(\rho u) = 0 soit : (\partial_t + u.\nabla)\rho + \rho\nabla.u = 0$$

$$(\partial_t + u.\nabla)b = (B.\nabla)u + \eta_m \Delta u$$ -b div(u)

$$B = B_0 + b, V = U_0 + u$$

$$p = f(\rho), p_m = B^2 / 2\mu_0, p_c \approx \rho\Omega_0^2 r^2 / 2$$

$$\nabla.u \neq 0, \nabla.b = 0$$

Equations de base linéarisées (L)

- par linéarisation des eqs NL : $U=U_0+u$...
- eqs(1-3), dans (1) ici grad(P_0) ρ/ρ_0^2 =0 au r.h.s , $P=\rho c_s^2$, $A_{ij}=S\,\delta_{i1}\,\delta_{j2}$(shear/cisaillement), eqs (4-6):

$$\rho_0(\partial_t + U_0.\nabla)u = -A.u - \nabla(p+p_m) - 2\Omega_0 \times u + (B.\nabla)B/\mu_0 - g\rho/\rho_0 \hat{e}_3 + \eta_e \Delta u$$

$$(\partial_t + U_0.\nabla)\rho + \rho\nabla.U_0 = -\nabla\rho_0.u - \rho_0\nabla.u$$

$$(\partial_t + U_0.\nabla)b = A.b + (B_0.\nabla)u + \eta_m \Delta u$$

$-B_0$ div(u)-bdiv(U0)=0

$$A.u = (u.\nabla)U_0, A.b = (b.\nabla)U_0$$

$$\text{here}: \nabla.U_0 = 0$$

$$D_t = (\partial_t + U_0.\nabla)$$

Equilibre : flot de base (U_0, P_0, ρ_0, B_0)

- (si stratification: pas de profil radial mais possible en z /boite cisaillée..)
- équilibre hydro pour $P_0(z)$: $dP_0/dz(+g)+ z\Omega_0^2=0$ (7)
- $S=-q\Omega_0$ sur Ox -> $U_0=(0,Sx,0)$; x =radial (1) , y=azimuthal (2), z=vertical (3)
- (flot admissible…)

Re-ecriture des équations L dans l'espace spectral (1)

- Choix des perturbations en:
- u= u(k,t) exp(ik(t).x)   RDT/SLT
- Normalisation des variables …xi/H ,H=cs/Ω_0, vi/c_s, bi/B_0 …
- D_t ->d/dt mais au prix de k(t) qui obéit à l'équation eikonale :
- dk/dt=-A.k (8), pour A(U_0) on obtient ici :

$k_x(t)$ (=k_1)=k_{10}-Sk_2t ,k_y(k_2)=k_{20}, k_z=(k_3)=k_{30} (9)

Equations dans l'espace spectral (2)

- cas stratifié (si $N_3 \neq 0$) eqs (10-17):

$$d\hat{u}_1 / dt = 2\Omega_0 \hat{u}_2 - ic_s^2 k_1 \rho + i\omega_a (\hat{b}_1 - \hat{b}_3 k_1 / k_3)$$

$$d\hat{u}_2 / dt = (q-2)\Omega_0 \hat{u}_1 - ic_s^2 k_2 \rho + \omega_a (\hat{b}_2 - \hat{b}_3 k_2 / k_3)$$

$$d\hat{u}_3 / dt = -ic_s^2 k_3 \rho + (N_3^2 \ldots)$$

$$d\hat{b}_1 / dt = i\omega_a \hat{u}_1$$

$$\sum_i k_i(t) \hat{b}_i = 0$$

$$d\hat{b}_2 / dt = i\omega_a \hat{u}_2 - q\Omega_0 \hat{b}_1$$

$$d\hat{b}_3 / dt = i\omega_a \hat{u}_3$$ $-i(B_0)\Sigma(k_i\, u_i)$

$$d\rho / dt = -i \sum_{i=1,2,3} k_i \hat{u}_i + (N_3^2 \ldots)$$

Croissance transitoire/modes non normaux

- Pour des modes spectralement linéairement stables on peut avoir une croissance des perturbations en loi de puissance sur un temps fini à cause ici de S .
- Cas des flots cisaillés/modes non normaux

  $k_x(t)$ ($=k_1$)$=k_{10}-Sk_2t$, $S=-q\Omega_{0.}$

Au lieu d'avoir une équation de dispersion on doit résoudre une équation à coefficients dépendant du temps.

Choix des variables :
eqs (18-20) : Système différentiel d'ordre 2 variables : $b_1, b_2, s(\rho, b_3) = b_3 - (B_{0z})\rho/\rho_0$

$$d^2s / dt^2 = c_s^2 k_3 (k_1(t) b_1 + k_2 b_2 - k_3 s)$$

$$d^2b_1 / dt^2 = 2\Omega_0 db_2 / dt - (\omega_{a3}^2 + (c_s^2 + v_{a3}^2) k_1^2 - 2q\Omega_0^2) b_1 - ...$$

$$.... - (c_s^2 + v_{a3}^2) k_1(t) k_2 b_2 + c_s^2 k_3 k_1(t) s$$

$$d^2b_2 / dt^2 = -2\Omega_0 db_1 / dt - (\omega_{a3}^2 + (c_s^2 + v_{a3}^2) k_2^2) b_2 - (c_s^2 + v_{a3}^2) k_1(t) k_2 b_1 + c_s^2 k_3 k_2 s$$

relation de dispersion dans le cas WKB/adiabatique

- Ici WKB+ régime adiabatique tel que d=/ky/kx/=
- /k2/k1(t)/<<1 (qW0 t>>1)

(variation temporelle faible donc négligée de k_x(t) et donc de ktot(t))

Modes WKB « like » en :

$$\exp(-i\int_{t_0}^{t}\omega(t')\,dt')\,/\,(\omega(t)^{1/2})$$

- condition d'adiabaticité telle que : /dω/dt/<< ω^2(t)

Relation de dispersion (suite)

$$\omega^6 - a_1\omega^4 + a_2\omega^2 + a_3 = 0 \quad (21)$$

$$a_1 = 2(2-q)\Omega_0^2 + (\mathrm{c}_s^2 + \mathrm{v}_{a3}^2)\mathrm{k}^2 + \omega_{a3}^2$$

$$a_2 = \mathrm{v}_{a3}^2(2\mathrm{c}_s^2 + \mathrm{v}_{a3}^2)(\mathrm{k}_3 k)^2 + ...$$

$$....2\Omega_0^2(2\mathrm{k}_3^2\mathrm{c}_s^2 - q^2(\mathrm{c}_s^2 + \mathrm{v}_{a3}^2)(\mathrm{k}_1^2 + \mathrm{k}_3^2))$$

$$a_3 = \omega_{a3}^2\mathrm{c}_s^2(2q^2\Omega_0^2(\mathrm{k}_1^2 + \mathrm{k}_3^2) - \mathrm{k}^2\omega_{a3}^2)$$

$$\omega_{a3}^2 = (\mathrm{k}_3^2\mathrm{v}_{a3}^2) \quad \text{Alfven frequency}$$

cas L= 3 modes propres approchés : cas WKB (1)

- 3 modes (en approximation WKB) avec $\omega^2(+\omega,-\omega)$

mode 1 en hf : mode acoustique (compressible) de dispersion :

$$\omega_s^2 \approx a_1, \omega_s^2 = (\mathrm{kc}_s)^2 + 2(2-q)\Omega^2 + (k\mathrm{v}_{a3})^2 + \omega_{a3}^2 \quad (22)$$

$$\text{for } \mathrm{v}_{a3} \ll \mathrm{c}_s : \omega_s^2 \approx (\mathrm{kc}_s)^2 + 2(2-q)\Omega^2$$

modes propres , propriétés

- cette dernière dispersion est celle bien connue des ondes (spirales) de densité (si on néglige donc B et l'autogravitation) , ondes importantes en astrophysique /phénomènes d'accrétion…
- Modes 2 et 3 bf et quasi incompressibles, solutions de :

$$-a_1\omega^4 + a_2\omega^2 + a_3 \approx 0 \quad (23)$$

modes propres , propriétés (suite)

- Onde d'Alfven-inertielle ω_2 (=onde d' Alfven pure $\omega_2 = \omega_{a3}$ sans rotation) (24a)
- Le mode 2 est stable si B=0 il donne une onde inertielle dans le disque de type : (24b)

$$\omega_2^2 \approx 2(2-q)\Omega^2 (k_3 / k)^2$$

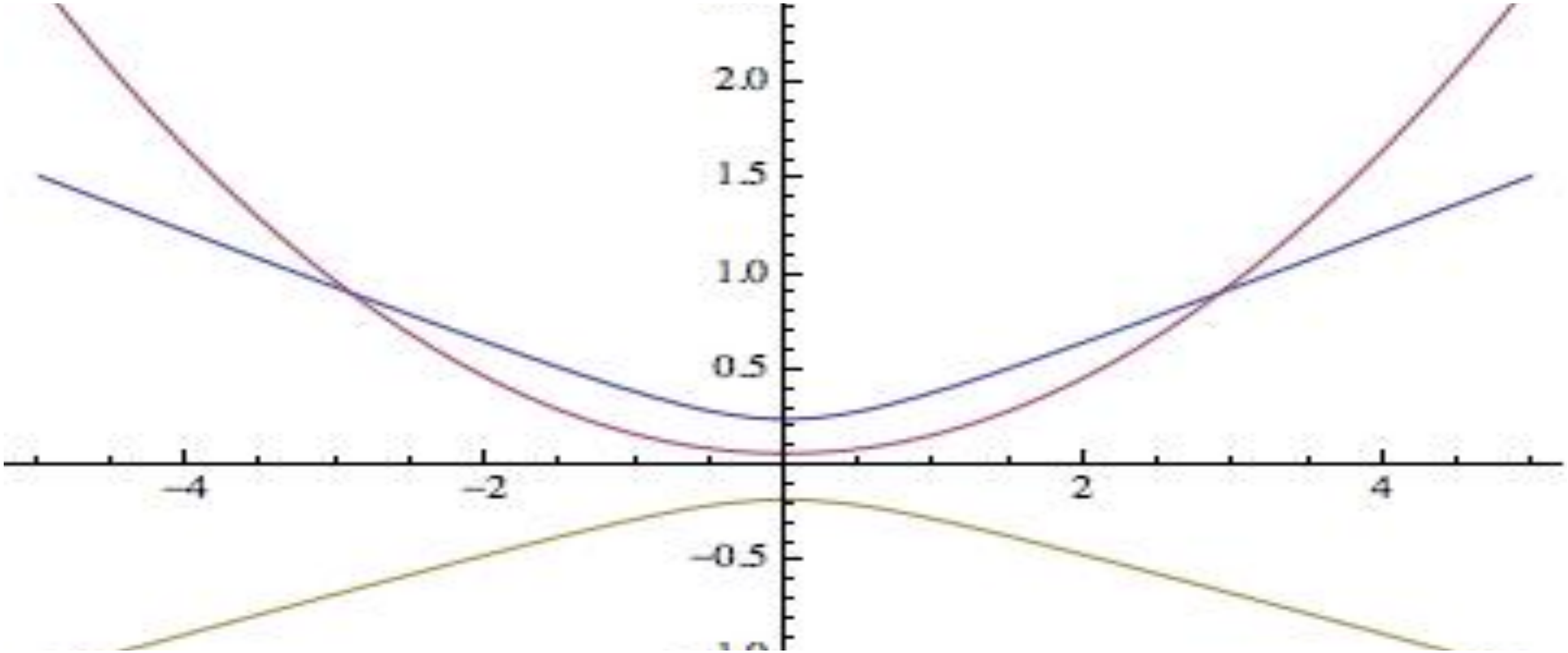
.mode 3 magnétique ω_3 de type onde magnéto-sonore lente à faible rotation. Instabilité MRI possible ($\omega_3^2 < 0$, pour $a_3 > 0$) pour ce mode si :

$\omega_{a3}{}^2 < 2q\Omega_0^2$ pour des valeurs de kx ($k_{1\ ici}$) telles que

$$k_1^2 < (2q\Omega^2 / \omega_{a3}^2 - 1)(k^2 - k_1^2) \quad (25)$$

Fréquences des modes $\omega_i(t)$: rouge $\omega 1(t)/10$, bleu $\omega 2(t)$, brun $\omega 3(t)/60$ ($\omega_3^2<0$: réel négatif instable MRI pour /k1/<54.7 (condition (25))

zone adia et zone non adia(à t « petit »)



Cas L mais non adiabatique : couplage linéaire des modes par S et système à coefficients (t) (2)

- Y''=E Y-> 1er ordre : diagonalisation de $dY/dt=B_{6\text{-}6}$ Y (26), Z=PY: $dZ/dt=D_{(6\text{-}6)}$ Z (27) , $D_{(6\text{-}6)}=P^{-1}B_{6\text{-}6}$ P
- $B_{6\text{-}6}$=…D_{66}=…matrice P….
- **Cas non adiabatique** : cas dépendant du temps avec d =1 ou >1 et dkx/dt=Sk2(ky) , nouveau système :
- dZ/dt= (D(t)+M(t)) Z (28) $M(t)=-P^{-1}dP/dt$ (29)
- dans D(t) on fait ωi->(t) avec kx->kx(t) idem pour k(t)
- Modes axi : (k2=0) ω->ω (t) mais ils ne sont pas modifiés par le shear S…
- Modes non axi : ωi->ωi-iM_{ii}: nouvelles instabilités possibles selon signe de Re (M_{ii}) et couplages à d'autres modes par M_{ij} (ex traité ici : cas des modes MRI non axi et des modes de densité)

Ex : Matrice de passage P(t) (1) (vp : $\lambda = i\omega$)

- Coefficients de P(t)

$$P = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \lambda_{(1)} & -\lambda_{(1)} & \lambda_{(2)} & -\lambda_{(2)} & \lambda_{(3)} & -\lambda_{(3)} \\ G_1(\lambda_{(1)}) & G_1(-\lambda_{(1)}) & . & . & . & . \\ \lambda_{(1)}G_1(\lambda_{(1)}) & -\lambda_{(1)}G_1(-\lambda_{(1)}) & . & . & . & . \\ G_2(\lambda_{(1)}) & G_2(-\lambda_{(1)}) & . & . & . & . \\ \lambda_{(1)}G_2(\lambda_{(1)}) & -\lambda_{(1)}G_2(-\lambda_{(1)}) & . & . & & \end{pmatrix}$$

Matrice de passage P (2)

- Coefficients de P :

$$G_1(\lambda_i^2) = (k_2(t)/k_3)((k_3 c_s)^2 (2 + (V_{a3}/c_s)^2) + \lambda_i^2 (1 + (V_{a3}/c_s)^2)/D$$

$$D = \lambda_i^2 + \omega_{a3}^2 - 2q\Omega_0^2$$

$$G_2(\lambda_i^2) = -((k_3^2 + \lambda_i^2/c_s^2) + k_2^2(t) G_1(\lambda_i^2))/k_1 k_3$$

Matrice M(t)

- $M=-P^{-1}dP/dt$ mais le cas 6-6 est déjà lourd
- Un calcul numérique complet (par RK) est possible pour le système complet : $dZ/dt= (D(t)+M(t)) Z$ (28)
- (Rem: cas asymptotique en t Th de Levinson…)
- Ici on voit que le mode 2 varie peu en temps et reste quasi découplé ; restriction à un système 4-4 pour les modes $\omega 1$,$-\omega 1$(modes 1 et 2), $\omega 3$,$-\omega 3$ (modes 5 et 6), mais dont les coefficients dépendent de la matrice 6-6 de départ :
- Soit : $dZ_4/dt= (D_{4\text{-}4}(t)+M_4(t)) Z_4$ (29)

Illustration de instabilités et couplages sur un CP : mode MRI-> ondes de densité

- Choix des conditions initiales : restriction à 2 modes pour illustrer les échanges possibles entre ici les modes 1 et 3.
- Soit génération de mode 3 (ω3,MRI) à partir du mode 1 (onde de densité) ou bien l'inverse
- Coefficients Mii (complexes)
- Coefficients Mij (complexes)
- Calcul des "Ptés de transitions"
- (mode i ->mode j)

Système couplé (1,3)

- Evolution du système (Z_1(mode1),Z_5(mode3))

$$dZ_1 / dt = -i\omega'_1(t)Z_1 + M_{15}(t)Z_5$$

$$dZ_5 / dt = -i\omega'_5(t)Z_5 + M_{51}(t)Z_1$$

$$\omega'_1(t) = (\omega_1(t) - iM_{11}(t)),\ \omega'_5(t) = (\omega_5(t) - iM_{55}(t))$$

if modes are decoupled (WKB) , used as a start :

$$Z_1(t) \approx Z_1(t_0)\exp - i\int_{t_0}^{t} \omega'_1(t')dt' \ \ (/\omega'_1(t)/^{1/2}$$

idem for mode 5 : $1 \rightarrow 5$

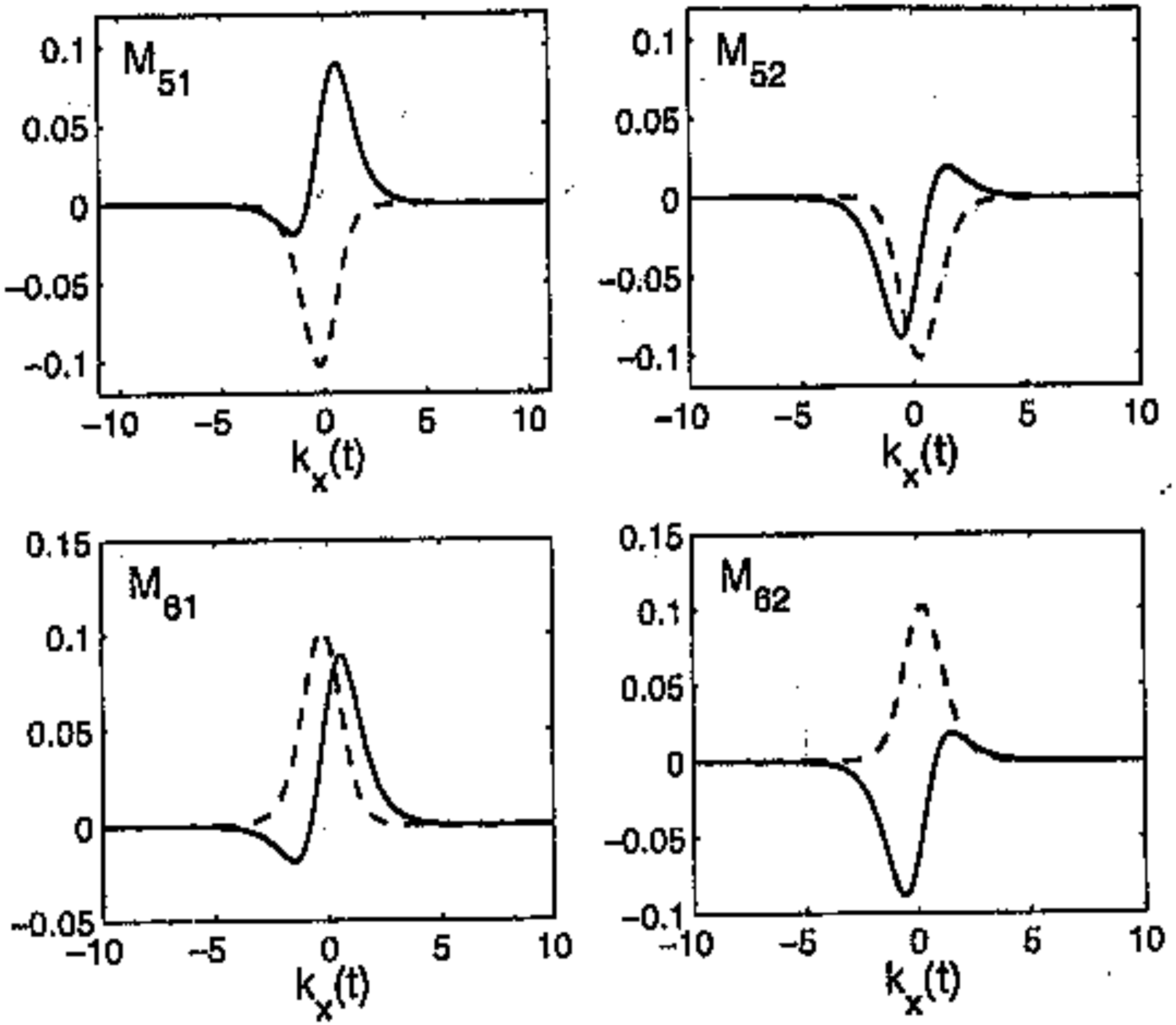
conservation (see over-reflection aside) :

$$/Z_1(t)/^2 + /Z_5(t)/^2 = cte(ci)$$

Calcul approché des « Ptés de transition »

- Ex de choix de ci : génération du mode 3, MRI (Z5) à partir du mode 1 de densité : $Z5(t_0)=0$ et $Z1(t_0)$ non nul
- ou bien l'inverse : génération du mode 1 de densité à partir du mode magnétique 3
- Calcul approché/cas couplé total : si Mij varie lentement ou pas /phase de $(\omega i-\omega j)t$ ou /phase intégrée
- calcul exact de Mij (cas 6-6) : voir diapo suivante
- –=Re( ), ---=Im( ); ky=2,kz=1, coupling zone if d<1

C



Formules : Pté de transition

- P_{13} et P_{31} approximate here for ci mainly as mode 1 (or mode 3) P_{13} $//Z_1(t_0)/^2$, P_{31} $//Z_5(t_0)/^2$

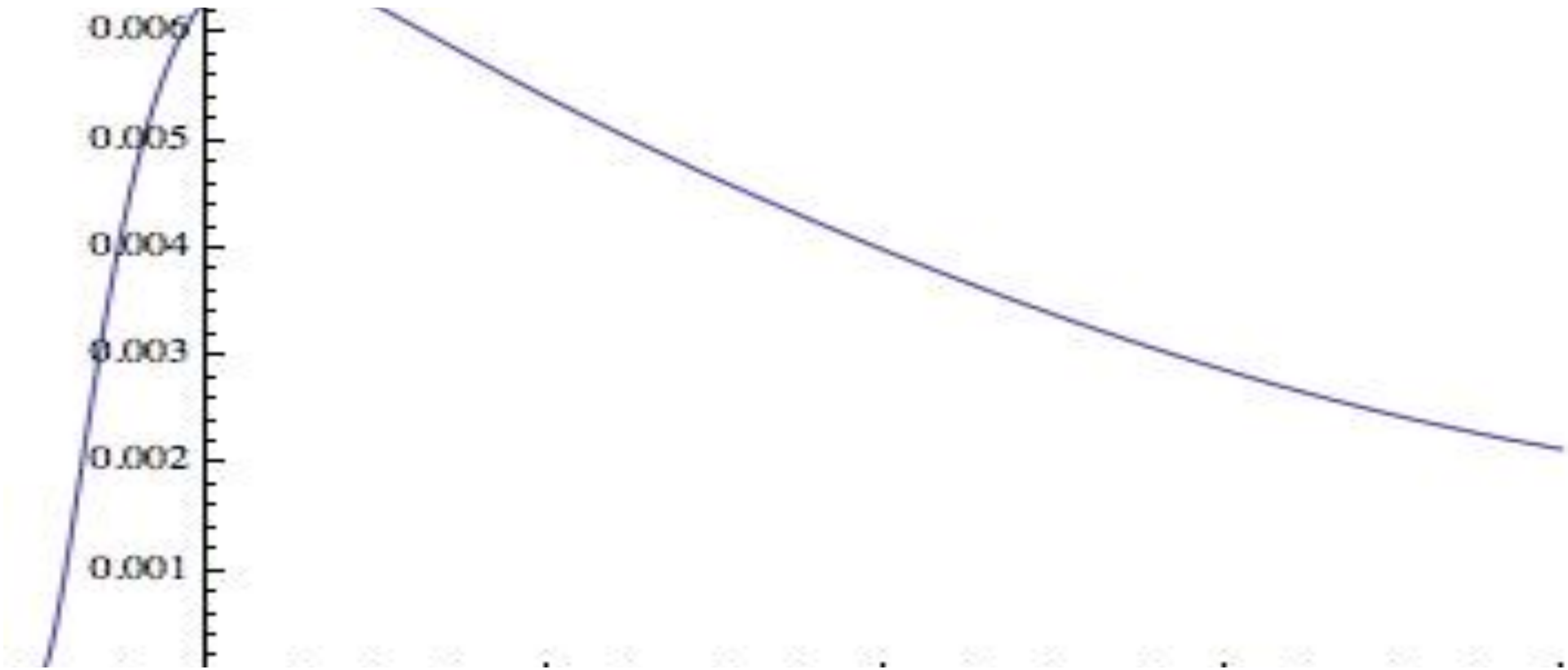
$$P(1 \rightarrow 3) \approx \left| \int_0^t M_{15}(t')(\omega'_3/\omega'_1)^{1/2}\, dt' \exp(-\phi(t') \right|^2$$

$$P(3 \rightarrow 1) \approx \left| \int_0^t M_{51}(t')(\omega'_3/\omega'_1)^{-1/2}\, dt' \exp(\phi(t') \right|^2$$

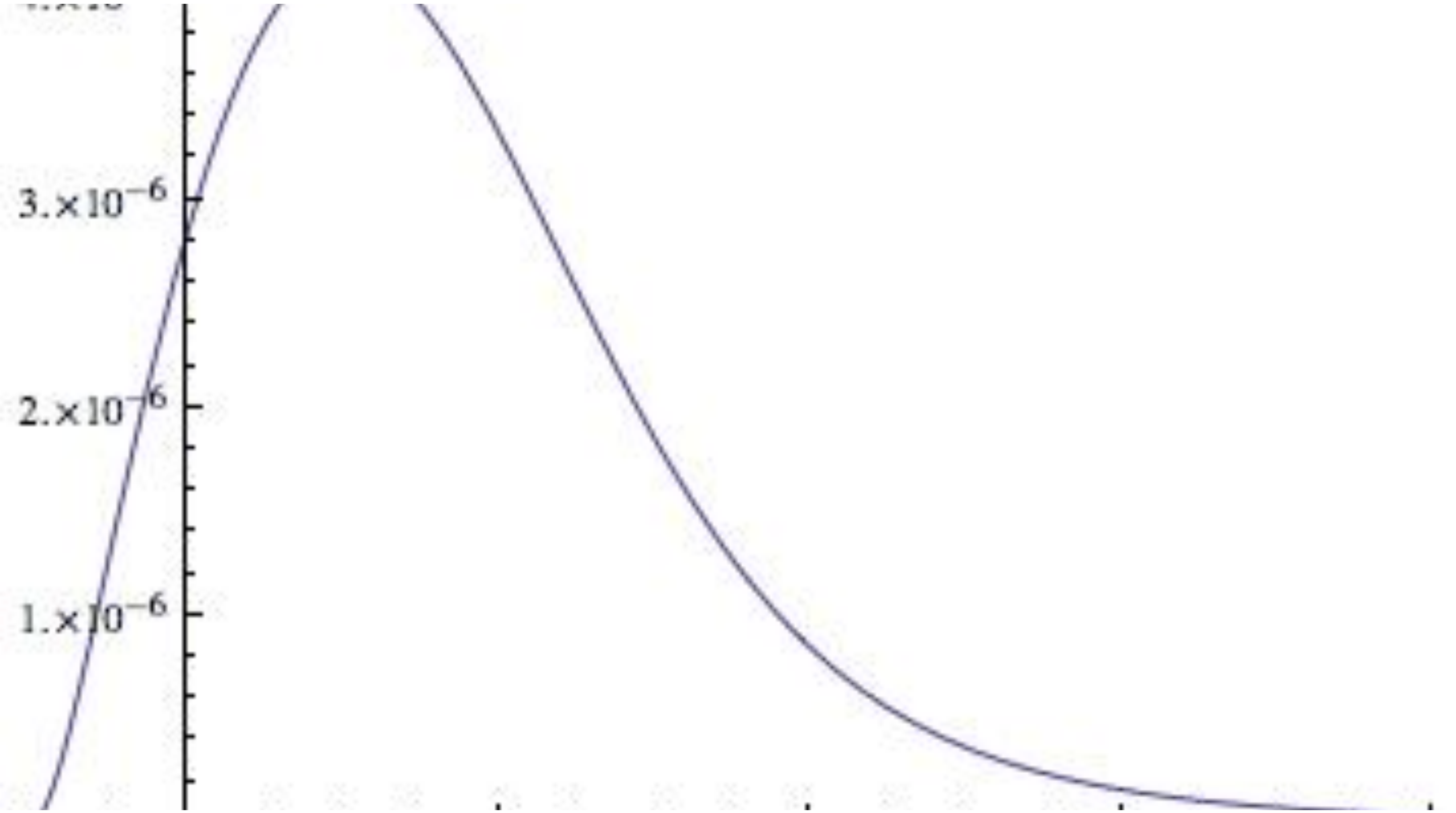
$$\phi(t') = i\int_{t_0}^{t'} (\omega'_1(t'') - \omega'_3(t''))dt'') - \ldots$$

$$\ldots((d\omega'_1/dt)/\omega'_1) - (1 \rightarrow 3))/2$$

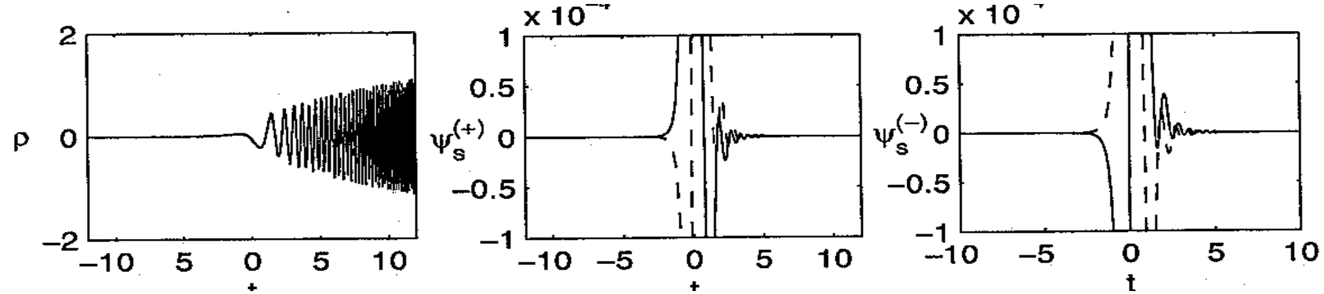
Pté 1->3(P_{15})



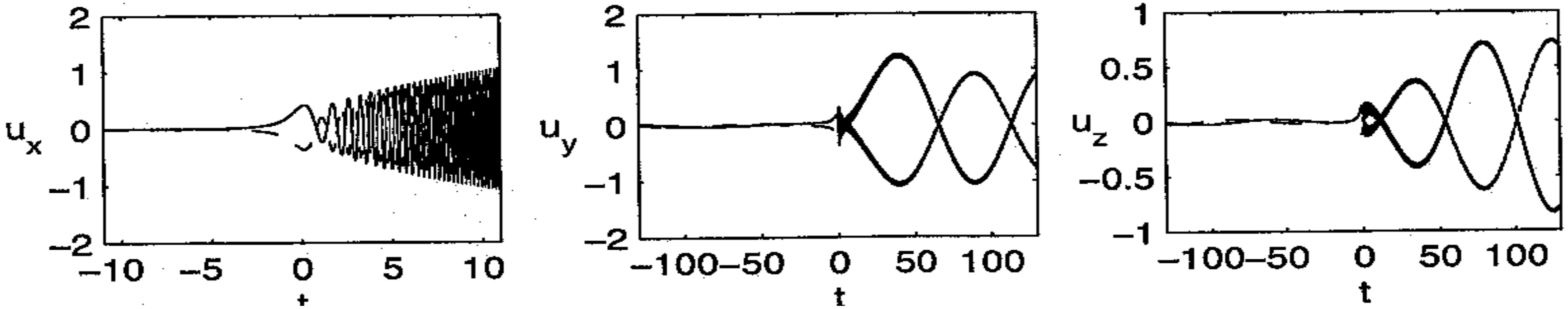
Pté 3->1 (P_{51})



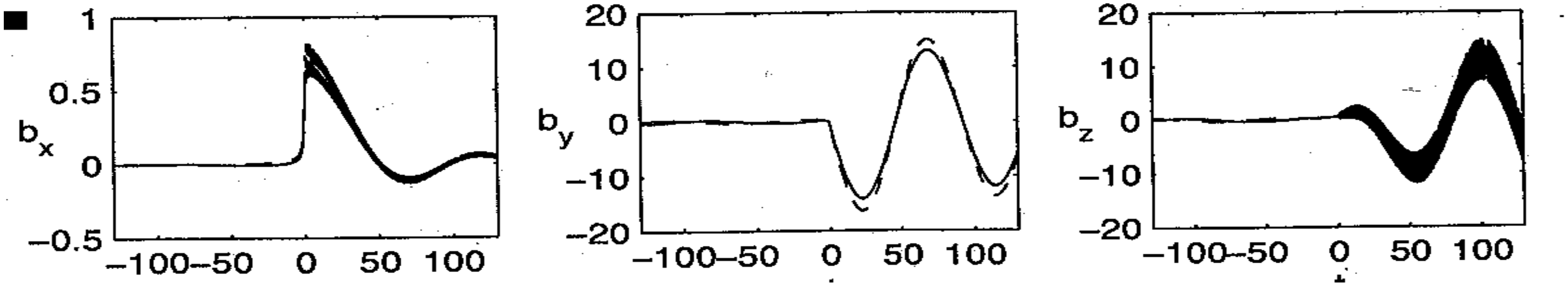
Evolution of the eigenfunction of density waves (SDW birth at fast oscillations, for d ≤1)



Evolution of the v components



Evolution of the b components



Conclusions/analyse L

- Résultats /résumé
- Etude des modes linéaires pour ce système et de leurs couplages linéaires /et instabilités possibles par le cisaillement.
- à suivre : balayage des paramètres dont k_2 et k_3 /zone de maximum de couplage …
- Régimes de croissance transitoire des modes MRI non axi/mode axi (cas N=0 ou non) , cf suite

Extensions (en cours) /analyse L

- cas stratifié (N) : nouveaux modes, dont notamment modification des ondes sonores par N (ex intérieurs stellaires), seuil MRI , Inst verticale cis… et étude de leurs couplages….
- à suivre : prédire les spectres en k , en t, et en fonction des paramètres …/GENERAL comparaison aux résultats des DNS…(anisotropie des transferts et cascades transversales à 3D)

Bilan d'énergie dans l'espace k : cas L+NL

Energies (NL) kinetic E_K, magnetic E_m, compressional E_ρ (and « reservoir of energy » by mean sheared flow) :

E_K (in L case : N_K=0)

$$(\hat{D}_t + 2k^2 / R_e)E_K = (q)I_K + I_{K-M} + N_K + R_{1\rho}$$

$$I_K = \hat{\mathrm{u}}_{\mathrm{x}}\hat{\mathrm{u}}_y^* + \hat{\mathrm{u}}_y\hat{\mathrm{u}}_{\mathrm{x}}^* \text{ (Reynolds stress), Re= Reynolds number}$$

$$I_{K-M} = i(B_{0z})k_z \sum_i \hat{\mathrm{u}}_i \hat{b}_i^* + cc$$

$$N_K = i\sum_i k_i \hat{\mathrm{u}}*_{\mathrm{xi}} \hat{N}_{\mathrm{xi,xi}} + cc \quad , \hat{N}_{\mathrm{xi,xi}} = TF(N_{\mathrm{xi,xi}}) = ...$$

$$R_{1\rho} = -i(\beta)\sum_i k_i (\hat{\mathrm{u}}*_i \hat{\rho} - \hat{\mathrm{u}}_i \hat{\rho}*)$$

Bilan d’énergie , suite 1

- E_m(in L case N_M=0)

$$(\hat{D}_t + 2k^2 / R_m + 2\mathrm{i}\overline{\mathrm{k}}.\hat{\mathrm{u}})E_m = (q)I_M + I_{M-K} + N_M$$

$(2\mathrm{i}\overline{\mathrm{k}}.\hat{\mathrm{u}})E_m$ new NL term...

$$I_M = -\hat{b}_{\mathrm{x}}\hat{b}_y^* - \hat{b}_y\hat{b}_{\mathrm{x}}^* \text{ (Maxwell stress) , Rm B Reynolds number}$$

$$I_{M-K} = -I_{K-M}$$

$$N_M = \sum_i \hat{b}*_{\mathrm{xi}}\hat{N}^m_{\mathrm{xi,xj}} + \mathrm{cc} \qquad \hat{N}^m_{\mathrm{xi,xj}} = TF(N^m_{\mathrm{xi,xj}}) = ...$$

Bilan d'énergie , suite 2

- $E^c(E\rho)$

$$\hat{D}_t E^c = R_{2\rho} + \hat{\rho}\hat{N}^{c*} + \hat{\rho}*\hat{N}^c$$

$$\hat{N}^c(\hat{\rho}\hat{)} = -i\sum_i k_i(\hat{u}_i \otimes \hat{\rho})$$

$$R_{2\rho} = -i(\hat{\rho}*\sum_i k_i \hat{\mathrm{u}}_i - \hat{\rho}\sum_i k_i \hat{\mathrm{u}}*_i)$$

Etot =EK+Em+Eρ
(L case)

$$\mathrm{dE_{tot}/dt{=}q(I_K{+}I_M){+}R_{1\rho}{+}R_{2\rho}} \quad (\nu' s{=}0)$$

Bilans d'énergie cas NL (eqs) (3)

- Etot (NL case)

$$dE_{tot}/dt + C = q(I_K + I_M) + R_{1\rho} + R_{2\rho} + N_K + N_m$$

$$C = (2k^2/Re)E_K + (2k^2/R_m)E_m + D,$$

$$D = (2ik.\hat{u})E_m, \nu's = 0:\ C = D$$

- Analyse des différents termes dans l'espace spectral (étude en cours, cas NL)

Bilan d’énergie dans l’espace k : cas NL (en cours/DNS)

- Redistribution anisotrope dans l‘espace k des énergies (pas de « transfert » comme pour les cascades usuelles) car somme des termes NL=0, Notion de cascade transverse…concentration des énergies dans l’espace spectral
- Opinion /motivation : “Régénération » par feedback des termes L par les termes NL en croissance transitoire : liée au succès des analyses linéaires /DNS…(dont ici TG MRI qui nourrit la turbulence ….)

Analysis of (NL) computational data

- Context (OK)
- Transient growth features
- Energy time evolution
- Concentration of energies and NL exchange terms in k space
- Caracteristics of acoustic waves in the NL regime

using the F eigenvector for SDW (work in progress) :

Compressibility effects (in course)

- F eigenvector for acoustic waves (explain)
- $F'' + \omega_+^2 F = rhs$…(data), cf compare source terms

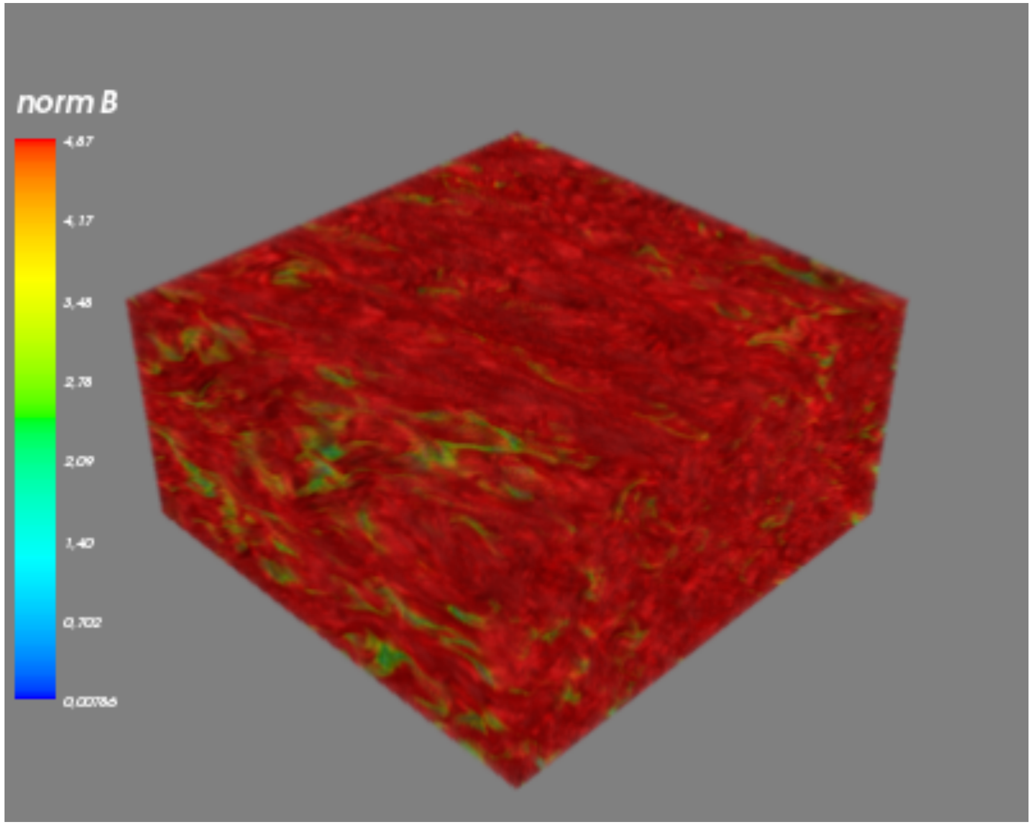
$$F = (\omega_+^2)\rho(t) - 2(i?)\Omega_0(ku_h(t)) = (\omega_+^2)\rho - 2a\Omega_0(ku_h) \text{ (a =free parameter for the moment)}$$

- $u_h = (k_1 u_2 - k_2 u_1)/k$ solenoidal horizontal velocity, $\omega_+^2 = (k(t)c_s)^2 - \omega_i^2$

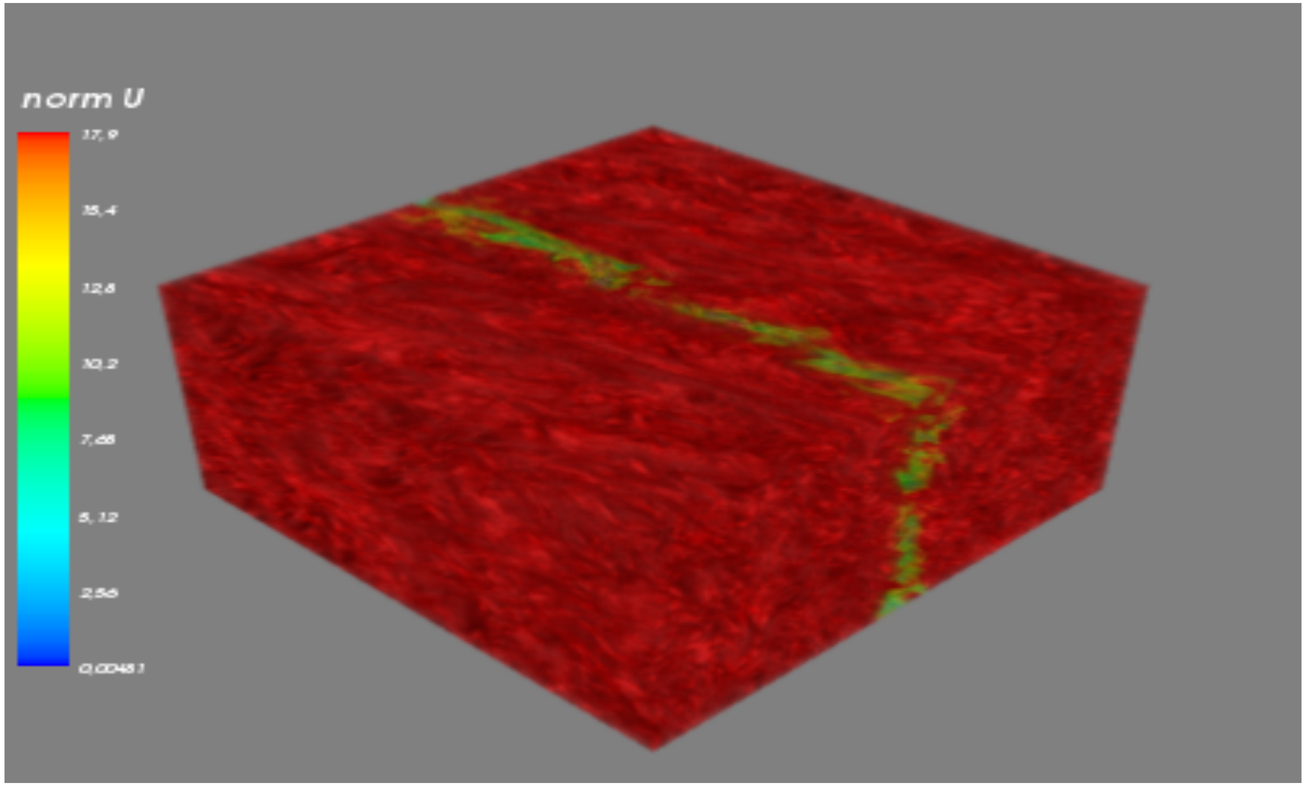
$$F' = (\omega_+^2)\rho' - 2a\Omega_0(ku_h)'(\text{WKB or } k_1\text{=cte}) + ((\omega_+^2)'\rho - 2a\Omega_0(k'_1(t)u_2) \text{ (second term if } k_1(t))$$

but if $k_1(t)$ ($q \neq 0$): ω_+ is no longer the exact eigenvalue...

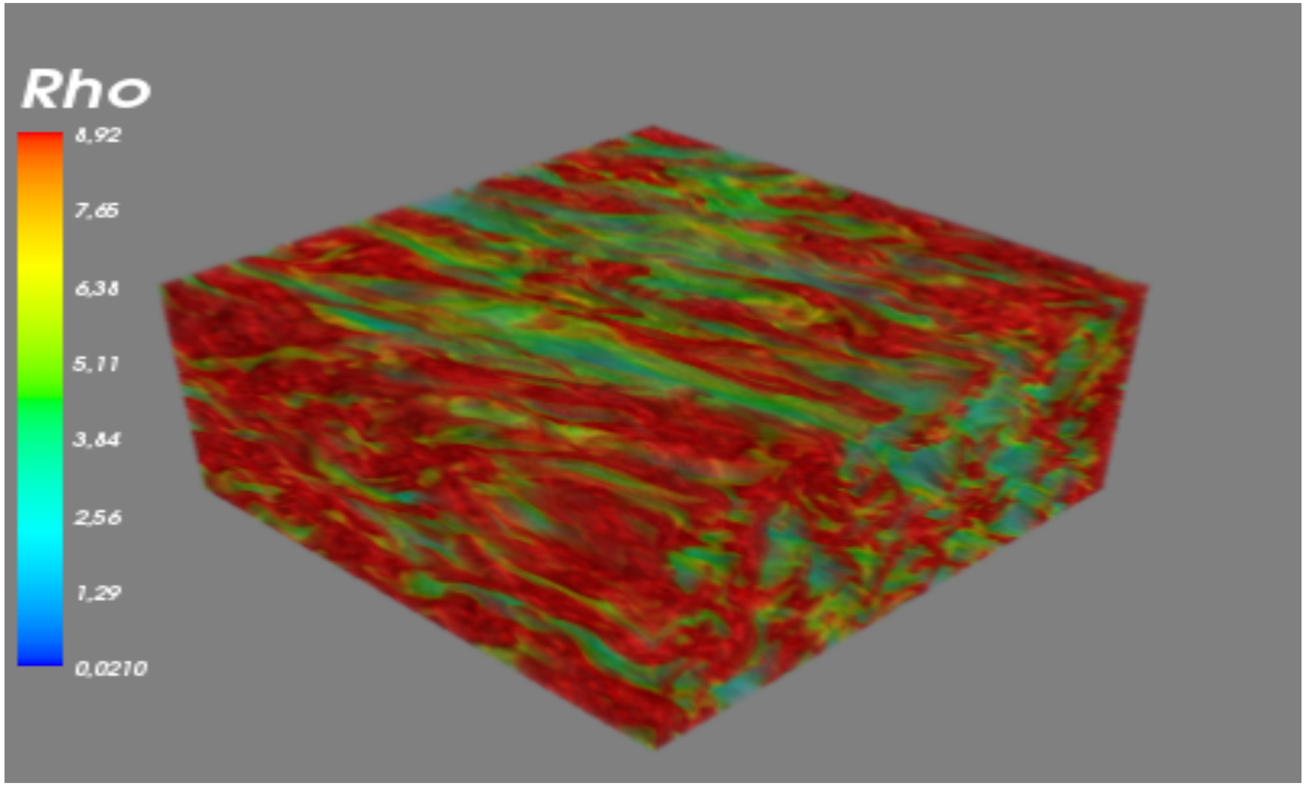
Some raw figures
B norm in space at fixed t



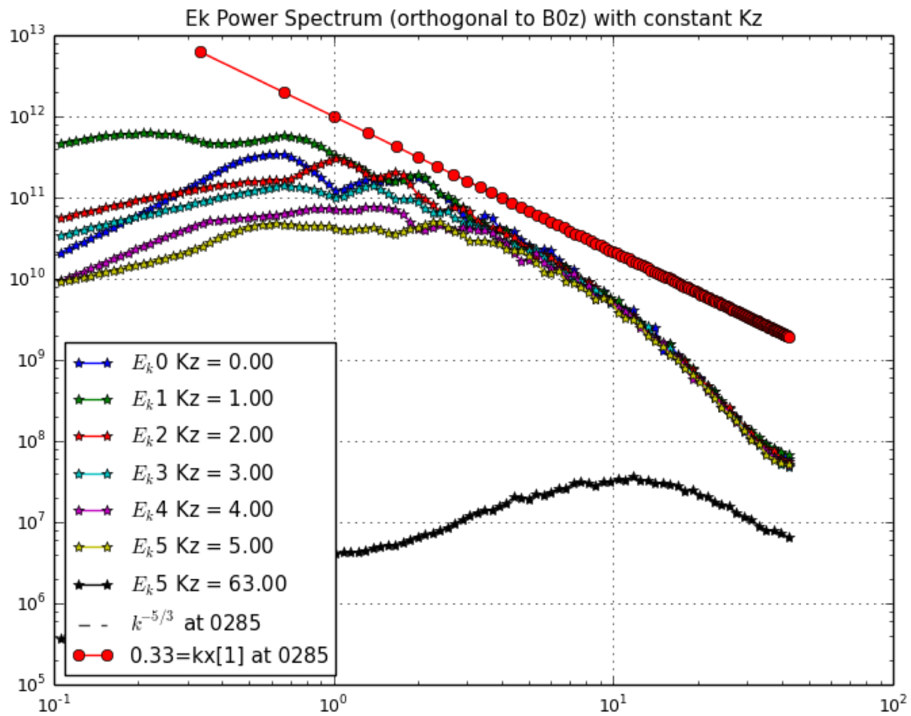
U norm in space at fixed time



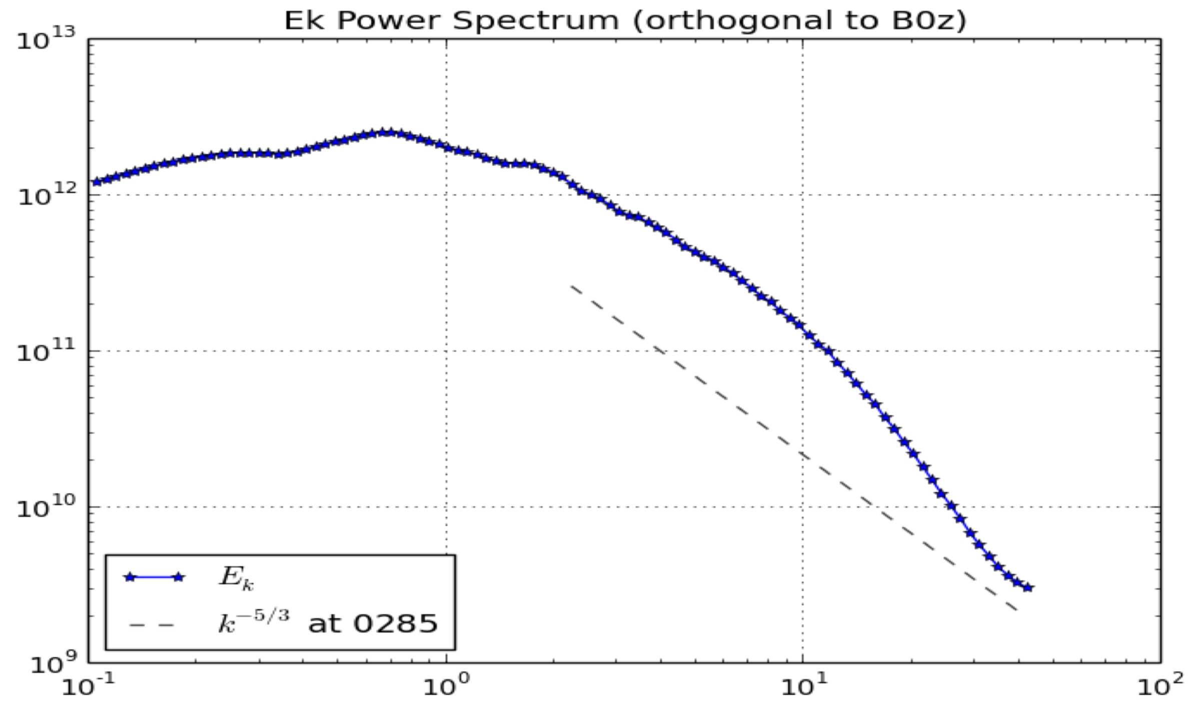
ρ norm in space at fixed time



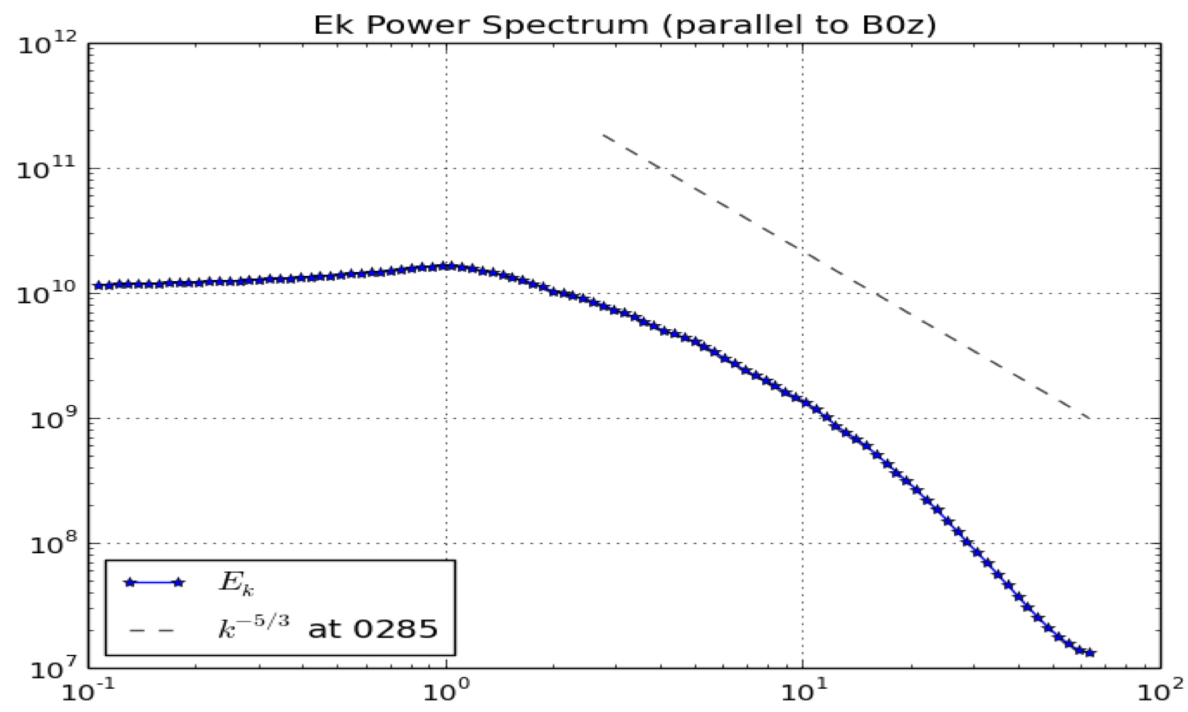
Some results all taken at t fixed in the saturated turbulent state
E_K power spectrum, slices at various kz



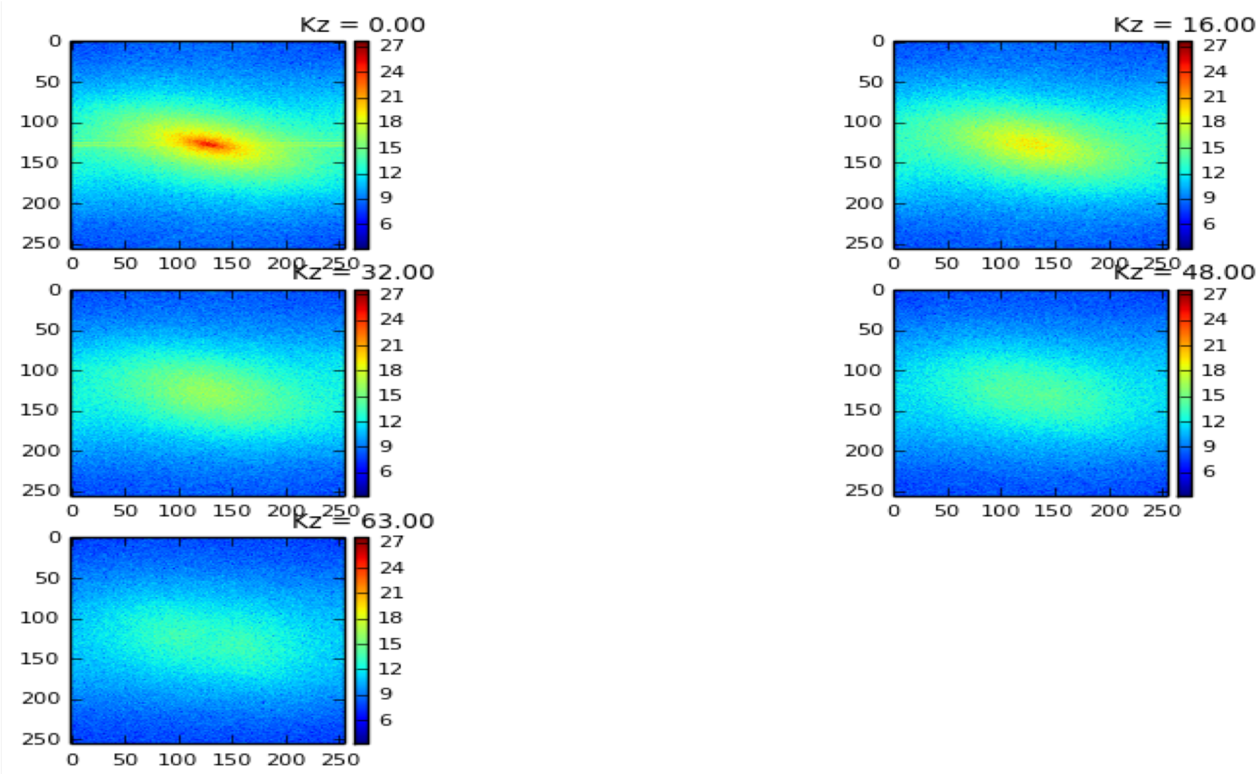
E_K power spectrum, orthogonal to B_{0z}



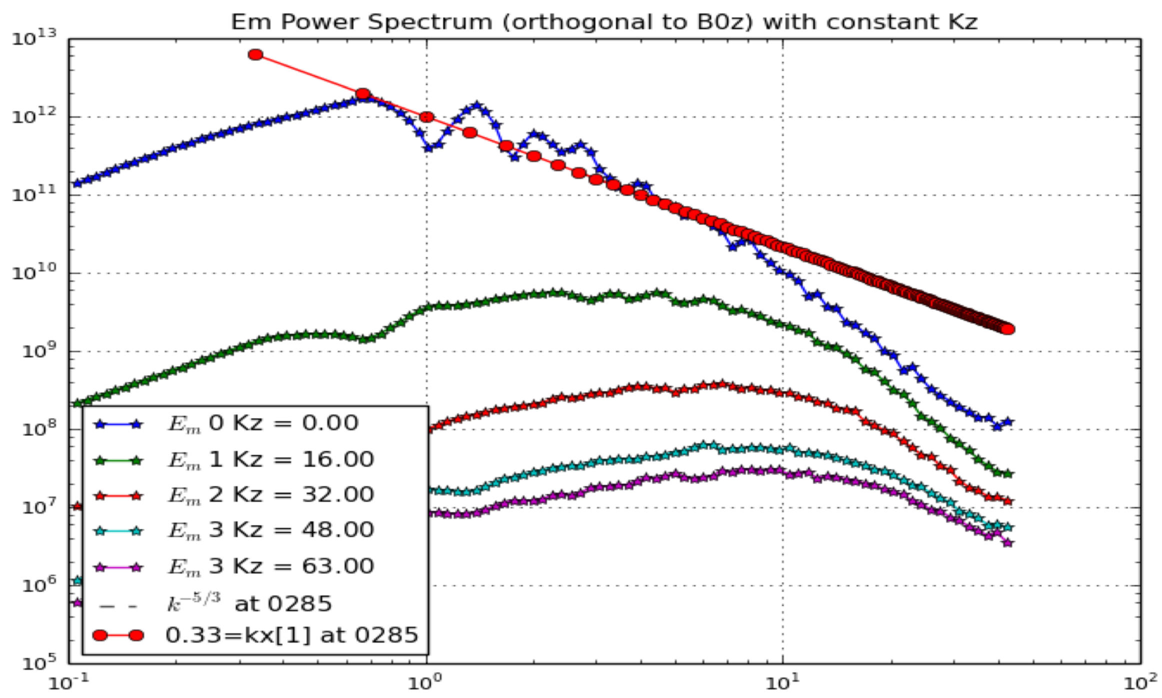
E_K power spectrum, parallel to B_{0z}



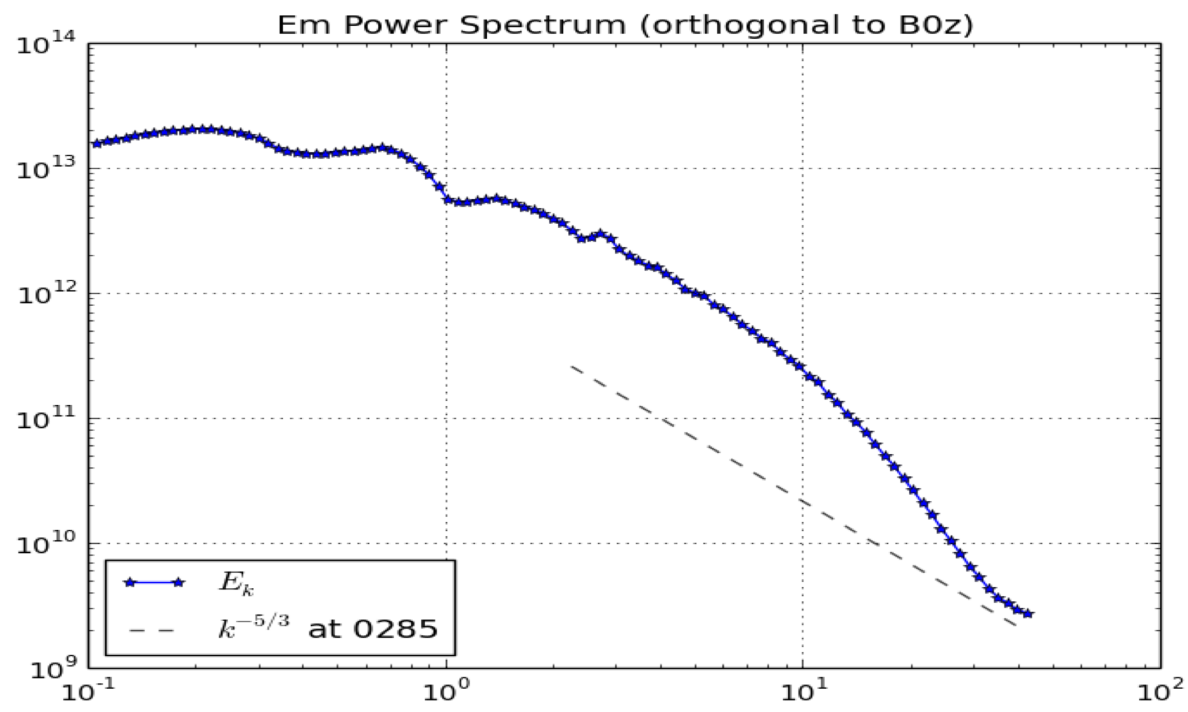
E_K slices in (kx,ky), at kz=cte



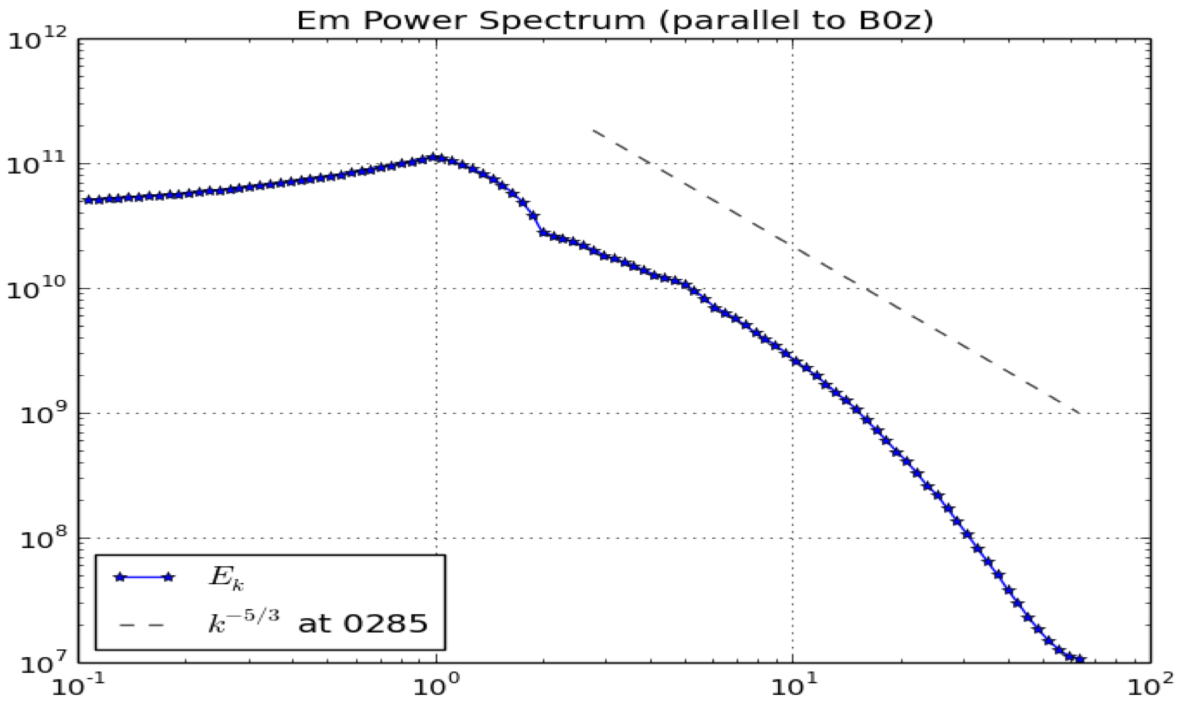
E_m power spectrum, slices at various kz



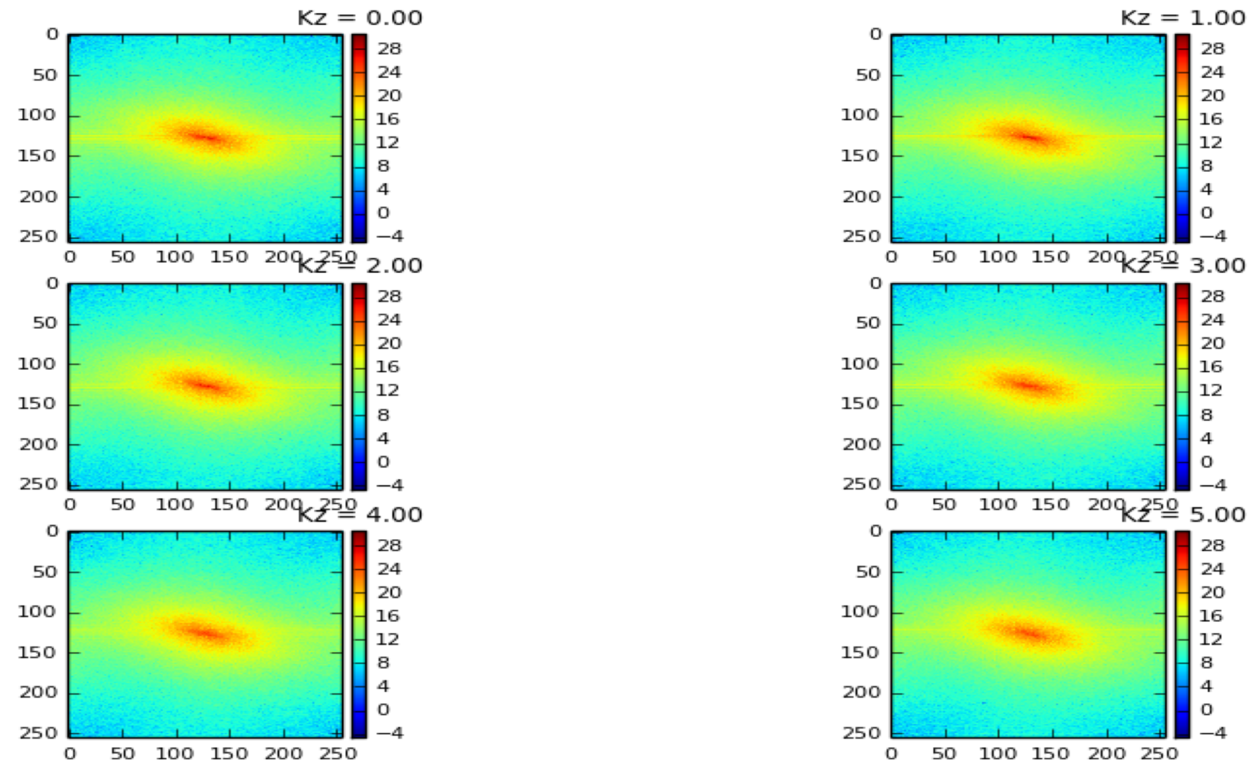
E_m power spectrum, orthogonal to B_{0z}



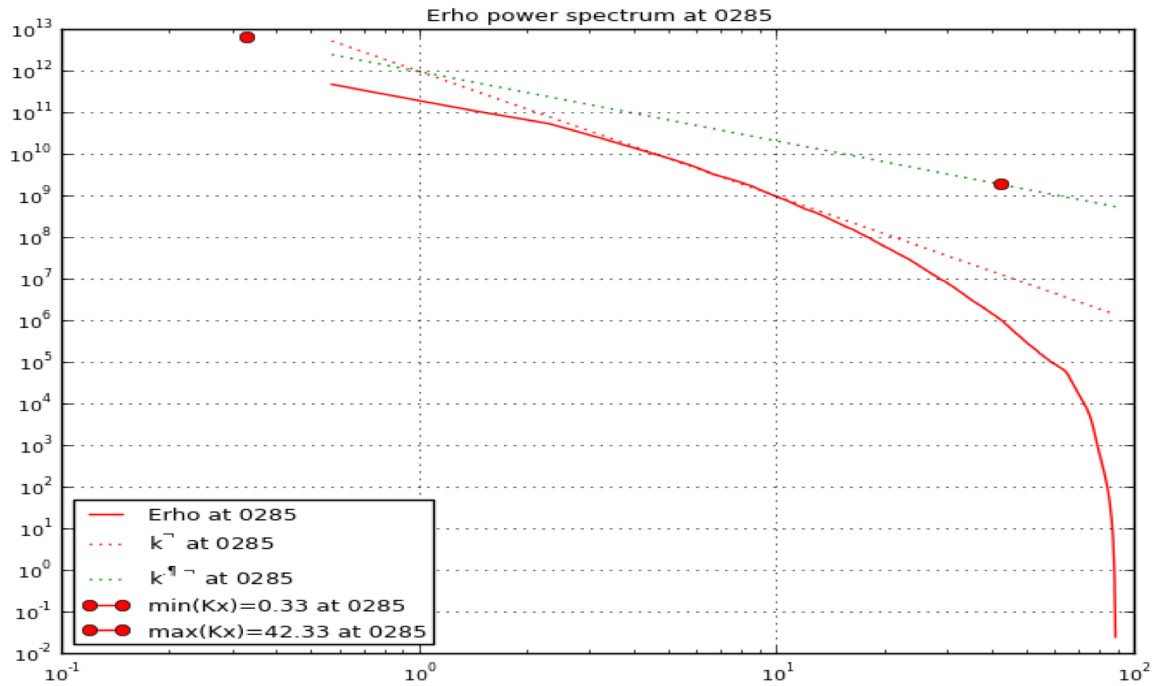
E_m power spectrum, parallel to B_{0z}



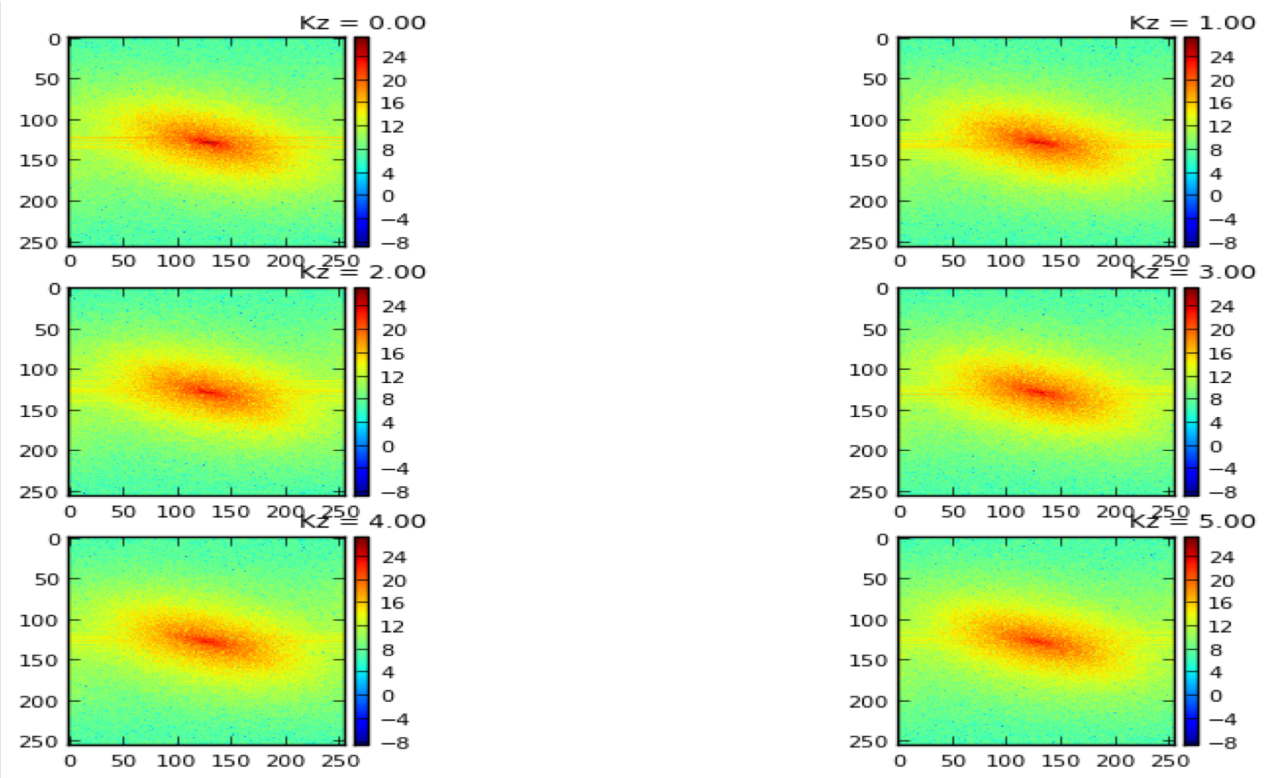
E_m slices in (kx,ky), at kz=cte



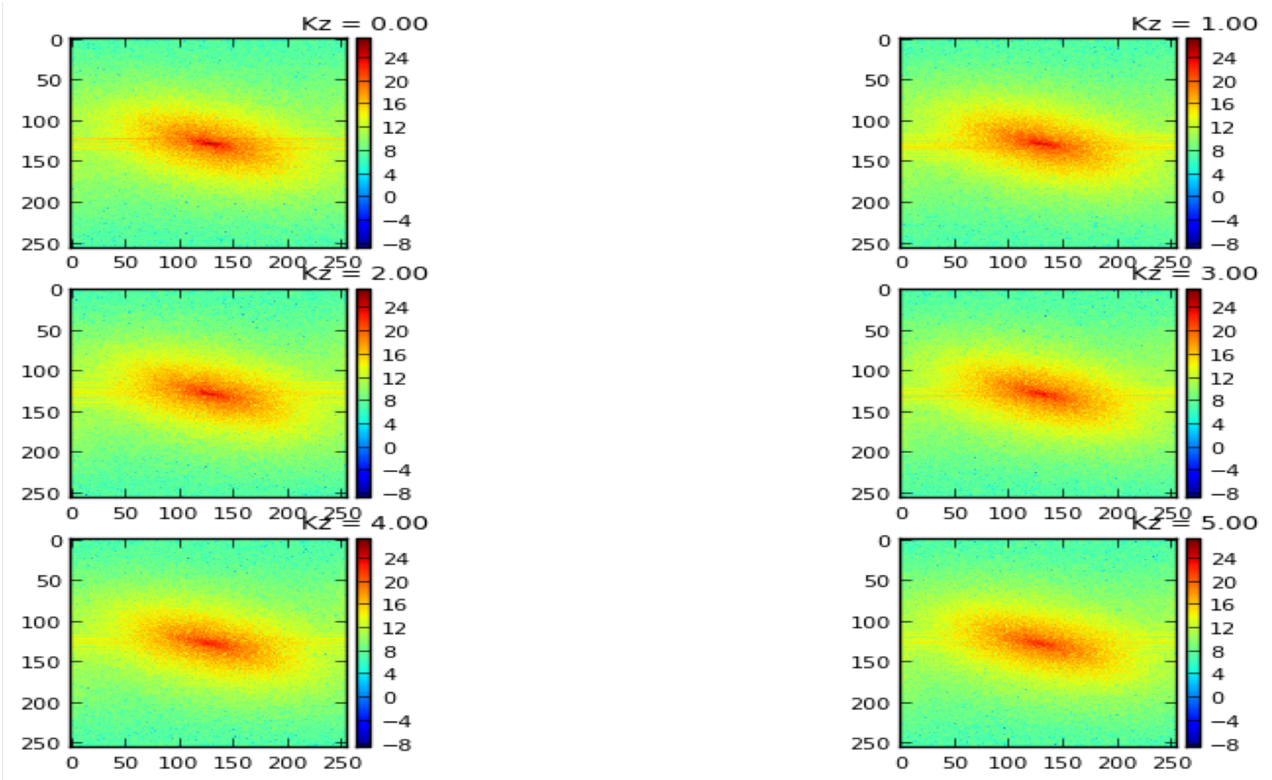
E_ρ (averaged) power spectrum,



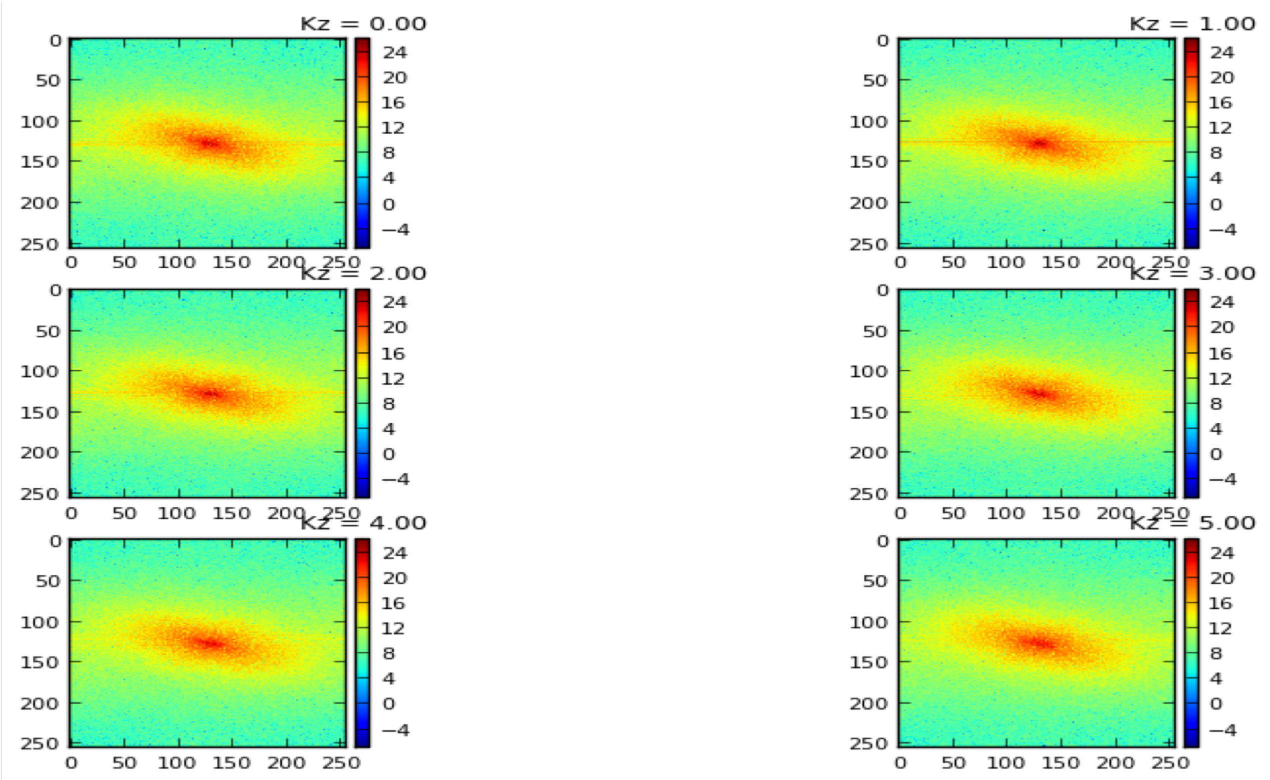
E_ρ slices in (kx,ky), at kz=cte



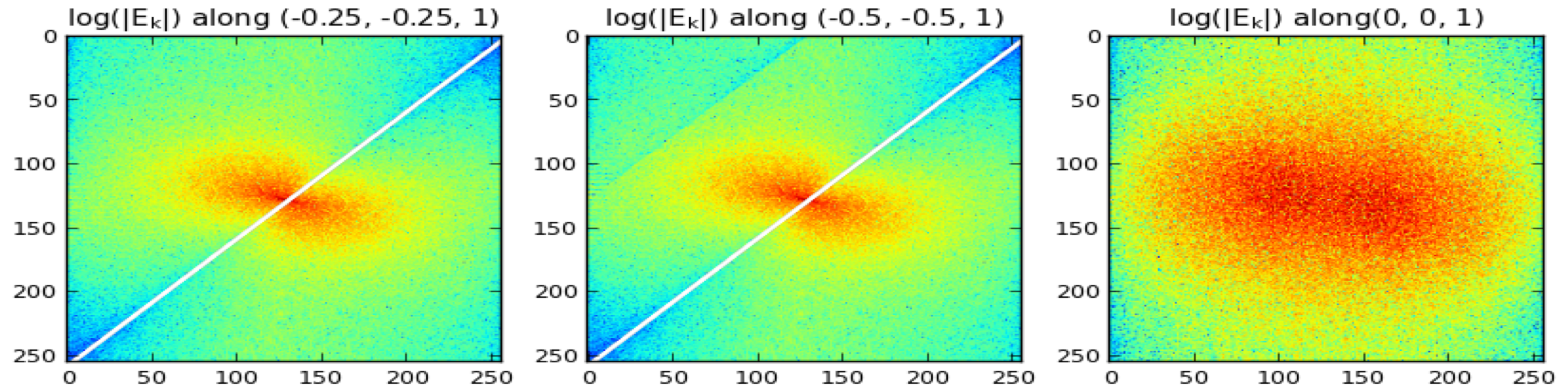
(idem)slices of rho(kx,ky,kz0,t0),at t fixed



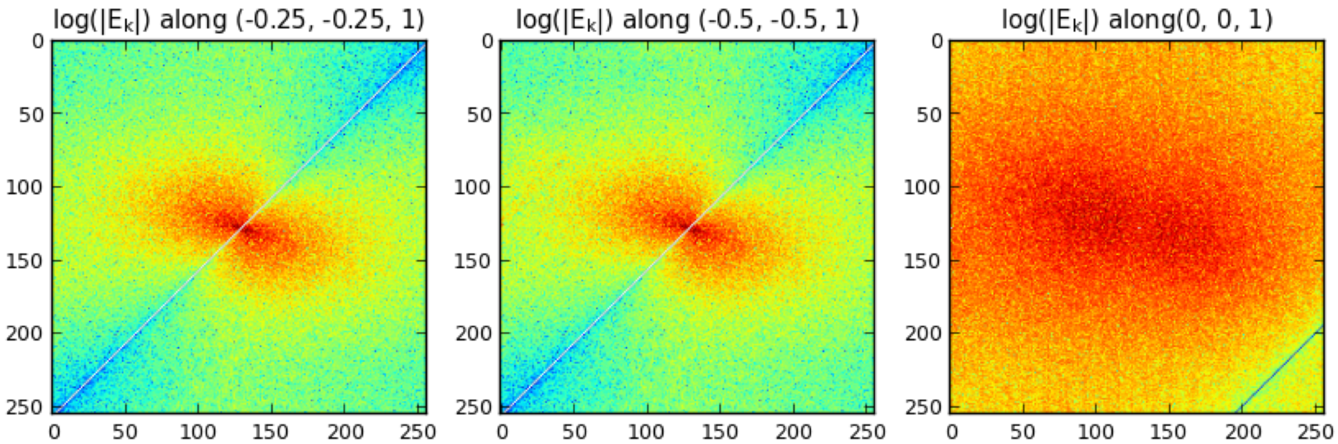
I_K (E_K injection term) slices in (kx,ky), at kz=cte



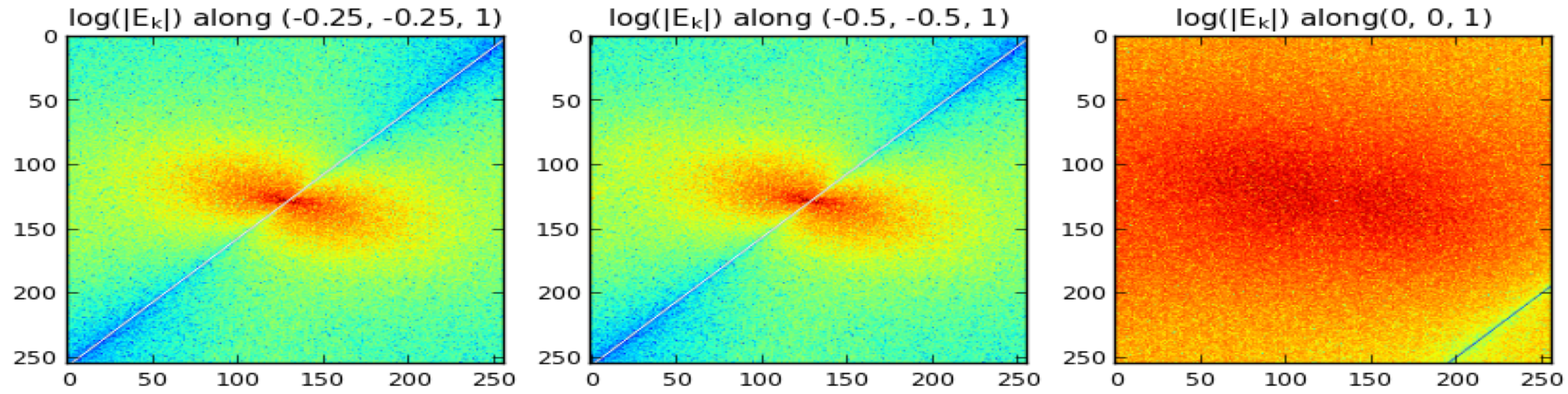
E_K source term I_{K-M} , cut plane



E_K source term N_K (errata legend) cut plane



E_K source term : $R_{1\rho}$ term, cut plane



To be analyzed more and for E_m and E_ρ source terms , comparison…

- In course of analysis …

Provisory Conclusions (NL data analysis part)

- Work in progress showing the features of TG for shear flows, to be continued
- See the main/dominant exchange terms among and inbetween Ek,Em and Eρ (new dl canal for exchange, decomposition wave-vortex)….

some of our related previous works/shear flows

- A. Salhi, S. Baklouti, F. Godeferd, T. Lehner and C. Cambon ; On the energy partition, scale by scale, in magnetic Archimedes Coriolis wave turbulence ; Physical Review E , 95, 023112, Feb 2017 ; DOI: 10.1103/PhysRevE.95.023112

. Secondary instability in horizontal shear flow under rapid rotation and strong stratification; S. Nasraoui, A. Salhi and T. Lehner, PRE, (2014)

- A. Salhi, T. Lehner, F. Godeferd, C. Cambon, Wave-Vortex mode coupling in astrophysical accretion disks under combined radial and vertical stratification , APJ , 771, n° 2 , (2013)
- A. Salhi, T. Lehner , F. Godeferd and C. Cambon, Magnetized stratified rotating shear waves, Phys.Rev E, 85, 026301,Février 2012.
- A. Salhi, T. Lehner and C. Cambon, Magnetohydrodynamic instabilities in rotating and precessing sheared flows: an asymptotic analysis ; Phys.Rev. E 82, 016315, 2010.